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Introduction[1]

ONE OF THE EARLIEST and most enduring models of the behavior of security prices is the Random Walk Hypothesis, an idea that was conceived in the sixteenth century as a model of games of chance.[2] Closely tied to the birth of probability theory, the Random Walk Hypothesis has had an illustrious history, with remarkable intellectual forbears such as Bachelier, Einstein, Lévy, Kolmogorov, and Wiener.

More recently, and as with so many of the ideas of modern economics, the first serious application of the Random Walk Hypothesis to financial markets can be traced back to Paul Samuelson (1965), whose contribution is neatly summarized by the title of his article: "Proof that Properly Anticipated Prices Fluctuate Randomly." In an informationally efficient market—not to be confused with an allocationally or Pareto-efficient market—price changes must be unforecastable if they are properly anticipated, i.e., if they fully incorporate the expectations and information of all market participants. Fama (1970) encapsulated this idea in his pithy dictum that "prices fully reflect all available information."

Unlike the many applications of the Random Walk Hypothesis in the natural and physical sciences in which randomness is assumed almost by default, because of the absence of any natural alternatives, Samuelson argues that randomness is achieved through the active participation of many investors seeking greater wealth. Unable to curtail their greed, an army of investors aggressively pounce on even the smallest informational advantages at their disposal, and in doing so, they incorporate their information into market prices and quickly eliminate the profit opportunities that gave rise to their aggression. If this occurs instantaneously, which it must in an idealized world of "frictionless" markets and costless trading, then prices must always fully reflect all available information and no profits can be gar-

[1]Parts of this introduction are adapted from Lo (1997a,b) and Lo and MacKinlay (1998).
[2]See, for example, Hald (1990, Chapter 4).

nered from information-based trading (because such profits have already been captured). This has a wonderfully counter-intuitive and seemingly contradictory flavor to it: the more efficient the market, the more random the sequence of price changes generated by such a market, and the most efficient market of all is one in which price changes are completely random and unpredictable.

For these reasons, the Random Walk Hypothesis and its close relative, the Efficient Markets Hypothesis, have become icons of modern financial economics that continue to fire the imagination of academics and investment professionals alike. The papers collected in this volume comprise our own foray into this rich literature, spanning a decade of research that we initiated in 1988 with our rejection of the Random Walk Hypothesis for US stock market prices, and then following a course that seemed, at times, to be self-propelled, the seeds of our next study planted by the results of the previous one.

If there is one central theme that organizes the papers contained in this volume, it is this: financial markets *are* predictable to some degree, but far from being a symptom of inefficiency or irrationality, predictability is the oil that lubricates the gears of capitalism. Indeed, quite by accident and rather indirectly, we have come face to face with an insight that Ronald Coase hit upon as an undergraduate over half a century ago: price discovery is neither instantaneous nor costless, and frictions play a major role in determining the nature of competition and the function of markets.

1.1 The Random Walk and Efficient Markets

One of the most common reactions to our early research was surprise and disbelief. Indeed, when we first presented our rejection of the Random Walk Hypothesis at an academic conference in 1986, our discussant—a distinguished economist and senior member of the profession—asserted with great confidence that we had made a programming error, for if our results were correct, this would imply tremendous profit opportunities in the stock market. Being too timid (and too junior) at the time, we responded weakly that our programming was quite solid thank you, and the ensuing debate quickly degenerated thereafter. Fortunately, others were able to replicate our findings exactly, and our wounded pride has healed quite nicely with the passage of time (though we still bristle at the thought of being prosecuted for programming errors without "probable cause"). Nevertheless, this experience has left an indelible impression on us, forcing us to confront the fact that the Random Walk Hypothesis was so fully ingrained into the canon of our profession that it was easier to attribute our empirical results to programming errors than to accept them at face value.

Is it possible for stock market prices to be predictable to some degree in an efficient market?

This question hints at the source of disbelief among our early critics: an implicit—and incorrect—link between the Random Walk Hypothesis and the Efficient Markets Hypothesis. It is not difficult to see how the two ideas might be confused. Under very special circumstances, e.g., risk neutrality, the two are equivalent. However, LeRoy (1973), Lucas (1978), and many others have shown in many ways and in many contexts that the Random Walk Hypothesis is neither a necessary nor a sufficient condition for rationally determined security prices. In other words, unforecastable prices need not imply a well-functioning financial market with rational investors, and forecastable prices need not imply the opposite.

These conclusions seem sharply at odds with Samuelson's "proof" that properly anticipated prices fluctuate randomly, an argument so compelling that it is reminiscent of the role that uncertainty plays in quantum mechanics. Just as Heisenberg's uncertainty principle places a limit on what we can know about an electron's position and momentum if quantum mechanics holds, Samuelson's version of the Efficient Markets Hypothesis places a limit on what we can know about future price changes if the forces of economic self-interest hold.

Nevertheless, one of the central insights of modern financial economics is the necessity of some trade-off between risk and expected return, and although Samuelson's version of the Efficient Markets Hypothesis places a restriction on expected returns, it does not account for risk in any way. In particular, if a security's expected price change is positive, it may be just the reward needed to attract investors to hold the asset and bear the associated risks. Indeed, if an investor is sufficiently risk averse, he might gladly *pay* to avoid holding a security that has unforecastable returns.

In such a world, the Random Walk Hypothesis—a purely statistical model of returns—need not be satisfied even if prices do fully reflect all available information. This was demonstrated conclusively by LeRoy (1973) and Lucas (1978), who construct explicit examples of informationally efficient markets in which the Efficient Markets Hypothesis holds but where prices do not follow random walks.

Grossman (1976) and Grossman and Stiglitz (1980) go even further. They argue that perfectly informationally efficient markets are an *impossibility*, for if markets are perfectly efficient, the return to gathering information is nil, in which case there would be little reason to trade and markets would eventually collapse. Alternatively, the degree of market *inefficiency* determines the effort investors are willing to expend to gather and trade on information, hence a non-degenerate market equilibrium will arise only when there are sufficient profit opportunities, i.e., inefficiencies, to compensate investors for the costs of trading and information-gathering. The profits earned by these industrious investors may be viewed as economic rents that accrue to those willing to engage in such activities. Who are the providers of these rents? Black (1986) gives us a provocative answer: noise traders, individuals who trade on what they think is information but is in fact merely noise. More generally, at any time there are always investors who trade for reasons other than information—for example, those with unexpected liquidity needs—and these investors are willing to "pay up" for the privilege of executing their trades immediately.

These investors may well be losing money on average when they trade with information-motivated investors, but there is nothing irrational or inefficient about either group's behavior. In fact, an investor may be trading for liquidity reasons one day and for information reasons the next, and losing or earning money depending on the circumstances surrounding the trade.

1.2 The Current State of Efficient Markets

There is an old joke, widely told among economists, about an economist strolling down the street with a companion when they come upon a $100 bill lying on the ground. As the companion reaches down to pick it up, the economist says "Don't bother—if it were a real $100 bill, someone would have already picked it up."

This humorous example of economic logic gone awry strikes dangerously close to home for students of the Efficient Markets Hypothesis, one of the most important controversial and well-studied propositions in all the social sciences. It is disarmingly simple to state, has far-reaching consequences for academic pursuits and business practice, and yet is surprisingly resilient to empirical proof or refutation. Even after three decades of research and literally thousands of journal articles, economists have not yet reached a consensus about whether markets—particularly financial markets—are efficient or not.

What can we conclude about the Efficient Markets Hypothesis? Amazingly, there is still no consensus among financial economists. Despite the many advances in the statistical analysis, databases, and theoretical models surrounding the Efficient Markets Hypothesis, the main effect that the large number of empirical studies have had on this debate is to harden the resolve of the proponents on each side.

One of the reasons for this state of affairs is the fact that the Efficient Markets Hypothesis, by itself, is not a well-defined and empirically refutable hypothesis. To make it operational, one must specify additional structure, e.g., investors' preferences, information structure, business conditions, etc. But then a test of the Efficient Markets Hypothesis becomes a test of several auxiliary hypotheses as well, and a rejection of such a joint hypothesis tells us little about which aspect of the joint hypothesis is inconsistent with the data. Are stock prices too volatile because markets are inefficient, or is it due to risk aversion, or dividend smoothing? All three inferences are consistent with the data. Moreover, new statistical tests designed to distinguish among them will no doubt require auxiliary hypotheses of their own which, in turn, may be questioned.

More importantly, tests of the Efficient Markets Hypothesis may not be the most informative means of gauging the efficiency of a given market. What is often of more consequence is the *relative* efficiency of a particular market, relative to other markets, e.g., futures vs. spot markets, auction vs. dealer markets, etc. The advantages of the concept of relative efficiency, as opposed to the all-or-nothing notion of absolute efficiency, are easy to spot by way of an analogy. Physical systems are often given an efficiency rating based on the relative proportion of energy or fuel converted to useful work. Therefore, a piston engine may be rated at 60% efficiency, meaning that on average 60% of the energy contained in the engine's fuel is used to turn the crankshaft, with the remaining 40% lost to other forms of work, e.g., heat, light, noise, etc.

Few engineers would ever consider performing a statistical test to determine whether or not a given engine is perfectly efficient—such an engine exists only in the idealized frictionless world of the imagination. But measuring relative efficiency—relative to a frictionless ideal—is commonplace. Indeed, we have come to expect such measurements for many household products: air conditioners, hot water heaters, refrigerators, etc. Therefore, from a practical point of view, and in light of Grossman and Stiglitz (1980), the Efficient Markets Hypothesis is an idealization that is economically unrealizable, but which serves as a useful benchmark for measuring relative efficiency.

A more practical version of the Efficient Markets Hypothesis is suggested by another analogy, one involving the notion of thermal equilibrium in statistical mechanics. Despite the occasional "excess" profit opportunity, on average and over time, it is not possible to earn such profits consistently without some type of competitive advantage, e.g., superior information, superior technology, financial innovation, etc. Alternatively, in an efficient market, the only way to earn positive profits *consistently* is to develop a competitive advantage, in which case the profits may be viewed as the economic rents that accrue to this competitive advantage. The consistency of such profits is an important qualification—in this version of the Efficient Markets Hypothesis, an occasional free lunch is permitted, but free lunch plans are ruled out.

To see why such an interpretation of the Efficient Markets Hypothesis is a more practical one, consider for a moment applying the classical version of the Efficient Markets Hypothesis to a non-financial market, say the market for biotechnology. Consider, for example, the goal of developing a vaccine for the AIDS virus. If the market for biotechnology is efficient in the classical sense, such a vaccine can *never* be developed—if it could, someone would have already done it! This is clearly a ludicrous presumption since it ignores the difficulty and gestation lags of research and development in biotechnology. Moreover, if a pharmaceutical company does succeed in developing such a vaccine, the profits earned would be measured in the billions of dollars. Would this be considered "excess" profits, or economic rents that accrue to biotechnology patents?

Financial markets are no different in principle, only in degrees. Consequently, the profits that accrue to an investment professional need not be a market *inefficiency*, but may simply be the fair reward to breakthroughs in financial technology. After all, few analysts would regard the hefty profits of Amgen over the past few years as evidence of an inefficient market for pharmaceuticals—Amgen's recent profitability is readily identified with the development of several new drugs (Epogen, for example, a drug that stimulates the production of red blood cells), some considered breakthroughs in biotechnology. Similarly, even in efficient financial markets there are very handsome returns to breakthroughs in financial technology.

Of course, barriers to entry are typically lower, the degree of competition is much higher, and most financial technologies are not patentable (though this may soon change) hence the "half life" of the profitability of financial innovation is considerably smaller. These features imply that financial markets should be relatively more efficient, and indeed they are. The market for "used securities" is considerably more efficient than the market for used cars. But to argue that financial markets must be perfectly efficient is tantamount to the claim that an AIDS vaccine cannot be found. In an efficient market, it is difficult to earn a good living, but not impossible.

1.3 Practical Implications

Our research findings have several implications for financial economists and investors. The fact that the Random Walk Hypothesis hypothesis can be rejected for recent US equity returns suggests the presence of predictable components in the stock market. This opens the door to superior long-term investment returns through disciplined active investment management. In much the same way that innovations in biotechnology can garner superior returns for venture capitalists, innovations in financial technology can garner equally superior returns for investors.

