Normal-Form Games

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Logistics

We are using part of the new book,**Multiagent Systems: Algorithmic, Game-Theoretic, and Logical Foundations**, by Yoav Shoham and Kevin Leyton-Brown, as the text for game theory.

Please find a copy of the draft book at http://www.eecs.harvard.edu/cs286r/files/SLB.pdf.

Today, we cover Chapter 3 (3.1, 3.2, 3.3.1, 3.3.2, 3.3.3, 3.4.3).

Problem Set 1, due Wednesday September 24 in class.

Outline

1. **Expected Utility Theory**

2. **Normal-Form Games**

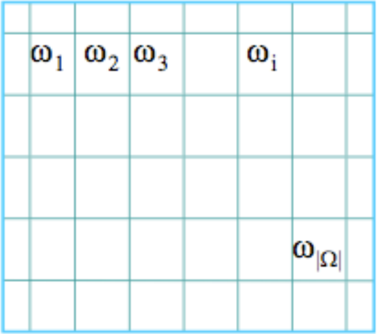
Agent's Preferences

- We talk about self-interested agents
- They have preferences over states of the world
  - $\omega_1 \succeq \omega_2$
    The agent weakly prefers ω_1 to ω_2.
  - $\omega_1 \sim \omega_2$ (defined as $\omega_1 \succeq \omega_2$ and $\omega_2 \succeq \omega_1$)
    The agent is indifferent between ω_1 and ω_2.
  - $\omega_1 \succ \omega_2$ (defined as $\omega_1 \succeq \omega_2$ and $\omega_2 \not\succeq \omega_1$)
    The agent is strongly prefers ω_1 to ω_2.

Utility of Agents

- Utility function $u(\omega_i)$ is a mapping from the state of the world to a real number.
- Utility function quantifies agent's preferences over the states of the world.
- If $\omega_1 \succeq \omega_2$, then $u(\omega_1) \geq u(\omega_2)$.
- It allows us to consider how uncertainty affects happiness of agents.

Uncertainty and Risk, in General



- ω are disjoint exhaustive states of the world
- Uncertainty: ω_1 or ω_2
- Risk: $\mathrm{P}(\omega_1)$ $\mathrm{P}(\omega_2)$

Decision Making Under Uncertainty

- Evaluate expected utility

$$E[u] = \sum_i \mathrm{P}(\omega_i) u(\omega_i)$$

- A rational agent maximizes expected utility
- Decisions or actions can affect $\mathrm{P}(\omega)$ or $u(\omega)$

Non-Cooperative Game Theory

- What is it?

  Mathematical study of interactions between rational and self-interested agents.

- Non-Cooperative

  Focus on agents who make their own individual decisions without coalition

Definition of Normal-Form Game

- A finite n-person game, $G =< N, A, u >$
  - $N = \{1, 2, ..., n\}$ is the set of players.
  - $A = \{A_1, A_2, ..., A_n\}$ is a set of available actions. $a = (a_1, a_2, ..., a_n) \in A$ is an action profile (or a pure strategy profile).
  - $u = \{u_1, u_2, ...u_n\}$ is a set of utility functions for n agents.
- A strategy is a complete contingent plan that defines the action an agent will take in all states of the world. Pure strategies (as opposed to mixed strategies) are the same as actions of agents.
- Players move simultaneously.
- Matrix representation for 2-person games

Example 1: Prisoner's Dilemma

- Two friends are arrested for crime. Each of them has two strategies (actions): Cooperate (keep silent) or Defect (talk). If both silent, serve a shor t term. If one talks, goes free other serves long time. If both talk, both serve moderate time.

	C	D
C	5, 5	0, 8
D	8, 0	1, 1

Player 1 is the row player. Player 2 is the column player. Utilities are in the form of (first player, second player).

Dominant and Dominated Strategies

- A strategy $a_i^* \in A_i$ is a strictly dominant strategy, if

$$u_i(a_i^*, a_{-i}) > u_i(a_i, a_{-i}) \quad \forall a_i \in A_i \text{ and } a_i \neq a_i^* \quad \forall a_{-i} \in A_{-i}$$

where $a_{-i} = (a_1, ..., a_{i-1}, a_{i+1}, ..., a_n)$. (weakly dominant, if $\geq$)

- A strategy $a_i \in A_i$ is a strictly dominated strategy, if there exists a strategy a_i' such that

$$u_i(a_i', a_{-i}) > u_i(a_i, a_{-i}) \forall a_{-i} \in A_{-i}.$$

(weakly dominated, if $\geq$)

Prisoner's Dilemma

	C	**D**
C	5, 5	0, 8
D	8, 0	1, 1

(Pure Strategy) Nash Equilibrium

- A pure strategy profile $a = (a_1, a_2, ..., a_n)$ is a weak Nash Equilibrium if, for all agents i and for all strategies $a'_i \neq a_i$, $u_i(a_i, a_{-i}) \geq u_i(a'_i, a_{-i})$.
- A pure strategy profile $a = (a_1, a_2, ..., a_n)$ is a dominant strategy Nash Equilibrium, if all a_i's are dominant strategy.

  In the previous Prisoner's Dilemma example, (D, D) is a strictly dominant strategy NE.

