CS285 Spring 2009 - Homework 1
(due March 5, in class)

Professor David Parkes

Points will be awarded for clarity, correctness and completeness of the answers. Typed answers will get a 5% bonus, but typing in not required. Submissions should be electronic or printed in class. Each student will be alloted 3 late days throughout the semester. Delays exceeding 3 days will cause penalties of 5% each to the overall homework grade. Submissions more than 4 days late on any one homework will not be accepted. Students may work **in pairs** *and submit only one solution, but they* **both** *have to understand it.*

Total points: 315, Extra points: 50

1. (*45 points*) Consider the game matrix given below:

	L	R
U	$(3,2)$	$(2,2)$
M	$(1,1)$	$(0,0)$
D	$(0,0)$	$(1,1)$

   (a) (*5 points*) Eliminate strongly dominated strategies. Find all (pure and mixed) Nash equilibria.

   (b) (*10 points*) Eliminate all (strongly and weakly) dominated strategies. Find all Nash equilibria. Confirm that the order of elimination matters.

   (c) (*15 points*) Find the minmax, maxmin, and minmax regret strategies for the row player. In a general game (not just this example), should we remove dominated strategies first to calculate each of the above three solution concepts? Discuss why.

   (d) (*15 points*) Prove that, if the process of iterated elimination of strictly dominated strategies in a Normal-form game $G =< N, (S_i)_{i\in N}, (u_i)_{i\in N} >$

results in a *unique* strategy profile, $s^* = (s_1^*, \ldots, s_n^*)$, that this is a Nash eq. of the game. (**Hint:** By contradiction, suppose there exists some agent i for which $s_i \neq s_i^*$ is preferred over s_i^*, and show a contradiction with the fact that s_i was eliminated.)

2. (*20 points*) In a 2-player game with $|S_1| = m$ and $|S_2| = n$ (where $|X|$ is the number of elements in set X), how many combinations of supports need to be searched for Nash equilibria? What about an n-player game where $|S_i| = m_i, \forall i \in N$?

3. (*35 points*) In a certain market there are two firms, which we label a and b. If the firms produce output q_a and q_b, then the price they will receive for their goods is given by $p = \max\{0, \alpha - \beta(q_a + q_b)\}$. Each firm has marginal cost $c > 0$ and no fixed costs. Suppose $\alpha > 3c$.

   (a) (*5 points*) Suppose the firms choose the quantity q_a and q_b independently in each period, without regard to their behavior in previous periods, and each maximizes profits $\pi_a = (p-c)q_a$ and $\pi_b = (p-c)q_b$. Find the unique pure strategy Nash equilibrium of this game. This is called the *Cournot duopoly* model. (**Hint:** In equilibrium, each choice of q_a, q_b, is a best response to the other choice—you may use a graphical approach if it helps you.)

   (b) (*5 points*) Suppose the two firms collude by agreeing that each will produce amount $q^* = q_a = q_b$, and they have some way to enforce the agreement. What should they choose for q^*? What are their profits? This is called the *monopoly* or *cartel* model.

   (c) (*5 points*) Suppose firm a reneges on its promise in the previous part but b does not. What should a choose for q_a? What are a's profits, and b's profits?

   (d) (*10 points*) Suppose firm b discovers a's intentions and chooses q_b to maximize its profits, given that a is going to renege. What is q_b now? What do you think will happen if a also discovers that its intentions have been discovered, and they go back-and-forth like this?

   (e) (*10 points*) Suppose firm a gets to choose its output first (which is announced) and only afterward does firm b get to choose its own output. Find the equilibrium choices of q_a and q_b in this case. This is the *Stackelberg duopoly* model. (**Hint:** Firm a will need to calculate b's response q_b to every choice of q_a and then optimize its own choice of q_a with respect to this response.)

4. (*10 points*) In many distributed constraint optimization problems, each agent may have a complex local problem with multiple local variables and constraints, the details of which may be unimportant to the rest of the system. Explain a naive way to handle such a setting with the existing ABT algorithm, and suggest also a sketch of a more elegant, and computationally useful, adaptation of ABT. (Just a few sentences, with support for your argument, is fine.)