However, several qualifications must be kept in mind when assessing which of the many active strategies currently being touted is appropriate for an particular investor. First, the riskiness of active strategies can be very different from passive strategies, and such risks do not necessarily "average out" over time. In particular, an investor's risk tolerance must be taken into account in selecting the long-term investment strategy that will best match the investor's goals. This is no simple task since many investors have little understanding of their own risk preferences, hence consumer education is perhaps the most pressing need in the near term. Fortunately, computer technology can play a major role in this challenge, providing scenario analyses, graphical displays of potential losses and gains, and realistic simulations of long-term investment performance that are user-friendly and easily incorporated into an investor's world view. Nevertheless, a good understanding of the investor's understanding of the nature of financial risks and rewards is the natural starting point for the investment process.

Second, there are a plethora of active managers vying for the privilege of managing institutional and pension assets, but they cannot all outperform the market every year (nor should we necessarily expect them to). Though often judged against a common benchmark, e.g., the S&P 500, active strategies can have very diverse risk characteristics and these must be weighed in assessing their performance. An active strategy involving high-risk venture-capital investments will tend to outperform the S&P 500 more often than a less aggressive "enhanced indexing" strategy, yet one is not necessarily better than the other.

In particular, past returns should not be the *sole* or even the *major* criterion by which investment managers are judged. This statement often surprises investors and finance professionals—after all, isn't this the bottom line? Put another way, "If it works, who cares why?". Selecting an investment manager this way is one of the surest paths to financial disaster. Unlike the experimental sciences such as physics and biology, financial economics (and most other social sciences) relies primarily on statistical inference to test its theories. Therefore, we can never know with perfect certainty that a particular investment strategy is successful since even the most successful strategy can always be explained by pure luck (see Chapter 8 for some concrete illustrations).

Of course, some kinds of success are easier to attribute to luck than others, and it is precisely this kind of attribution that must be performed in deciding on a particular active investment style. Is it luck, or is it genuine?

While statistical inference can be very helpful in tackling this question, in the final analysis the question is not about statistics, but rather about economics and financial innovation. Under the practical version of the Efficient Markets Hypothesis, it is difficult—but not impossible—to provide investors with consistently superior investment returns. So what are the sources of superior performance promised by an active manager and why have other competing managers not recognized these opportunities? Is it better mathematical models of financial markets? Or more accurate statistical methods for identifying investment opportunities? Or more timely data in a market where minute delays can mean the difference between profits and losses? Without a compelling argument for *where* an active manager's value-added is coming from, one must be very skeptical about the prospects for future performance. In particular, the concept of a "black box"—a device that performs a known function reliably but obscurely—may make sense in engineering applications where repeated experiments can validate the reliability of the box's performance, but has no counterpart in investment management where performance attribution is considerably more difficult. For analyzing investment strategies, it matters a great deal *why* a strategy is supposed to work.

Finally, despite the caveats concerning performance attribution and proper motivation, we *can* make some educated guesses about where the likely sources of value-added might be for active investment management in the near future.

- The revolution in computing technology and datafeeds suggest that highly computation-intensive strategies—ones that could not have been implemented five years ago—that exploit certain regularities in securities prices, e.g., clientele biases, tax opportunities, information lags, can add value.
- Many studies have demonstrated the enormous impact that transactions costs can have on long-term investment performance. More sophisticated methods for measuring and controlling transactions costs—methods which employ high-frequency data, economic models of price impact, and advanced optimization techniques—can add value. Also, the introduction of financial instruments that reduce transactions costs, e.g., swaps, options, and other derivative securities, can add value.
- Recent research in psychological biases inherent in human cognition suggest that investment strategies exploiting these biases can add value. However, contrary to the recently popular "behavioral" approach to investments which proposes to take advantage of individual "irrationality," I suggest that value-added comes from creating investments with more attractive risk-sharing characteristics suggested by psychological models. Though the difference may seem academic, it has far-reaching consequences for the long-run performance of such strategies: taking advantage of individual irrationality cannot be a recipe for long-term success, but providing a better set of opportunities that more closely matches what investors desire seems more promising.

Of course, forecasting the sources of future innovations in financial technology is a treacherous business, fraught with many half-baked successes and some embarrassing failures. Perhaps the only reliable prediction is that the innovations of future are likely to come from unexpected and underappreciated sources. No one has illustrated this principal so well as Harry Markowitz, the father of modern portfolio theory and a winner of the 1990 Nobel Prize in economics. In describing his experience as a Ph.D. student on the eve of his graduation in the following way, he wrote in his Nobel address:

> . . . [W]hen I defended my dissertation as a student in the Economics Department of the University of Chicago, Professor Milton Friedman argued that portfolio theory was not Economics, and that they could not award me a Ph.D. degree in Economics for a dissertation which was not Economics. I assume that he was only half serious, since they did award me the degree without long debate. As to the merits of his arguments, at this point I am quite willing to concede: at the time I defended my dissertation, portfolio theory was not part of Economics. But now it is.

It is our hope and conceit that the research contained in this volume will be worthy of the tradition that Markowitz and others have so firmly established.

Part I

THE FIVE CHAPTERS IN THIS FIRST PART focus squarely on whether the Random Walk Hypothesis is a plausible description of recent US stock market prices. At the time we started our investigations—in 1985, just a year after we arrived at the Wharton School—the Random Walk Hypothesis was taken for granted as gospel truth. A number of well-known empirical studies had long since established the fact that markets were "weak-form efficient" in Roberts's (1967) terminology, implying that past prices could not be used to forecast future prices changes (see, for example, Cowles and Jones (1973), Kendall (1953), Osborne (1959, 1962), Roberts (1959, 1967), Larson (1960), Cowles (1960), Working (1960), Alexander (1961, 1964), Granger and Morgenstern (1963), Mandelbrot (1963), Fama (1965), and Fama and Blume (1966)). And although some of these studies did find evidence against the random walk, e.g., Cowles and Jones (1973), they were largely dismissed as statistical anomalies or not economically meaningful after accounting for transactions costs, e.g., Cowles (1960). For example, after conducting an extensive empirical analysis of the "runs' of US stock returns from 1956 to 1962, Fama (1965) concludes that, "... there is no evidence of important dependence from either an investment or a statistical point of view."

It was in this milieu that we decided to revisit the Random Walk Hypothesis. Previous studies had been unable to reject the random walk, hence we surmised that perhaps a more sensitive statistical test was needed, one capable of detecting small but significant departures from pure randomness. In the jargon of statistical inference, we hoped to develop a more "powerful" test, a test that has a higher probability of rejecting the Random Walk Hypothesis if it is indeed false. Motivated partly by an insight of Merton's (1980), that variances can be estimated more accurately than means when data is sampled at finer intervals, we proposed a test of the random walk based on a comparison of variances at different sampling intervals. And by casting the comparison as a Hausman (1978) specification test, we were able to obtain an asymptotic sampling theory for the variance ratio statistic almost immediately, which we later generalized and extended in many ways. These results and their empirical implementation are described in Chapter 2.

In retrospect, our motivation for the variance ratio test was completely unnecessary.

Although Merton's (1980) observation holds quite generally, the overwhelming rejections of the Random Walk Hypothesis that we obtained for weekly US stock returns from 1962 to 1985 implied that a more powerful test was not needed—the random walk could have been rejected on the basis of the simple first-order autocorrelation coefficient, which we estimated to be 30 percent for the equal-weighted weekly returns index! We were taken completely by surprise (and carefully re-checked our programs several times for coding errors before debuting these results in a November 1986 conference). How could such compelling evidence against the random walk be overlooked by the vast literature we were fed as graduate students?

At first, we attributed this to our using weekly returns—prior studies used either daily or monthly. We chose a weekly sampling interval to balance the desire for a large sample size against the problems associated with high-frequency financial data, e.g., nonsynchronous prices, bid/ask "bounce," etc. But we soon discovered that the case against the random walk was equally compelling with daily returns.

This puzzling state of affairs sparked the series of studies contained in Chapters 3 to 6, studies that attempted to reconcile what we, and many others, viewed as a sharp contradiction between our statistical inferences and the voluminous literature that came before us. We checked the accuracy of our statistical methods (Chapter 3), we quantified the potential biases introduced by nonsynchronous prices (Chapter 4), we investigated the sources of the rejections of the random walk and traced them to large positive cross-autocorrelations and lead/lag effects (Chapter 5), and we considered statistical fractals as an alternative to the random walk (Chapter 6). Despite our best efforts, we were unable to explain away the evidence against the Random Walk Hypothesis.

With the benefit of hindsight and a more thorough review of the literature, we have come to the conclusion that the apparent inconsistency between the broad support for the Random Walk Hypothesis and our empirical findings is largely due to the common misconception that the Random Walk Hypothesis is equivalent to the Efficient Markets Hypothesis, and the near religious devotion of economists to the latter (see Chapter 1). Once we saw that we, and our colleagues, had been trained to study the data through the filtered lenses of classical market efficiency, it became clear that the problem lay not with our empirical analysis, but with the economic implications that others incorrected attributed to our results—unbounded profit opportunities, irrational investors, and the like.

We also discovered that ours was not the first study to reject the random walk, and that the departures from the random walk uncovered by Osborne (1962), Larson (1960), Cootner (1962), Steiger (1964), Niederhoffer and Osborne (1966), and Schwartz and Whitcomb (1977), to name just a few examples, were largely ignored by the academic community and unknown to us until after our own papers were published.[3] We were all in a collective fog regarding the validity of the Random Walk Hypothesis, but as we confronted the empirical evidence from every angle and began to rule out other explanations, slowly the fog lifted for us.

In Niederhoffer's (1997) entertaining and irreverent autobiography, he sheds some light on the kind of forces at work in creating this fog. In describing the Random Walk Hypothesis as it developed at the University of Chicago in the 1960's, he writes:

> This theory and the attitude of its adherents found classic expression in one incident I personally observed that deserves memorialization. A team of four of the most respected graduate students in finance had joined forces with two professors, now considered venerable enough to have won or to have been considered for a Nobel prize, but at that time feisty as Hades and insecure as a kid on his first date. This elite group was studying the possible impact of volume on stock price movements, a subject I had researched. As I was coming down the steps from the library on the third floor of Haskell Hall, the main business building, I could see this Group of Six gathered together on a stairway landing, examining some computer output. Their voices wafted up to me, echoing off the stone walls of the building. One of the students was pointing to some output while querying the professors, "Well, what if we really do find something? We'll be up the creek. It won't be consistent with the random walk model." The younger professor replied, "Don't worry, we'll cross that bridge in the unlikely event we come to it."
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> I could hardly believe my ears—here were six scientists openly hoping to find no departures from ignorance. I couldn't hold my tongue, and blurted out, "I sure am glad you are all keeping an open mind about your research." I could hardly refrain from grinning as I walked past them. I heard muttered imprecations in response.

[3]In fact, both Alexander (1961) and Schwartz and Whitcomb (1977) use variance ratios to test the Random Walk Hypothesis, and although they do not employ the kind of rigorous statistical inference that we derived in our study, nevertheless it was our mistake to have overlooked their contributions. Our only defense is that none of our colleagues were aware of these studies either, for no one pointed out these references to us either before or after our papers were published.

From this, Niederhoffer (1997) concludes that "As usual, academicians are way behind the form" and with respect to the Random Walk Hypothesis, we are forced to agree.

But beyond the interesting implications that this cognitive dissonance provides for the sociology of science, we think there is an even more important insight to be gleaned from all of this. In a recent update of our original variance ratio test for weekly US stock market indexes, we discovered that the most current data (1986–1996) conforms more closely to the random walk than our original 1962–1985 sample period. Moreover, upon further investigation, we learned that over the past decade several investment firms—most notably, Morgan Stanley and D.E. Shaw—have been engaged in high-frequency equity trading strategies specifically designed to take advantage of the kind of patterns we uncovered in 1988. Previously known as "pairs trading" and now called "statistical arbitrage," these strategies have fared reasonably well until recently, and are now regarded as a very competitive and thin-margin business because of the proliferation of hedge funds engaged in these activities. This provides a plausible explanation for the trend towards randomness in the recent data, one that harkens back to Samuelson's "Proof that Properly Anticipated Prices Fluctuate Randomly."