Example 2: Battle of the Sexes

- A husband and wife want to go to movies. They can select between "Devils wear Parada" and "Iron Man". They prefer to go to the same movie, but while the wife prefers "Devils wear Parada" the husband prefers "Iron Man". They need to make the decision independently.

Husband

		DWP	IM
Wife	DWP	10, 5	0, 0
	IM	0, 0	5, 10

Example 3

	L	C	R
U	0, 2	0, 1	3, 0
M	1, 0	1, 1	2, 0
D	0, 1	4, 2	0, 0

Iteratively Eliminate Strictly Dominated Strategies

- Choose an agent i and remove a strictly dominated strategy
- In the remaining game, repeat the process
- If a unique strategy profile survives, it is the unique Nash Equilibrium
- If there is a Nash Equilibrium, it will survive this procedure.

Example 3

	L	C	R
U	0, 2	0, 1	3, 0
M	1, 0	1, 1	2, 0
D	0, 1	4, 2	0, 0

Best-Response Correspondences

- The best-response correspondence $\text{BR}_i(a_{-i}) \in A_i$ are the set of strategies that maximizes agent i's utility given other agents' strategy a_{-i}.

- Compute every agent's best-response correspondences. The fixed points of the best-response correspondences, i.e. $a^* \in \text{BR}(a^*)$, are the NEs of the game.

Example 4: Continuous Strategy Space

Cournot Competition

- Two suppliers producing a homogeneous good need to choose their production quantity, q_i and q_2. The demand that they are facing is $p(Q) = 1000 - Q$, where $Q = q_1 + q_2$. Unit cost $c > 0$.
- Utility: $u_1(q_1, q_2) = q_1 * [p(q_1 + q_2) - c]$
- Best-response of supplier 1 is $\mathrm{BR}_1(q_2) = \arg\max_{q_1} u_1(q_1, q_2)$

Example 5: Matching Pennies

- Each of the two players has a penny. They independently choose to display either heads or tails. If the two pennies are the same, player 1 takes both pennies. If they are different, player 2 takes both pennies.

	Heads	Tails
Heads	1, -1	-1, 1
Tails	-1, 1	1, -1

Example 6: Rock, Paper, Scissors

	Rock	Paper	Scissors
Rock	0, 0	-1, 1	1, -1
Paper	1, -1	0, 0	-1, 1
Scissors	-1, 1	1, -1	0, 0

Mixed Strategies

- A mixed strategy of agent i, $\sigma_i \in \mathbf{\Delta}(A_i)$, defines a probability, $\sigma_i(a_i)$ for each pure strategy $a_i \in A_i$.
- Agent i's expected utility of following strategy σ_i is

$$u_i(\sigma) = \sum_{a \in A_i} P_\sigma(a) u_i(a)$$

- The support of σ_i is the set of pure strategies $\{a_i : \sigma(a_i) > 0\}$.
- Pure strategies are special cases of mixed strategies.

Mixed Strategy Nash Equilibrium

- Mixed-strategy prole $\sigma^* = \{\sigma_1^*, \sigma_2^*, ..., \sigma_n^*\}$ is a Nash equilibrium in a game if, for all i,

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*), \quad \forall \sigma_i \in \mathbf{\Delta}(A_i).$$

- Theorem (Nash 1951): Every game with a finite number of players and action profiles has at least one Nash equilibrium.

- All pure strategies in the support of agent i at a mixed strategy Nash Equilibrium have the same expected utility.

Finding Mixed Strategy Nash Equilibrium

	Heads	Tails
Heads	1, -1	-1, 1
Tails	-1, 1	1, -1

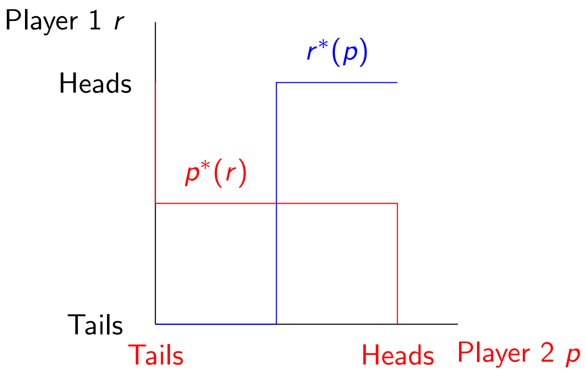
- Use the fact that all pure strategies in the support at a mixed strategy NE has the same expected utility.

Finding Mixed Strategy Nash Equilibrium

	Heads	Tails
Heads	1, -1	-1, 1
Tails	-1, 1	1, -1

- Find the fixed point of the Best-Response correspondences.

Fixed Point



Pareto Optimality

Now we can use s to represent a generic strategy profile, either pure or mixed.

- Strategy profiles s Pareto dominates strategy profile s' if for all $i \in N$, $u_i(s) \geq u_i(s')$ and there exists some $j \in N$ for which the inequality is strict.

- Strategy profile s is Pareto optimal, or strictly Pareto efficient, if there does not exist another strategy profile s' that Pareto dominates s.

Example 2: Prisoner's Dilemma

	C	**D**
C	5, 5	0, 8
D	8, 0	1, 1