5. (*15 points*) Give a simple example to show why LRTA* finding the same path on two sequential trials need not imply that LRTA* has identified the shortest path. How would you fix the termination criteria? (**Hint:** this was discussed in lecture!) Also: what property of LRTA* is admissibility important for, and how would LRTA* fail if the heuristic was inadmissible?

6. (*10 points*) Give an example of a useful social law in human society (other than the one in the textbook), and one in artificial multi-agent systems. A couple of sentences for each is fine.

7. (*50 points*) Consider a single item allocation problem with 3 agents, with values 4, 8 and 10 respectively.

    (a) (*10 points*) What is the range of competitive equilibrium (CE) prices? [**Hint:** a feasible assignment and a price on the item is in competitive equilibrium in this problem if every agent maximizes its utility with the assignment at the price, and as long as the item is allocated whenever the price is non-zero.]

    (b) (*15 points*) Design a simple auction algorithm that terminates at the minimal CE price (or within ϵ of the price).

    (c) (*10 points*) Does the naive auction algorithm in the textbook (adapted to this special case of a single item) also terminate with ϵ of this minimal CE price?

    (d) (*15 points*) Is your algorithm manipulable? What about the naive auction algorithm? (Provide just a few sentences of justification for your answer—a technical proof is appreciated, but not required.)

8. (*15 points*) Provide an analog to Figure 3.11 (right hand side), giving another simplicial subdivision with a proper labeling, and draw on your simplicial subdivision all of the "Sperner's lemma walks" from one of the 2-faces of your simplex. Confirm that you succeed in identifying an odd number of completely labeled subsimplexes, and also draw in any walks that are "paired up" on the interior of the simplex between two completely-labeled subsimplexes.

9. (*15 points*) Define two other fixpoint theorems, in addition to Brouwer's, and provide a very brief discussion to explain how they differ from Brouwer's fixpoint theorem. [*Cite any sources that you use.*]

10. (*50 points*) Axiomatic decision theory: Consider a binary relation $\succ$. Such a relation is said to be *asymmetric* if $x \succ y$ implies $y \not\succ x$. It is called *negatively transitive* if $x \not\succ y$ and $y \not\succ z$ implies $x \not\succ z$.

    (a) (*5 points*) Do you think that these two properties are reasonable for preference relations?

    (b) (*10 points*) Prove that if a relation is asymmetric and negatively transitive, it is also transitive (i.e., $x \succ y$ and $y \succ z$ imply $x \succ z$).

(c) (*35 points*) Prove that if $\succ$ is negatively transitive, then $x \succ z$ implies that, for all y, either $x \succ y$ or $y \succ z$.

11. (*50 points*) Discussion: How close do you think game theory and the study of multi-agent systems lie (or should lie)? Which aspects of a "real" system does a game-theoretic analysis overlook? Conversely, how bad is the effect of "real" systems not being designed with game-theoretic considerations in mind? Please try to include in your analysis aspects such as: (*i*) technical limitations (hardware failures, computational capacity, time constraints, etc.), (*ii*) considerations of whether "playing Nash" is the best thing to do, (*iii*) uncertainty, dynamic environments and learning, (*iv*) large-scale systems, (*v*) human behavior, and others. Your answers need not be technical, but should not be simplistic either.

12. (Extra credit *50 points*) Consider the 1-simplex on the real line, defined on points $x^0 = 0$ and $x^1 = 1$. Consider a continuous function $f : [0, 1] \rightarrow [0, 1]$. Draw the 1-simplex and a simplicial subdivision. Above this annotated line, illustrate the $f(x)$ function by using the "y-axis" direction on your paper to plot the value of $f(x)$ as x varies from 0 to 1. Now add proper labels to the vertices of your subsimplexes, following the rule used in the proof of Brouwer's fixpoint theorem that requires $L(v) \in \chi(v) \cap \{i : f_i(v) \leq v\}$. Finally, show that the fixed point of f is within the one completely-labeled subsimplex.

Good luck!