But if Morgan Stanley and D.E. Shaw were profiting in the 1980's from the predictability in stock returns that is now waning because of competition, can we conclude that markets were inefficient in the 1980's? Not without additional information about the cost and risk of their trading operations, and the novelty of their trading strategies relative to their competitors'.

In particular, the profits earned by the early statistical arbitrageurs may be viewed as economic rents that accrued to their innovation, creativity, perseverance, and appetite for risk. Now that others have begun to reverse engineer and mimick their technologies, profit margins are declining. Therefore, neither the evidence against the random walk, nor the more recent trend towards the random walk, are inconsistent with the practical version of the Efficient Markets Hypothesis. Market opportunities need not be market inefficiencies.

2

Stock Market Prices Do Not Follow Random Walks: Evidence from a Simple Specification Test

SINCE KEYNES' (1936) NOW FAMOUS PRONOUNCEMENT that most investors' decisions "can only be taken as a result of animal spirits—of a spontaneous urge to action rather than inaction, and not as the outcome of a weighted average of benefits multiplied by quantitative probabilities," a great deal of research has been devoted to examining the efficiency of stock market price formation. In Fama's (1970) survey, the vast majority of those studies were unable to reject the "efficient markets" hypothesis for common stocks. Although several seemingly anomalous departures from market efficiency have been well documented,[1] many financial economists would agree with Jensen's (1978a) belief that "there is no other proposition in economics which has more solid empirical evidence supporting it than the Efficient Markets Hypothesis."

Although a precise formulation of an empirically refutable efficient markets hypothesis must obviously be model-specific, historically the majority of such tests have focused on the forecastability of common stock returns. Within this paradigm, which has been broadly categorized as the "random walk" theory of stock prices, few studies have been able to reject the random walk model statistically. However, several recent papers have uncovered empirical evidence which suggests that stock returns contain predictable components. For example, Keim and Stambaugh (1986) find statistically significant predictability in stock prices by using forecasts based on certain predetermined variables. In addition, Fama and French (1988) show that

[1] See, for example, the studies in Jensen's (1978b) volume on anomalous evidence regarding market efficiency.

long holding-period returns are significantly negatively serially correlated, implying that 25 to 40 percent of the variation of longer-horizon returns is predictable from past returns.

In this chapter we provide further evidence that stock prices do not follow random walks by using a simple specification test based on variance estimators. Our empirical results indicate that the random walk model is generally not consistent with the stochastic behavior of weekly returns, especially for the smaller capitalization stocks. However, in contrast to the negative serial correlation that Fama and French (1988) found for longer-horizon returns, we find significant positive serial correlation for weekly and monthly holding-period returns. For example, using 1216 weekly observations from September 6, 1962, to December 26, 1985, we compute the weekly first-order autocorrelation coefficient of the equal-weighted Center for Research in Security Prices (CRSP) returns index to be 30 percent! The statistical significance of our results is robust to heteroskedasticity. We also develop a simple model which indicates that these large autocorrelations cannot be attributed solely to the effects of infrequent trading. This empirical puzzle becomes even more striking when we show that autocorrelations of individual securities are generally negative.

Of course, these results do not necessarily imply that the stock market is inefficient or that prices are not rational assessments of "fundamental" values. As Leroy (1973) and Lucas (1978) have shown, rational expectations equilibrium prices need not even form a martingale sequence, of which the random walk is a special case. Therefore, without a more explicit economic model of the price-generating mechanism, a rejection of the random walk hypothesis has few implications for the efficiency of market price formation. Although our test results may be interpreted as a rejection of *some* economic model of efficient price formation, there may exist other plausible models that are consistent with the empirical findings. Our more modest goal in this study is to employ a test that is capable of distinguishing among several interesting alternative stochastic price processes. Our test exploits the fact that the variance of the increments of a random walk is linear in the sampling interval. If stock prices are generated by a random walk (possibly with drift), then, for example, the variance of monthly sampled log-price relatives must be 4 times as large as the variance of a weekly sample. Comparing the (per unit time) variance estimates obtained from weekly and monthly prices may then indicate the plausibility of the random walk theory.[2] Such a comparison

[2]The use of variance ratios is, of course, not new. Most recently, Campbell and Mankiw (1987), Cochrane (1987b, 1987c), Fama and French (1988), French and Roll (1986), and Huizinga (1987) have all computed variance ratios in a variety of contexts; however, these studies do not provide any formal sampling theory for our statistics. Specifically, Cochrane (1988), Fama and French (1988), and French and Roll (1986) all rely on Monte Carlo simulations to obtain standard errors for their variance ratios under the null. Campbell and Mankiw (1987)

is formed quantitatively along the lines of the Hausman (1978) specification test and is particularly simple to implement.

In Section 2.1 we derive our specification test for both homoskedastic and heteroskedastic random walks. Our main results are given in Section 2.2, where rejections of the random walk are extensively documented for weekly returns indexes, size-sorted portfolios, and individual securities. Section 2.3 contains a simple model which demonstrates that infrequent trading cannot fully account for the magnitude of the estimated autocorrelations of weekly stock returns. In Section 2.4 we discuss the consistency of our empirical rejections with a mean-reverting alternative to the random walk model. We summarize briefly and conclude in Section 2.5.

2.1 The Specification Test

Denote by P_t the stock price at time t and define $X_t \equiv \ln P_t$ as the log-price process. Our maintained hypothesis is given by the recursive relation

$$X_t = \mu + X_{t-1} + \epsilon_t \qquad (2.1.1)$$

where μ is an arbitrary drift parameter and ϵ_t is the random disturbance term. We assume throughout that for all t, $\mathrm{E}[\epsilon_t] = 0$, where $\mathrm{E}[\cdot]$ denotes the

and Cochrane (1987c) do derive the asymptotic variance of the variance ratio but only under the assumption that the aggregation value q grows with (but more slowly than) the sample size T. Specifically, they use Priestley's (1981, page 463) expression for the asymptotic variance of the estimator of the spectral density of ΔX_t at frequency 0 (with a Bartlett window) as the appropriate asymptotic variance of the variance ratio. But Priestley's result requires (among other things) that $q \to \infty$, $T \to \infty$, and $q/T \to 0$. In this chapter we develop the formal sampling theory of the variance-ratio statistics for the more general case.

Our variance ratio may, however, be related to the spectral-density estimates in the following way. Letting $f(0)$ denote the spectral density of the increments ΔX_t at frequency 0, we have the following relation:

$$\pi f(0) = \gamma(0) + 2\sum_{k=1}^{\infty} \gamma(k)$$

where $\gamma(k)$ is the autocovariance function. Dividing both sides by the variance $\gamma(0)$ then yields

$$\pi f^*(0) = 1 + 2\sum_{k=1}^{\infty} \rho(k)$$

where f^* is the normalized spectral density and $\rho(k)$ is the autocorrelation function. Now in order to estimate the quantity $\pi f^*(0)$, the infinite sum on the right-hand side of the preceding equation must obviously be truncated. If, in addition to truncation, the autocorrelations are weighted using Newey and West's (1987) procedure, then the resulting estimator is formally equivalent to our $M_r(q)$-statistic. Although he does not explicitly use this variance ratio, Huizinga (1987) does employ the Newey and West (1987) estimator of the normalized spectral density.

expectations operator. Although the traditional random walk hypothesis restricts the ϵ_t's to be independently and identically distributed (IID) Gaussian random variables, there is mounting evidence that financial time series often possess time-varying volatilities and deviate from normality. Since it is the unforecastability, or uncorrelatedness, of price changes that is of interest, a rejection of the IID Gaussian random walk because of heteroskedasticity or nonnormality would be of less import than a rejection that is robust to these two aspects of the data. In Section 2.1.2 we develop a test statistic which is sensitive to correlated price changes but which is otherwise robust to many forms of heteroskedasticity and nonnormality. Although our empirical results rely solely on this statistic, for purposes of clarity we also present in Section 2.1.1 the sampling theory for the more restrictive IID Gaussian random walk.

2.1.1 Homoskedastic Increments

We begin with the null hypothesis H that the disturbances ϵ_t are independently and identically distributed normal random variables with variance σ_o^2; thus,

$$\text{H: } \epsilon_t \text{ IID } \mathcal{N}(0, \sigma_o^2). \tag{2.1.2}$$

In addition to homoskedasticity, we have made the assumption of independent Gaussian increments. An example of such a specification is the exact discrete-time process X_t obtained by sampling the following well-known continuous-time process at equally spaced intervals:

$$dX(t) = \mu\, dt + \sigma_o\, dW(t) \tag{2.1.3}$$

where $dW(t)$ denotes the standard Wiener differential. The solution to this stochastic differential equation corresponds to the popular lognormal diffusion price process.

One important property of the random walk X_t is that the variance of its increments is linear in the observation interval. That is, the variance of $X_t - X_{t-2}$ is twice the variance of $X_t - X_{t-1}$. Therefore, the plausibility of the random walk model may be checked by comparing the variance estimate of $X_t - X_{t-1}$ to, say, one-half the variance estimate of $X_t - X_{t-2}$. This is the essence of our specification test; the remainder of this section is devoted to developing the sampling theory required to compare the variances quantitatively.

Suppose that we obtain $2n + 1$ observations $X_0, X_1, \ldots, X_{2n}$ of X_t at equally spaced intervals and consider the following estimators for the unknown parameters μ and σ_o^2:

$$\hat{\mu} \equiv \frac{1}{2n} \sum_{k=1}^{2n} (X_k - X_{k-1}) = \frac{1}{2n} (X_{2n} - X_0) \tag{2.1.4a}$$

$$\hat{\sigma}_a^2 \equiv \frac{1}{2n} \sum_{k=1}^{2n} (X_k - X_{k-1} - \hat{\mu})^2 \tag{2.1.4b}$$

$$\hat{\sigma}_b^2 \equiv \frac{1}{2n} \sum_{k=1}^{n} (X_{2k} - X_{2k-2} - 2\hat{\mu})^2. \tag{2.1.4c}$$

The estimators $\hat{\mu}$ and $\hat{\sigma}_a^2$ correspond to the maximum-likelihood estimators of the μ and σ_o^2 parameters; $\hat{\sigma}_b^2$ is also an estimator of σ_o^2 but uses only the subset of $n+1$ observations $X_0, X_2, X_4, \ldots, X_{2n}$ and corresponds formally to $\frac{1}{2}$ times the variance estimator for increments of even-numbered observations. Under standard asymptotic theory, all three estimators are strongly consistent; that is, holding all other parameters constant, as the total number of observations $2n$ increases without bound the estimators converge almost surely to their population values. In addition, it is well known that both $\hat{\sigma}_a^2$ and $\hat{\sigma}_b^2$ possess the following Gaussian limiting distributions:

$$\sqrt{2n}\,(\hat{\sigma}_a^2 - \sigma_o^2) \overset{a}{\sim} \mathcal{N}(0, 2\sigma_o^4) \tag{2.1.5a}$$

$$\sqrt{2n}\,(\hat{\sigma}_b^2 - \sigma_o^2) \overset{a}{\sim} \mathcal{N}(0, 4\sigma_o^4) \tag{2.1.5b}$$

where $\overset{a}{\sim}$ indicates that the distributional equivalence is asymptotic. Of course, it is the limiting distribution of the *difference* of the variances that interests us. Although it may readily be shown that such a difference is also asymptotically Gaussian with zero mean, the variance of the limiting distribution is not apparent since the two variance estimators are clearly not asymptotically uncorrelated. However, since the estimator $\hat{\sigma}_a^2$ is asymptotically efficient under the null hypothesis H, we may apply Hausman's (1978) result, which shows that the asymptotic variance of the difference is simply the difference of the asymptotic variances.[3] If we define $J_d \equiv \hat{\sigma}_b^2 - \hat{\sigma}_a^2$, then we have the result

$$\sqrt{2n}\, J_d \overset{a}{\sim} \mathcal{N}(0, 2\sigma_o^4). \tag{2.1.6}$$

Using any consistent estimator of the asymptotic variance of J_d, a standard significance test may then be performed. A more convenient alternative

[3]Briefly, Hausman (1978) exploits the fact that any asymptotically efficient estimator of a parameter θ, say $\hat{\theta}_e$, must possess the property that it is asymptotically uncorrelated with the difference $\hat{\theta}_a - \hat{\theta}_e$, where $\hat{\theta}_a$ is any other estimator of θ. If not, then there exists a linear combination of $\hat{\theta}_e$ and $\hat{\theta}_a - \hat{\theta}_e$ that is more efficient than $\hat{\theta}_e$, contradicting the assumed efficiency of $\hat{\theta}_e$. The result follows directly, then, since

$$\begin{aligned} \mathrm{aVar}(\hat{\theta}_a) = \mathrm{aVar}(\hat{\theta}_e + \hat{\theta}_a - \hat{\theta}_e) &= \mathrm{aVar}(\hat{\theta}_e) + \mathrm{aVar}(\hat{\theta}_a - \hat{\theta}_e) \\ \Rightarrow \mathrm{aVar}(\hat{\theta}_a - \hat{\theta}_e) &= \mathrm{aVar}(\hat{\theta}_a) - \mathrm{aVar}(\hat{\theta}_e) \end{aligned}$$

where aVar$(\cdot)$ denotes the asymptotic variance operator.

test statistic is given by the ratio of the variances, J_r:[4]

$$J_r \equiv \frac{\hat{\sigma}_b^2}{\hat{\sigma}_a^2} - 1 \qquad \sqrt{2n}\, J_r \overset{a}{\sim} \mathcal{N}(0, 2). \tag{2.1.7}$$

Although the variance estimator $\hat{\sigma}_b^2$ is based on the differences of every other observation, alternative variance estimators may be obtained by using the differences of every qth observation. Suppose that we obtain $nq+1$ observations $X_0, X_1, \ldots, X_{nq}$, where q is any integer greater than 1. Define the estimators:

$$\hat{\mu} \equiv \frac{1}{nq} \sum_{k=1}^{nq} (X_k - X_{k-1}) = \frac{1}{nq}(X_{nq} - X_0) \tag{2.1.8a}$$

$$\hat{\sigma}_a^2 \equiv \frac{1}{nq} \sum_{k=1}^{nq} (X_k - X_{k-1} - \hat{\mu})^2 \tag{2.1.8b}$$

$$\hat{\sigma}_b^2(q) \equiv \frac{1}{nq} \sum_{k=1}^{n} (X_{qk} - X_{qk-q} - q\hat{\mu})^2 \tag{2.1.8c}$$

$$J_d(q) \equiv \hat{\sigma}_b^2(q) - \hat{\sigma}_a^2, \qquad J_r(q) \equiv \frac{\hat{\sigma}_b^2(q)}{\hat{\sigma}_a^2} - 1. \tag{2.1.8d}$$

The specification test may then be performed using Theorem 2.1.[5]

Theorem 2.1. *Under the null hypothesis* H, *the asymptotic distributions of* $J_d(q)$ *and* $J_r(q)$ *are given by*

$$\sqrt{nq}\, J_d(q) \overset{a}{\sim} \mathcal{N}(0, 2(q-1)\sigma_o^4) \tag{2.1.9a}$$

$$\sqrt{nq}\, J_r(q) \overset{a}{\sim} \mathcal{N}(0, 2(q-1)). \tag{2.1.9b}$$

Two further refinements of the statistics J_d and J_r result in more desirable finite-sample properties. The first is to use *overlapping* qth differences of X_t in estimating the variances by defining the following estimator of σ_o^2:

$$\hat{\sigma}_c^2(q) = \frac{1}{nq^2} \sum_{k=q}^{nq} (X_k - X_{k-q} - q\hat{\mu})^2. \tag{2.1.10}$$

[4]Note that if $(\hat{\sigma}_a^2)^2$ is used to estimate σ_o^4, then the standard t-test of $J_d = 0$ will yield inferences identical to those obtained from the corresponding test of $J_r = 0$ for the ratio, since

$$\frac{J_d}{\sqrt{2\hat{\sigma}_a^4}} = \frac{\hat{\sigma}_b^2 - \hat{\sigma}_a^2}{\sqrt{2}\,\hat{\sigma}_a^2} = \frac{J_r}{\sqrt{2}} \sim \mathcal{N}(0, 1).$$

[5]Proofs of all the theorems are given in the Appendices.

This differs from the estimator $\hat{\sigma}_b^2(q)$ since this sum contains $nq-q+1$ terms, whereas the estimator $\hat{\sigma}_b^2(q)$ contains only n terms. By using overlapping qth increments, we obtain a more efficient estimator and hence a more powerful test. Using $\hat{\sigma}_c^2(q)$ in our variance-ratio test, we define the corresponding test statistics for the difference and the ratio as

$$M_d(q) \equiv \hat{\sigma}_c^2(q) - \hat{\sigma}_a^2 \qquad M_r(q) \equiv \frac{\hat{\sigma}_c^2(q)}{\hat{\sigma}_a^2} - 1. \tag{2.1.11}$$

The second refinement involves using unbiased variance estimators in the calculation of the M-statistics. Denote the unbiased estimators as $\bar{\sigma}_a^2$ and $\bar{\sigma}_c^2(q)$, where

$$\bar{\sigma}_a^2 = \frac{1}{nq-1}\sum_{k=1}^{nq}(X_k - X_{k-1} - \hat{\mu})^2 \tag{2.1.12a}$$

$$\bar{\sigma}_c^2 = \frac{1}{m}\sum_{k=q}^{nq}(X_k - X_{k-q} - q\hat{\mu})^2$$

$$m = q(nq-q+1)\left(1 - \frac{q}{nq}\right) \tag{2.1.12b}$$

and define the statistics:

$$\bar{M}_d(q) \equiv \bar{\sigma}_c^2(q) - \bar{\sigma}_a^2, \qquad \bar{M}_r(q) \equiv \frac{\bar{\sigma}_c^2(q)}{\bar{\sigma}_a^2} - 1. \tag{2.1.13}$$

Although this does not yield an unbiased variance ratio, simulation experiments show that the finite-sample properties of the test statistics are closer to their asymptotic counterparts when this bias adjustment is made.[6] Inference for the overlapping variance differences and ratios may then be performed using Theorem 2.2.

Theorem 2.2. *Under the null hypothesis* H, *the asymptotic distributions of the statistics* $M_d(q)$, $M_r(q)$, $\bar{M}_d(q)$, *and* $\bar{M}_r(q)$ *are given by*

$$\sqrt{nq}\,M_d(q) \overset{a}{\sim} \sqrt{nq}\,\bar{M}_d(q) \overset{a}{\sim} \mathcal{N}\left(0, \frac{2(2q-1)(q-1)}{3q}\sigma_o^4\right) \tag{2.1.14a}$$

$$\sqrt{nq}\,M_r(q) \overset{a}{\sim} \sqrt{nq}\,\bar{M}_r(q) \overset{a}{\sim} \mathcal{N}\left(0, \frac{2(2q-1)(q-1)}{3q}\right). \tag{2.1.14b}$$

[6] According to the results of Monte Carlo experiments in Lo and MacKinlay (1989a), the behavior of the bias-adjusted M-statistics (which we denote as $\bar{M}_d(q)$ and $\bar{M}_r(q)$) does not depart significantly from that of their asymptotic limits even for small sample sizes. Therefore, all our empirical results are based on the $\bar{M}_r(q)$-statistic.

In practice, the statistics in Equations (2.1.14) may be standardized in the usual manner (e.g., define the (asymptotically) standard normal test statistic $z(q) \equiv \sqrt{nq}\,\bar{M}_r(q)(2(2q-1)(q-1)/3q)^{-1/2} \overset{a}{\sim} \mathcal{N}(0,1)$).

To develop some intuition for these variance ratios, observe that for an aggregation value q of 2, the $M_r(q)$-statistic may be reexpressed as

$$M_r(2) \;=\; \hat{\rho}(1) - \frac{1}{4n\hat{\sigma}_a^2}\,[(X_1 - X_0 - \hat{\mu})^2 + (X_{2n} - X_{2n-1} - \hat{\mu})^2] \;\simeq\; \hat{\rho}(1). \tag{2.1.15}$$

Hence, for $q = 2$ the $M_r(q)$-statistic is approximately the first-order autocorrelation coefficient estimator $\hat{\rho}(1)$ of the differences. More generally, it may be shown that

$$M_r(q) \;\simeq\; \frac{2(q-1)}{q}\,\hat{\rho}(1) + \frac{2(q-2)}{q}\,\hat{\rho}(2) + \cdots + \frac{2}{q}\,\hat{\rho}(q-1) \tag{2.1.16}$$

where $\hat{\rho}(k)$ denotes the kth-order autocorrelation coefficient estimator of the first differences of X_t.[7] Equation (2.1.16) provides a simple interpretation for the variance ratios computed with an aggregation value q: They are (approximately) linear combinations of the first $q-1$ autocorrelation coefficient estimators of the first differences with arithmetically declining weights.[8]

2.1.2 Heteroskedastic Increments

Since there is already a growing consensus among financial economists that volatilities do change over time,[9] a rejection of the random walk hypothesis because of heteroskedasticity would not be of much interest. We therefore wish to derive a version of our specification test of the random walk model that is robust to changing variances. As long as the increments are uncorrelated, even in the presence of heteroskedasticity the variance ratio must still approach unity as the number of observations increase without bound, for the variance of the sum of uncorrelated increments must still equal the sum of the variances. However, the asymptotic variance of the variance ratios will clearly depend on the type and degree of heteroskedasticity present. One possible approach is to assume some specific form of heteroskedasticity and then to calculate the asymptotic variance of $\bar{M}_r(q)$ under this null

[7] See Equation (A.1.6a) in the Appendix.

[8] Note the similarity between these variance ratios and the Box-Pierce Q-statistic, which is a linear combination of *squared* autocorrelations with all the weights set identically equal to unity. Although we may expect the finite-sample behavior of the variance ratios to be comparable to that of the Q-statistic under the null hypothesis, they can have very different power properties under various alternatives. See Lo and MacKinlay (1989a) for further details.

[9] See, for example, Merton (1980), Poterba and Summers (1986), and French, Schwert, and Stambaugh (1987).

hypothesis. However, to allow for more general forms of heteroskedasticity, we employ an approach developed by White (1980) and by White and Domowitz (1984). This approach also allows us to relax the requirement of Gaussian increments, an especially important extension in view of stock returns' well-documented empirical departures from normality.[10] Specifically, we consider the null hypothesis H*:[11]

(A1) For all t, $\mathrm{E}(\epsilon_t) = 0$, and $\mathrm{E}(\epsilon_t \epsilon_{t-\tau}) = 0$ for any $\tau \neq 0$.

(A2) $\{\epsilon_t\}$ is ϕ-mixing with coefficients $\phi(m)$ of size $r/(2r-1)$ or is α-mixing with coefficients $\alpha(m)$ of size $r/(r-1)$, where $r > 1$, such that for all t and for any $\tau \geq 0$, there exists some $\delta > 0$ for which

$$\mathrm{E}|\epsilon_t \epsilon_{t-\tau}|^{2(r+\delta)} < \Delta < \infty. \tag{2.1.17}$$

(A3) $\lim_{nq\to\infty} \frac{1}{nq} \sum_{t=1}^{nq} \mathrm{E}(\epsilon_t^2) = \sigma_o^2 < \infty.$

(A4) For all t, $\mathrm{E}(\epsilon_t \epsilon_{t-j} \epsilon_t \epsilon_{t-k}) = 0$ for any nonzero j and k where $j \neq k$.

This null hypothesis assumes that X_t possesses uncorrelated increments but allows for quite general forms of heteroskedasticity, including deterministic changes in the variance (due, for example, to seasonal factors) and Engle's (1982) ARCH processes (in which the conditional variance depends on past information).

Since $\bar{M}_r(q)$ still approaches zero under H*, we need only compute its asymptotic variance (call it $\theta(q)$) to perform the standard inferences. We do this in two steps. First, recall that the following equality obtains asymptotically:

$$\bar{M}_r(q) \stackrel{a}{=} \sum_{j=1}^{q-1} \frac{2(q-j)}{q} \hat{\rho}(j). \tag{2.1.18}$$

Second, note that under H* (condition 2.1.2) the autocorrelation coefficient estimators $\hat{\rho}(j)$ are asymptotically uncorrelated.[12] If we can obtain

[10] Of course, second moments are still assumed to finite; otherwise, the variance ratio is no longer well defined. This rules out distributions with infinite variance, such as those in the stable Pareto-Levy family (with characteristic exponents that are less than 2) proposed by Mandelbrot (1963) and Fama (1965). We do, however, allow for many other forms of leptokurtosis, such as that generated by Engle's (1982) autoregressive conditionally heteroskedastic (ARCH) process.

[11] Condition 2.1.2 is the essential property of the random walk that we wish to test. Conditions 2.1.2 and 2.1.2 are restrictions on the maximum degree of dependence and heterogeneity allowable while still permitting some form of the law of large numbers and the central limit theorem to obtain. See White (1984) for the precise definitions of ϕ- and α-mixing random sequences. Condition 2.1.2 implies that the sample autocorrelations of ϵ_t are asymptotically uncorrelated; this condition may be weakened considerably at the expense of computational simplicity (see note 12).

[12] Although this restriction on the fourth cross-moments of ϵ_t may seem somewhat unintuitive, it is satisfied for any process with independent increments (regardless of heterogeneity)

asymptotic variances $\delta(j)$ for each of the $\hat{\rho}(j)$ under H*, we may readily calculate the asymptotic variance $\theta(q)$ of $\bar{M}_r(q)$ as the weighted sum of the $\delta(j)$, where the weights are simply the weights in relation (2.1.18) squared. More formally, we have:

Theorem 2.3. *Denote by $\delta(j)$ and $\theta(q)$ the asymptotic variances of $\hat{\rho}(j)$ and $\bar{M}_r(q)$, respectively. Then under the null hypothesis* H**:*

1. *The statistics $J_d(q)$, $J_r(q)$, $M_d(q)$, $M_r(q)$, $\bar{M}_d(q)$, and $\bar{M}_r(q)$ all converge almost surely to zero for all q as n increases without bound.*
2. *The following is a heteroskedasticity-consistent estimator of $\delta(j)$:*

$$\hat{\delta}(j) = \frac{nq \sum_{k=j+1}^{nq} (X_k - X_{k-1} - \hat{\mu})^2 (X_{k-j} - X_{k-j-1} - \hat{\mu})^2}{\left[\sum_{k=1}^{nq} (X_k - X_{k-1} - \hat{\mu})^2\right]^2}. \tag{2.1.19}$$

3. *The following is a heteroskedasticity-consistent estimator of $\theta(q)$:*

$$\hat{\theta}(q) \equiv \sum_{j=1}^{q-1} \left[\frac{2(q-j)}{q}\right]^2 \hat{\delta}(j). \tag{2.1.20}$$

Despite the presence of general heteroskedasticity, the standardized test statistic $z^*(q) \equiv \sqrt{nq}\,\bar{M}_r(q)/\sqrt{\hat{\theta}}$ is still asymptotically standard normal. In Section 2.2 we use the $z^*(q)$ statistic to test empirically for random walks in weekly stock returns data.

2.2 The Random Walk Hypothesis for Weekly Returns

To test for random walks in stock market prices, we focus on the 1216-week time span from September 6, 1962, to December 26, 1985. Our choice of a weekly observation interval was determined by several considerations. Since our sampling theory is based wholly on asymptotic approximations, a large number of observations is appropriate. While daily sampling yields many

and also for linear Gaussian ARCH processes. This assumption may be relaxed entirely, requiring the estimation of the asymptotic covariances of the autocorrelation estimators in order to estimate the limiting variance θ of $\bar{M}_r(q)$ via relation (2.1.18). Although the resulting estimator of θ would be more complicated than Equation (2.1.20), it is conceptually straightforward and may readily be formed along the lines of Newey and West (1987). An even more general (and possibly more exact) sampling theory for the variance ratios may be obtained using the results of Dufour (1981) and Dufour and Roy (1985). Again, this would sacrifice much of the simplicity of our asymptotic results.

observations, the biases associated with nontrading, the bid-ask spread, asynchronous prices, etc., are troublesome. Weekly sampling is the ideal compromise, yielding a large number of observations while minimizing the biases inherent in daily data.

The weekly stock returns are derived from the CRSP daily returns file. The weekly return of each security is computed as the return from Wednesday's closing price to the following Wednesday's close. If the following Wednesday's price is missing, then Thursday's price (or Tuesday's if Thursday's is missing) is used. If both Tuesday's and Thursday's prices are missing, the return for that week is reported as missing.[13]

In Section 2.2.1 we perform our test on both equal- and value-weighted CRSP indexes for the entire 1216-week period, as well as for 608-week subperiods, using aggregation values q ranging from 2 to 16.[14] Section 2.2.2 reports corresponding test results for size-sorted portfolios, and Section 2.2.3 presents results for individual securities.

2.2.1 Results for Market Indexes

Tables 2.1a and 2.1b report the variance ratios and the test statistics $z^*(q)$ for CRSP NYSE-AMEX market-returns indexes. Table 2.1a presents the results for a one-week base observation period, and Table 2.1b reports similar results for a four-week base observation period. The values reported in the main rows are the actual variance ratios $[\bar{M}_r(q)+1]$, and the entries enclosed in parentheses are the $z^*(q)$ statistics.[15]

Panel A of Table 2.1a displays the results for the CRSP equal-weighted index. The first row presents the variance ratios and test statistics for the entire 1216-week sample period, and the next two rows give the results for the two 608-week subperiods. The random walk null hypothesis may be rejected at all the usual significance levels for the entire time period and all subperiods. Moreover, the rejections are not due to changing variances since the $z^*(q)$ statistics are robust to heteroskedasticity. The estimates of the variance ratio are *larger* than 1 for all cases. For example, the entries in the first column of panel A correspond to variance ratios with an aggregation value q of 2. In view of Equation (2.1.15), ratios with $q = 2$ are approximately equal to 1 plus the first-order autocorrelation coefficient estimator of weekly returns; hence, the entry in the first row, 1.30, implies that the

[13]The average fraction (over all securities) of the entire sample where this occurs is less than 0.5 percent of the time for the 1216-week sample period.

[14]Additional empirical results (304-week subperiods, larger q values, etc.) are reported in Lo and MacKinlay (1987b).

[15]Since the values of $z^*(q)$ are always smaller than the values of $z(q)$ in our empirical results, to conserve space we report only the more conservative statistics. Both statistics are reported in Lo and MacKinlay (1987b).

Table 2.1a. Variance-ratio test of the random walk hypothesis for CRSP equal- and value-weighted indexes, for the sample period from September 6, 1962, to December 26, 1985, and subperiods. The variance ratios $1+\bar{M}_r(q)$ are reported in the main rows, with the heteroskedasticity-robust test statistics $z^*(q)$ given in parentheses immediately below each main row. Under the random walk null hypothesis, the value of the variance ratio is 1 and the test statistics have a standard normal distribution (asymptotically). Test statistics marked with asterisks indicate that the corresponding variance ratios are statistically different from 1 at the 5 percent level of significance.

Time period	Number nq of base observations	Number q of base observations aggregated to form variance ratio			
		2	4	8	16
A. Equal-weighted CRSP NYSE-AMEX index					
620906–851226	1216	1.30	1.64	1.94	2.05
		(7.51)*	(8.87)*	(8.48)*	(6.59)*
620906–740501	608	1.31	1.62	1.92	2.09
		(5.38)*	(6.03)*	(5.76)*	(4.77)*
740502–851226	608	1.28	1.65	1.93	1.91
		(5.32)*	(6.52)*	(6.13)*	(4.17)*
B. Value-weighted CRSP NYSE-AMEX index					
620906–851226	1216	1.08	1.16	1.22	1.22
		(2.33)*	(2.31)*	(2.07)*	(1.38)
620906–740501	608	1.15	1.22	1.27	1.32
		(2.89)*	(2.28)*	(1.79)	(1.46)
740502–851226	608	1.05	1.12	1.18	1.10
		(0.92)	(1.28)	(1.24)	(0.46)

first-order autocorrelation for weekly returns is approximately 30 percent. The random walk hypothesis is easily rejected at common levels of significance. The variance ratios increase with q, but the magnitudes of the $z^*(q)$ statistics do not. Indeed, the test statistics seem to decline with q; hence, the significance of the rejections becomes weaker as coarser-sample variances are compared to weekly variances. Our finding of *positive* autocorrelation for weekly holding-period returns differs from Fama and French's (1988) finding of negative serial correlation for long holding-period returns. This positive correlation is significant not only for our entire sample period but also for all subperiods.

The rejection of the random walk hypothesis is much weaker for the value-weighted index, as panel B indicates; nevertheless, the general patterns persist: the variance ratios exceed 1, and the $z^*(q)$ statistics decline as

Table 2.1b. Market index results for a four-week base observation period

Time period	Number nq of base observations	Number q of base observations aggregated to form variance ratio			
		2	4	8	16
A. Equal-weighted CRSP NYSE-AMEX index					
620906–851226	304	1.15	1.19	1.30	1.30
		(2.26)*	(1.54)	(1.52)	(1.07)
620906–740501	152	1.13	1.23	1.40	
		(1.39)	(1.32)	(1.46)	
740502–851226	152	1.15	1.11	1.02	
		(1.68)	(0.64)	(0.09)	
B. Value-weighted CRSP NYSE-AMEX index					
620906–851226	304	1.05	1.00	1.11	1.07
		(0.75)	(0.00)	(0.57)	(0.26)
620906–740501	152	1.02	1.04	1.12	
		(0.26)	(0.26)	(0.46)	
740502–851226	152	1.05	0.95	0.89	
		(0.63)	(−0.31)	(−0.42)	

Variance-ratio test of the random walk hypothesis for CRSP equal- and value-weighted indexes, for the sample period from September 6, 1962, to December 26, 1985, and subperiods. The variance ratios $1 + \bar{M}_r(q)$ are reported in the main rows, with the heteroskedasticity-robust test statistics $z^*(q)$ given in parentheses immediately below each main row. Under the random walk null hypothesis, the value of the variance ratio is 1 and the test statistics have a standard normal distribution (asymptotically). Test statistics marked with asterisks indicate that the corresponding variance ratios are statistically different from 1 at the 5 percent level of significance.

q increases. The rejections for the value-weighted index are due primarily to the first 608 weeks of the sample period.

Table 2.1b presents the variance ratios using a base observation period of four weeks; hence, the first entry of the first row, 1.15, is the variance ratio of eight-week returns to four-week returns. With a base interval of four weeks, we generally do not reject the random walk model even for the equal-weighted index. This is consistent with the relatively weak evidence against the random walk that previous studies have found when using monthly data.

Although the test statistics in Tables 2.1a and 2.1b are based on nominal stock returns, it is apparent that virtually the same results would obtain with real or excess returns. Since the volatility of weekly nominal returns is so much larger than that of the inflation and Treasury-bill rates, the use of nominal, real, or excess returns in a volatility-based test will yield practically identical inferences.

2.2.2 Results for Size-Based Portfolios

An implication of the work of Keim and Stambaugh (1986) is that, conditional on stock and bond market variables, the logarithms of wealth relatives of portfolios of smaller stocks do not follow random walks. For portfolios of larger stocks, Keim and Stambaugh's results are less conclusive. Consequently, it is of interest to explore what evidence our tests provide for the random walk hypothesis for the logarithm of size-based portfolio wealth relatives.

We compute weekly returns for five size-based portfolios from the NYSE-AMEX universe on the CRSP daily returns file. Stocks with returns for any given week are assigned to portfolios based on which quintile their market value of equity is in. The portfolios are equal-weighted and have a continually changing composition.[16] The number of stocks included in the portfolios varies from 2036 to 2720.

Table 2.2 reports the $\bar{M}_r(q)$ test results for the size-based portfolios, using a base observation period of one week. Panel A reports the results for the portfolio of small firms (first quintile), panel B reports the results for the portfolio of medium-size firms (third quintile), and panel C reports the results for the portfolio of large firms (fifth quintile). Evidence against the random walk hypothesis for small firms is strong for all time periods considered; in panel A all the $z^*(q)$ statistics are well above 2.0, ranging from 6.12 to 11.92. As we proceed through the panels to the results for the portfolio of large firms, the $z^*(q)$ statistics become smaller, but even for the large-firms portfolio the evidence against the null hypothesis is strong. As in the case of the returns indexes, we may obtain estimates of the first-order autocorrelation coefficient for returns on these size-sorted portfolios simply by subtracting 1 from the entries in the $q = 2$ column. The values in Table 2.2 indicate that the portfolio returns for the smallest quintile have a 42 percent weekly autocorrelation over the entire sample period! Moreover, this autocorrelation reaches 49 percent in subperiod 2 (May 2, 1974, to December 26, 1985). Although the serial correlation for the portfolio returns of the largest quintile is much smaller (14 percent for the entire sample period), it is statistically significant.

[16]We also performed our tests using value-weighted portfolios and obtained essentially the same results. The only difference appeared in the largest quintile of the value-weighted portfolio, for which the random walk hypothesis was generally not rejected. This, of course, is not surprising, given that the largest value-weighted quintile is quite similar to the value-weighted market index.

Table 2.2. *Variance-ratio test of the random walk hypothesis for size-sorted portfolios, for the sample period from September 6, 1962, to December 26, 1985, and subperiods. The variance ratios* $1 + \bar{M}_r(q)$ *are reported in the main rows, with the heteroskedasticity-robust test statistics* $z^*(q)$ *given in parentheses immediately below each main row. Under the random walk null hypothesis, the value of the variance ratio is 1 and the test statistics have a standard normal distribution (asymptotically). Test statistics marked with asterisks indicate that the corresponding variance ratios are statistically different from 1 at the 5 percent level of significance.*

Time period	Number nq of base observations	Number q of base observations aggregated to form variance ratio			
		2	4	8	16
A. Portfolio of firms with market values in smallest NYSE-AMEX quintile					
620906–851226	1216	1.42	1.97	2.49	2.68
		(8.81)*	(11.58)*	(11.92)*	(9.65)*
620906–740501	608	1.37	1.83	2.27	2.52
		(6.12)*	(7.83)*	(7.94)*	(6.68)*
740502–851226	608	1.49	2.14	2.76	2.87
		(6.44)*	(8.66)*	(9.06)*	(7.06)*
B. Portfolio of firms with market values in central NYSE-AMEX quintile					
620906–851226	1216	1.28	1.60	1.84	1.91
		(7.38)*	(8.37)*	(7.70)*	(5.78)*
620906–740501	608	1.30	1.59	1.85	2.01
		(5.31)*	(5.73)*	(5.33)*	(4.42)*
740502–851226	608	1.27	1.59	1.80	1.69
		(5.31)*	(5.73)*	(5.33)*	(4.42)*
C. Portfolio of firms with market values in largest NYSE-AMEX quintile					
620906–851226	1216	1.14	1.27	1.36	1.34
		(3.82)*	(3.99)*	(3.45)*	(2.22)*
620906–740501	608	1.21	1.36	1.45	1.44
		(4.04)*	(3.70)*	(2.96)*	(2.02)*
740502–851226	608	1.09	1.20	1.27	1.18
		(1.80)	(2.18)*	(1.95)	(0.87)

Using a base observation interval of four weeks, much of the evidence against the random walk for size-sorted portfolios disappears. Although the smallest-quintile portfolio still exhibits a serial correlation of 23 percent with a $z^*(2)$ statistic of 3.09, none of the variance ratios for the largest-quintile portfolio is significantly different from 1. In the interest of brevity, we do not report those results here but refer interested readers to Lo and MacKinlay (1987b).

The results for size-based portfolios are generally consistent with those for the market indexes. The patterns of (1) the variance ratios increasing in q and (2) the significance of rejections decreasing in q that we observed for the indexes also obtain for these portfolios. The evidence against the random walk hypothesis for the logarithm of wealth relatives of small-firms portfolios is strong in all cases considered. For larger firms and a one-week base observation interval, the evidence is also inconsistent with the random walk; however, as the base observation interval is increased to four weeks, our test does not reject the random walk model for larger firms.

2.2.3 Results for Individual Securities

For completeness, we performed the variance-ratio test on all individual stocks that have complete return histories in the CRSP database for our entire 1216-week sample period, yielding a sample of 625 securities. Owing to space limitations, we report only a brief summary of these results in Table 2.3. Panel A contains the cross-sectional means of variance ratios for the entire sample as well as for the 100 smallest, 100 intermediate, and 100 largest stocks. Cross-sectional standard deviations are given in parentheses below the main rows. Since the variance ratios are clearly not cross-sectionally independent, these standard deviations cannot be used to form the usual tests of significance; they are reported only to provide some indication of the cross-sectional dispersion of the variance ratios.

The average variance ratio for individual securities is less than unity when $q = 2$, implying that there is negative serial correlation on average. For all stocks, the average serial correlation is -3 percent, and -6 percent for the smallest 100 stocks. However, the serial correlation is both statistically and economically insignificant and provides little evidence against the random walk hypothesis. For example, the largest average $z^*(q)$ statistic over all stocks occurs for $q = 4$ and is -0.90 (with a cross-sectional standard deviation of 1.19); the largest average $z^*(q)$ for the 100 smallest stocks is -1.67 (for $q = 2$, with a cross-sectional standard deviation of 1.75). These results complement French and Roll's (1986) finding that daily returns of individual securities are slightly negatively autocorrelated.

For comparison, panel B reports the variance ratios of equal- and value-weighted portfolios of the 625 securities. The results are consistent with those in Tables 2.1 and 2.2; significant positive autocorrelation for the equal-weighted portfolio, and less significant positive autocorrelation for the value-weighted portfolio.

That the returns of individual securities have statistically insignificant autocorrelation is not surprising. Individual returns contain much company-

Table 2.3. Means of variance ratios over all individual securities with complete return histories from September 2, 1962, to December 26, 1985 (625 stocks). Means of variance ratios for the smallest 100 stocks, the intermediate 100 stocks, and the largest 100 stocks are also reported. For purposes of comparison, panel B reports the variance ratios for equal- and value-weighted portfolios, respectively, of the 625 stocks. Parenthetical entries for averages of individual securities (panel A) are standard deviations of the cross-section of variance ratios. Because the variance ratios are not cross-sectionally independent, the standard deviation cannot be used to perform the usual significance tests; they are reported only to provide an indication of the variance ratios' cross-sectional dispersion. Parenthetical entries for portfolio variance ratios (panel B) are the heteroskedasticity-robust $z^*(q)$ statistics. Asterisks indicate variance ratios that are statistically different from 1 at the 5 percent level of significance.

Sample	Number nq of base observations	Number q of base observations aggregated to form variance ratio			
		2	4	8	16
A. Averages of variance ratios over individual securities					
All stocks	1216	0.97	0.94	0.92	0.89
(625 stocks)		(0.05)*	(0.08)	(0.11)	(0.15)
Small stocks	1216	0.94	0.91	0.90	0.88
(100 stocks)		(0.06)	(0.10)	(0.13)	(0.18)
Medium stocks	1216	0.98	0.97	0.96	0.93
(100 stocks)		(0.05)	(0.09)	(0.12)	(0.15)
Large stocks	1216	0.97	0.94	0.86	0.86
(100 stocks)		(0.04)	(0.07)	(0.11)	(0.17)
B. Variance ratios of equal- and value-weighted portfolios of all stocks					
Equal-weighted portfolio	1216	1.21	1.64	1.65	1.76
(625 stocks)		(5.94)*	(6.71)*	(6.06)*	(4.25)*
Value-weighted portfolio	1216	1.04	1.08	1.12	1.12
(625 stocks)		(1.30)	(1.24)	(1.16)	(0.76)

specific, or "idiosyncratic," noise that makes it difficult to detect the presence of predictable components. Since the idiosyncratic noise is largely attenuated by forming portfolios, we would expect to uncover the predictable "systematic" component more readily when securities are combined. Nevertheless, the negativity of the individual securities' autocorrelations is an interesting contrast to the positive autocorrelation of the portfolio returns. Since this is a well-known symptom of infrequent trading, we consider such an explanation in Section 2.3.

2.3 Spurious Autocorrelation Induced by Nontrading

Although we have based our empirical results on weekly data to minimize the biases associated with market microstructure issues, this alone does not ensure against the biases' possibly substantial influences. In this section we explicitly consider the conjecture that infrequent or nonsynchronous trading may induce significant spurious correlation in stock returns.[17] The common intuition for the source of such artificial serial correlation is that small capitalization stocks trade less frequently than larger stocks. Therefore, new information is impounded first into large-capitalization stock prices and then into smaller-stock prices with a lag. This lag induces a positive serial correlation in, for example, an equal-weighted index of stock returns. Of course, this induced positive serial correlation would be less pronounced in a value-weighted index. Since our rejections of the random walk hypothesis are most resounding for the equal-weighted index, they may very well be the result of this nontrading phenomenon. To investigate this possibility, we consider the following simple model of nontrading.[18]

Suppose that our universe of stocks consists of N securities indexed by i, each with the return-generating process

$$R_{it} = R_{Mt} + \epsilon_{it} \qquad i = 1, \ldots, N \tag{2.3.1}$$

where R_{Mt} represents a factor common to all returns (e.g., the market) and is assumed to be an independently and identically distributed (IID) random variable with mean μ_M and variance σ_M^2. The ϵ_{it} term represents the idiosyncratic component of security i's return and is also assumed to be IID (over *both* i and t), with mean 0 and variance σ_M^2. The return-generating process may thus be identified with N securities each with a unit beta such that the theoretical R^2 of a market-model regression for each security is 0.50.

Suppose that in each period t there is some chance that security i does not trade. One simple approach to modeling this phenomenon is to distinguish between the observed returns process and the virtual returns process. For example, suppose that security i has traded in period $t-1$; consider its behavior in period t. If security i does not trade in period t, we define its vir-

[17]See, for example, Scholes and Williams (1977) and Cohen, Hawawini, Maier, Schwartz, and Whitcomb (1983a).

[18]Although our model is formulated in discrete time for simplicity, it is in fact slightly more general than the Scholes and Williams (1977) continuous-time model of nontrading. Specifically, Scholes and Williams implicitly assume that each security trades at least once within a given time interval by "ignoring periods over which no trades occur" (page 311), whereas our model requires no such restriction. As a consequence, it may be shown that, ceteris paribus, the magnitude of spuriously induced autocorrelation is lower in Scholes and Williams (1977) than in our framework. However, the qualitative predictions of the two models of nontrading are essentially the same. For example, both models imply that returns for individual securities will exhibit negative serial correlation but that portfolio returns will be positively autocorrelated.

tual return as R_{it} (which is given by Equation (2.3.1)), whereas its observed return R^o_{it} is zero. If security i then trades at $t+1$, its observed return R^o_{it+1} is defined to be the sum of its virtual returns R_{it} and R_{it+1}; hence, nontrading is assumed to cause returns to cumulate. The cumulation of returns over periods of nontrading captures the essence of spuriously induced correlations due to the nontrading lag.

To calculate the magnitude of the positive serial correlation induced by nontrading, we must specify the probability law governing the nontrading event. For simplicity, we assume that whether or not a serucity trades may be modeled by a Bernoulli trial, so that in each period and for each security there is a probability p that it trades and a probability $1-p$ that is does not. It is assumed that these Bernoulli trials are IID across securities and, for each security, are IID over time. Now consider the observed return R^o_t at time t of an equal-weighted portfolio:

$$R^o_t \equiv \frac{1}{N}\sum_i^N R^o_{it}. \tag{2.3.2}$$

The observed return R^o_{it} for security i may be expressed as

$$R^o_{it} = X_{it}(0)R_{it} + X_{it}(1)R_{it-1} + X_{it}(2)R_{it-2} + \cdots \tag{2.3.3}$$

where $X_{it}(j)$, $j = 1, 2, 3, \ldots$ are random variables defined as

$$X_{it}(0) \equiv \begin{cases} 1 & \text{if } i \text{ trades at } t \\ 0 & \text{otherwise} \end{cases} \tag{2.3.4a}$$

$$X_{it}(1) \equiv \begin{cases} 1 & \text{if } i \text{ does not trade at } t-1 \text{ and } i \text{ trades at } t \\ 0 & \text{otherwise} \end{cases} \tag{2.3.4b}$$

$$X_{it}(2) \equiv \begin{cases} 1 & \text{if } i \text{ trades at } t \text{ and does not trade at } t-1 \text{ and } t-2 \\ 0 & \text{otherwise} \end{cases} \tag{2.3.4c}$$

$$\vdots$$

The $X_{it}(j)$ variables are merely indicators of the number of *consecutive* periods before t in which security j has not traded. Using this relation, we have

$$R^o_t = \frac{1}{N}\sum_i^N X_{it}(0)R_{it} + \frac{1}{N}\sum_i^N X_{it}(1)R_{it-1} + \frac{1}{N}\sum_i^N X_{it}(2)R_{it-2} + \cdots \tag{2.3.5}$$

For large N, it may readily be shown that because the ϵ_{it} component of each security's return is idiosyncratic and has zero expectation, the following approximation obtains:

$$R_t^o \simeq \frac{1}{N}\sum_i^N X_{it}(0)R_{Mt} + \frac{1}{N}\sum_i^N X_{it}(1)R_{Mt-1} + \frac{1}{N}\sum_i^N X_{it}(2)R_{Mt-2} + \cdots \tag{2.3.6}$$

It is also apparent that the averages $(1/N)\sum_i^N X_{it}(j)$ become arbitrarily close, again for large N, to the probability of j consecutive no-trades followed by a trade; that is,

$$\operatorname{plim}_{N\to\infty}\frac{1}{N}\sum_i^N X_{it}(j) = p(1-p)^j. \tag{2.3.7}$$

The observed equal-weighted return is then given by the approximation

$$R_t^o \simeq pR_{Mt} + p(1-p)R_{Mt-1} + p(1-p)^2R_{Mt-2} + \cdots \tag{2.3.8}$$

Using this expression, the general jth-order autocorrelation coefficient $\rho(j)$ may be readily computed as

$$\rho(j) \equiv \frac{\mathrm{Cov}(R_t^o, R_{t-j}^o)}{\mathrm{Var}(R_t^o)} = (1-p)^j. \tag{2.3.9}$$

Assuming that the implicit time interval corresponding to our single period is one trading day, we may also compute the weekly (five-day) first-order autocorrelation coefficient of R_t^o as

$$\rho^W(1) = \frac{\rho(1) + 2\rho(2) + \cdots + 5\rho(5) + 4\rho(6) + \cdots + \rho(9)}{5 + 8\rho(1) + 6\rho(2) + 4\rho(3) + 2\rho(4)}. \tag{2.3.10}$$

By specifying reasonable values for the probability of nontrading, we may calculate the induced autocorrelation using Equation (2.3.10). To develop some intuition for the parameter p, observe that the total number of secutities that trade in any given period t is given by the sum $\sum_i^N X_{it}(0)$. Under our assumptions, this random variable has a binomial distribution with parameters (N, p); hence, its expected value and variance are given by Np and $Np(1-p)$, respectively. Therefore, the probability p may be interpreted as the fraction of the total number of N securities that trades on average in any given period. A value of .90 implies that, on average, 10 percent of the securities do not trade in a single period.

Table 2.4 presents the theoretical daily and weekly autocorrelations induced by nontrading for nontrading probabilities of 10 to 50 percent. The first row shows that when (on average) 10 percent of the stocks do not trade each day, this induces a weekly autocorrelation of only 2.1 percent! Even when the probability of nontrading is increased to 50 percent (which is quite

Table 2.4. Spuriously induced autocorrelations are reported for nontrading probabilities $1-p$ of 10 to 50 percent. In the absence of the nontrading phenomenon, the theoretical values of daily jth-order autocorrelations $\rho(j)$ and the weekly first-order autocorrelation $\rho^W(1)$ are all zero.

$1-p$	$\rho(1)$	$\rho(2)$	$\rho(3)$	$\rho(4)$	$\rho(5)$	$\rho^W(1)$
.10	.1000	.0100	.0010	.0001	.0000	.0211
.20	.2000	.0400	.0080	.0016	.0003	.0454
.30	.3000	.0900	.0270	.0081	.0024	.0756
.40	.4000	.1600	.0640	.0256	.0102	.1150
.50	.5000	.2500	.1250	.0625	.0312	.1687

unrealistic), the induced weekly autocorrelation is 17 percent.[19] We conclude that our rejection of the random walk hypothesis cannot be attributed solely to infrequent trading.

The positive autocorrelation of portfolio returns and the negative autocorrelation of individual securities is puzzling. Although our stylized model suggests that infrequent trading cannot fully account for the 30 percent autocorrelation of the equal-weighted index, the combination of infrequent

[19]Several other factors imply that the actual sizes of the spurious autocorrelations induced by infrequent trading are lower than those given in Table 2.4. For example, in calculating the induced correlations using Equation (2.3.9), we have ignored the idiosyncratic components in returns because diversification makes these components trivial in the limit; in practice, perfect diversification is never achieved. But any residual risk increases the denominator of Equation (2.3.9) and does not necessarily increase the numerator (since the ϵ_{it}'s are cross-sectionally uncorrelated). To see this explicitly, we simulated the returns for 1000 stocks over 5120 days, calculated the weekly autocorrelations for the virtual returns and for the observed returns, computed the difference of those autocorrelations, repeated this procedure 20 times, and then averaged the differences. With a (daily) nontrading probability of 10 percent, the simulations yield a difference in *weekly* autocorrelations of 2.1 percent, of 4.3 percent for a nontrading probability of 20 percent, and of 7.6 percent for a nontrading probability of 30 percent.

Another factor that may reduce the spurious positive autocorrelation empirically is that, within the CRSP files, if a security does not trade, its price is reported as the average of the bid-ask spread. As long as the specialist adjusts the apread to reflect the new information, even if no trade occurs the reported CRSP price will reflect the new information. Although there may still be some delay before the bid-ask spread is adjusted, it is presumably less than the lag between trades.

Also, if it is assumed that the probability of no-trades depends upon whether or not the security has traded recently, it is natural to suppose that the likelihood of a no-trade tomorrow is lower if there is a no-trade today. In this case, it may readily be shown that the induced autocorrelation is even lower than that computed in our IID framework.

trading and Roll's (1984a) bid-ask effect may explain a large part of the small negative autocorrelation in individual returns.

One possible stochastic model that is loosely consistent with these observations is to let returns be the sum of a positively autocorrelated common component and an idiosyncratic white-noise component. The common component induces significant positive autocorrelation in portfolios since the idiosyncratic component is trivialized by diversification. The white-noise component reduces the positive autocorrelation of individual stock returns, and the combination of infrequent trading and the bid-ask spread effects drives the autocorrelation negative. Of course, explicit statistical estimation is required in order to formalize such heuristics and, ultimately, what we seek is an economic model of asset prices that might give rise to such empirical findings. This is beyond the scope of this chapter, but it is the focus of current investigation.

2.4 The Mean-Reverting Alternative to the Random Walk

Although the variance-ratio test has shown weekly stock returns to be incompatible with the random walk model, the rejections do not offer any explicit guidance toward a more plausible model for the data. However, the *patterns* of the test's rejections over different base observation intervals and aggregation values q do shed considerable light on the relative merits of competing alternatives to the random walk. For example, one currently popular hypothesis is that the stock-returns process may be described by the sum of a random walk and a stationary mean-reverting component, as in Summers (1986) and in Fama and French (1988).[20] One implication of this alternative is that returns are negatively serially correlated for all holding periods. Another implication is that, up to a certain holding period, the serial correlation becomes more negative as the holding period increases.[21] If returns are in fact generated by such a process, then their variance ratios

[20]Shiller and Perron (1985) propose only a mean-reverting process (the Ornstein-Uhlenbeck process), whereas Poterba and Summers (1988) propose the sum of a random walk and a stationary mean-reverting process. Although neither study offers any theoretical justification for its proposal, both studies motivate their alternatives as models of investors' fads.

[21]If returns are generated by the sum of a random walk and a stationary mean-reverting process, their serial correlation will be a U-shaped function of the holding period; the first-order autocorrelation becomes more negative as shorter holding periods lengthen, but it graudally returns to zero for longer holding periods because the random walk component dominates. The curvature of this U-shaped function depends on the relative variability of the random walk and mean-reverting components. Fama and French's (1988) parameter estimates imply that the autocorrelation coefficient is monotonically decreasing for holding periods up to three years; that is, the minimum of the U-shaped curve occurs at a holding period greater than or equal to three years.

should be less than unity when $q = 2$ (since negative serial correlation is implied by this process). Also, the rejection of the random walk should be stronger as q increases (larger $z^*(q)$ values for larger q).[22] But Tables 2.1 and 2.2 and those in Lo and MacKinlay (1987b) show that both these implications are contradicted by the empirical evidence.[23] Weekly returns do not follow a random walk, but they do not fit a stationary mean-reverting alternative any better.

Of course, the negative serial correlation in Fama and French's (1988) study for long (three- to five-year) holding-period returns is, on purely theoretical grounds, not necessarily inconsistent with positive serial correlation for shorter holding-period returns. However, our results do indicate that the sum of a random walk and a mean-reverting process cannot be a complete description of stock-price behavior.

2.5 Conclusion

We have rejected the random walk hypothesis for weekly stock market returns by using a simple volatility-based specification test. These rejections cannot be explained completely by infrequent trading or time-varying volatilities. The patterns of rejections indicate that the stationary mean-reverting models of Shiller and Perron (1985), Summers (1986), Poterba and Summers (1988), and Fama and French (1988) cannot account for the departures of weekly returns from the random walk.

As we stated in the introduction, the rejection of the random walk model does not necessarily imply the inefficiency of stock-price formation. Our results do, however, impose restrictions upon the set of plausible economic models for asset pricing; any structural paradigm of rational price formation must now be able to explain this pattern of serial correlation present in weekly data. As a purely descriptive tool for examining the stochastic evolution of prices through time, our specification test also serves a useful purpose, especially when an empirically plausible statistical model of the price process is more important than a detailed economic paradigm of equilibrium. For example, the pricing of complex financial claims often depends critically upon the specific stochastic process driving underlying asset returns. Since such models are usually based on arbitrage considerations, the particular economic equilibrium that generates prices is of less consequence. One specific implication of our empirical findings is that

[22]This pattern of stronger rejections with larger q is also only true up to a certain value of q. In view of Fama and French's (1988) results, this upper limit for q is much greater than 16 when the base observation interval is one week. See note 21.

[23]See Lo and MacKinlay (1989a) for explicit power calculations against this alternative and against a more empirically relevant model of stock prices.

the standard Black-Scholes pricing formula for stock index options is misspecified.

Although our variance-based test may be used as a diagnostic check for the random walk specification, it is a more difficult task to determine precisely which stochastic process best fits the data. The results of French and Roll (1986) for return variances when markets are open versus when they are closed add yet another dimension to this challenge. The construction of a single stochastic process that fits both short and long holding-period returns data is one important direction for further investigation. However, perhaps the more pressing problem is to specify an economic model that might give rise to such a process for asset prices, and this will be pursued in subsequent research.

Appendix A2
Proof of Theorems

Proof of Theorem 2.1

Under the IID Gaussian distributional assumption of the null hypothesis H, $\hat{\sigma}_a^2$ and $\hat{\sigma}_b^2$ are maximum-likelihood estimators of σ_o^2 with respect to data sets consisting of every observation and of every qth observation, respectively (the dependence of $\hat{\sigma}_b^2$ on q is suppressed for notational simplicity). Therefore, it is well known that

$$\sqrt{nq}\,(\hat{\sigma}_a^2 - \hat{\sigma}_0^2) \overset{a}{\sim} \mathcal{N}(0, 2\sigma_o^4) \tag{A2.1}$$

$$\sqrt{nq}\,(\hat{\sigma}_b^2 - \hat{\sigma}_0^2) \overset{a}{\sim} \mathcal{N}(0, 2\sigma_o^4)\,. \tag{A2.2}$$

Since, under the null hypothesis H, $\hat{\sigma}_a^2$ is the maximum-likelihood estimator of σ_o^2 using *every* observation, it is asymptotically efficient. Therefore, following Hausman's (1978) approach, we conclude that the asymptotic variance of $\sqrt{nq}(\hat{\sigma}_b^2 - \hat{\sigma}_a^2)$ is simply the difference of the asymptotic variances of $\sqrt{nq}(\hat{\sigma}_b^2 - \sigma_o^2)$ and $\sqrt{nq}(\hat{\sigma}_a^2 - \sigma_o^2)$. Thus, we have

$$\sqrt{nq}\,J_d(r) \equiv \sqrt{nq}(\hat{\sigma}_b^2 - \hat{\sigma}_a^2) \overset{a}{\sim} \mathcal{N}(0, 2(q-1)\sigma_o^4)\,. \tag{A2.3}$$

The asymptotic distribution of the ratio then follows by applying the "delta method" to the quantity $\sqrt{nq}(g(\hat{\sigma}_a^2, \hat{\sigma}_b^2) - g(\sigma_o^2, \sigma_o^2))$, where the bivariate function g is defined as $g(u, v) \equiv v/u$; hence,

$$\sqrt{nq}\,J_r(q) = \sqrt{nq}\left(\frac{\hat{\sigma}_b^2}{\hat{\sigma}_a^2} - 1\right) \overset{a}{\sim} \mathcal{N}(0, 2(q-1))\,. \tag{A2.4}$$

Q.E.D.

Proof of Theorem 2.2

To derive the limiting distributions of $\sqrt{nq}\, M_d$ and $\sqrt{nq}\, M_r$, we require the asymptotic distribution of $\sqrt{nq}(\hat{\sigma}_c^2 - \sigma_o^2)$ (the dependence of $\hat{\sigma}_c^2$ on q is suppressed for notational convenience). Our approach is to reexpress this variance estimator as a function of the autocovariances of the $(X_k - X_{k-q})$ terms and then employ well-known limit theorems for autocovariances. Consider the quantity

$$
\begin{aligned}
\hat{\sigma}_c^2 &= \frac{1}{nq^2} \sum_{k=q}^{nq} (X_k - X_{k-q} - q\hat{\mu})^2 \\
&= \frac{1}{nq^2} \sum_{k=q}^{nq} \left[\sum_{j=1}^{q} (X_{k-j+1} - X_{k-j} - \hat{\mu}) \right]^2 \qquad \text{(A2.5a)} \\
&= \frac{1}{nq^2} \sum_{k=q}^{nq} \left(\sum_{j=1}^{q} \hat{\epsilon}_{k-j+1} \right)^2 \qquad \text{(A2.5b)}
\end{aligned}
$$

where $\hat{\epsilon}_{k-j+1} \equiv X_{k-j+1} - X_{k-j} - \hat{\mu}$. But then we have

$$
\begin{aligned}
\hat{\sigma}_c^2 &= \frac{1}{nq^2} \sum_{k=q}^{nq} \left(\sum_{j=1}^{q} \hat{\epsilon}_{k-j+1}^2 + 2\sum_{j=1}^{q-1} \hat{\epsilon}_{k-j+1}\, \hat{\epsilon}_{k-j} \right. \\
&\quad \left. + 2\sum_{j=1}^{q-2} \hat{\epsilon}_{k-j+1}\, \hat{\epsilon}_{k-j-1} + \cdots + 2\hat{\epsilon}_k\, \hat{\epsilon}_{k-q+1} \right) \qquad \text{(A2.6a)} \\
&= \frac{1}{nq^2} \left\{ q \sum_{k=1}^{nq} \hat{\epsilon}_k^2 - \sum_{k=1}^{q-1} [(q-k)\hat{\epsilon}_k^2 + k\hat{\epsilon}_{nq-q+k+1}^2] \right. \\
&\quad + 2(q-1) \sum_{k=2}^{nq} \hat{\epsilon}_k\, \hat{\epsilon}_{k-1} - 2\sum_{k=2}^{q-1} [(q-k)\hat{\epsilon}_k\, \hat{\epsilon}_{k-1} \\
&\quad + (k-1)\hat{\epsilon}_{nq-q+k+1}\, \hat{\epsilon}_{nq-q+k}] \\
&\quad + 2(q-2) \sum_{k=3}^{nq} \hat{\epsilon}_k\, \hat{\epsilon}_{k-2} - 2\sum_{k=3}^{q-1} [(q-k)\hat{\epsilon}_k\, \hat{\epsilon}_{k-2} \\
&\quad + (k-2)\hat{\epsilon}_{nq-q+k+1}\, \hat{\epsilon}_{nq-q+k-1}] \\
&\quad \left. + \cdots + 2\sum_{k=q}^{nq} \hat{\epsilon}_k\, \hat{\epsilon}_{k-q+1} \right\} \qquad \text{(A2.6b)}
\end{aligned}
$$

$$= \hat{\gamma}(0) - o_p(n^{-1/2}) + \frac{2(q-1)}{q}\,\hat{\gamma}(1) - o_p(n^{-1/2})$$
$$+\frac{2(q-2)}{q}\,\hat{\gamma}(2) - o_p(n^{-1/2}) + \cdots + \frac{2}{q}\,\hat{\gamma}(q-1) \qquad \text{(A2.6c)}$$

where $\hat{\gamma}(j) \equiv (1/nq)\sum_{k=j+1}^{nq} \hat{\epsilon}_k \hat{\epsilon}_{k-j}$ and $o_p(n^{-1/2})$ denotes a quantity that is of an order smaller than $n^{-1/2}$ in probability. Now define the $(q \times 1)$ vector $\hat{\gamma} \equiv [\hat{\gamma}(0)\,\hat{\gamma}(1)\ldots\hat{\gamma}(q-1)]'$. A standard limit theorem for sample autocovariances $\hat{\gamma}$ of a stationary time series with independent Gaussian increments is (see, for example, Fuller, 1976, chap. 6.3)

$$\sqrt{nq}(\hat{\gamma} - \sigma_o^2 e_1) \overset{a}{\sim} \mathcal{N}[0, \sigma_o^4(I_q + e_1 e_1')] \qquad \text{(A2.7)}$$

where e_1 is the $(q \times 1)$ vector $[1\,0\ldots0]'$ and I_q is the identity matrix of order q. Returning to the quantity $\sqrt{nq}(\hat{\sigma}_c^2 - \sigma_o^2)$, we have

$$\sqrt{nq}(\hat{\sigma}_c^2 - \sigma_o^2) = \sqrt{nq}\Big[(\hat{\gamma}(0) - \sigma_o^2) + \frac{2(q-1)}{q}\,\hat{\gamma}(1) + \cdots$$
$$+ \frac{2}{q}\,\hat{\gamma}(q-1)\Big] - \sqrt{nq}\, o_p(n^{-1/2})\,. \qquad \text{(A2.8)}$$

Combining Equations (A2.7) and (A2.8) then yields the following result:

$$\sqrt{nq}(\hat{\sigma}_c^2 - \sigma_o^2) \overset{a}{\sim} \mathcal{N}(0, V_c) \qquad \text{(A2.9a)}$$

where

$$V_c \equiv 2\sigma_o^4 + \left(\frac{2(q-1)}{q}\right)^2 \sigma_o^4 + \cdots + \left(\frac{2}{q}\right)^2 \sigma_o^4 = 2\sigma_o^4\left(\frac{2q}{3} + \frac{1}{3q}\right). \qquad \text{(A2.9b)}$$

Given the asymptotic distributions (A2.1) and (A2.5), Hausman's (1978) method may be applied in precisely the same manner as in Theorem 2.1 to yield the desired result:

$$\sqrt{nq}\, M_d(q) \overset{a}{\sim} \mathcal{N}\left(0, \frac{2(2q-1)(q-1)}{3q}\,\sigma_o^4\right)$$
$$\sqrt{nq}\, M_r(q) \overset{a}{\sim} \mathcal{N}\left(0, \frac{2(2q-1)(q-1)}{3q}\right).$$

The distributional results for $\overline{M}_d(q)$ and $\overline{M}_r(q)$ follow immediately since asymptotically these statistics are equivalent to $M_d(q)$ and $M_r(q)$, respectively.

Q.E.D.

Proof of Theorem 2.3

1. We prove the result for $\overline{M}_r(q)$; the proofs for the other statistics follow almost immediately from this case. Define the increments process as $Y_t \equiv X_t - X_{t-1}$ and define $\hat{\rho}(\tau)$ as

$$\hat{\rho}(\tau) \equiv \frac{\frac{1}{nq}\sum_{t=\tau}^{nq}(Y_t-\hat{\mu})(Y_{t-\tau}-\hat{\mu})}{\frac{1}{T}\sum_{t=1}^{T}(Y_t-\hat{\mu})^2} \equiv \frac{A(\tau)}{B(\tau)}\,. \tag{A2.10}$$

Consider first the numerator $A(\tau)$ of $\hat{\rho}(\tau)$:

$$\begin{aligned} A(\tau) &\equiv \frac{1}{nq}\sum_{t=\tau}^{nq}(Y_t-\hat{\mu})(Y_{t-\tau}-\hat{\mu}) \\ &= \frac{1}{nq}\sum_{t=\tau}^{nq}(\mu-\hat{\mu}+\epsilon_t)(\mu-\hat{\mu}+\epsilon_{t-\tau}) && \text{(A2.11a)} \\ &= \frac{nq-\tau+1}{nq}(\mu-\hat{\mu})^2+(\mu-\hat{\mu})\frac{1}{nq}\sum_{t=\tau}^{nq}\epsilon_t \\ &\quad +(\mu-\hat{\mu})\frac{1}{nq}\sum_{t=\tau}^{nq}\epsilon_{t-\tau}+\frac{1}{nq}\sum_{t=\tau}^{nq}\epsilon_t\epsilon_{t-\tau}\,. && \text{(A2.11b)} \end{aligned}$$

Since $\hat{\mu}$ converges almost surely (a.s.) to μ, the first term of Equation (A2.11b) converges a.s. to zero as $nq \to \infty$. Moreover, under condition 2.1.2 it is apparent that $\{\epsilon_t\}$ satisfies the conditions of White's (1984) corollary 3.48; hence, H*'s condition 2.1.2 implies that the second and third terms of (A2.11b) also vanish a.s. Finally, because $\epsilon_t\,\epsilon_{t-\tau}$ is clearly a measurable function of the ϵ_t's, $\{\epsilon_t\,\epsilon_{t-\tau}\}$, is also mixing with coefficients of the same size as $\{\epsilon_t\}$. Therefore, under condition 2.1.2, corollary 3.48 of White (1984) may also be applied to $\{\epsilon_t\,\epsilon_{t-\tau}\}$, for which condition 2.1.2 implies that the fourth term of Equation (A2.11b) converges a.s. to zero as well. By similar arguments, it may also be shown that

$$B(\tau) \equiv \frac{1}{nq}\sum_{t=1}^{nq}(Y_t-\hat{\mu})^2 \overset{\text{a.s.}}{\to} \sigma_o^2\,. \tag{A2.12}$$

Therefore, we have $\hat{\rho}(\tau) \overset{\text{a.s.}}{\to} 0$ for all $\tau \neq 0$; hence, we conclude that

$$\overline{M}_r(q) \overset{\text{a.s.}}{\to} 0 \quad \text{as} \quad nq \to \infty\,.$$

2. By considering the regression of increments ΔX_t on a constant term and lagged increments ΔX_{t-j}, this follows directly from White and Domowitz

(1984). Taylor (1984) also obtains this result under the assumption that the multivariate distribution of the sequence of disturbances is symmetric.

3. This result follows trivially from Equation (2.1.14a) and condition 2.1.2.

Q.E.D.