# CS286r Fall 2009: Homework 1

**Due: September 16, in class.**

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*Points will be awarded for clarity, correctness and completeness of the answers. Submissions should be brought to class.* **There is one late day, to be used over homework 1 and 2.** *You may work in a pair and submit only one solution. But you must understand your solution.* Students with a solid background in game theory can skip Q2 completely. Students that have taken CS 285 or CS 286r last year can skip Q2 (a)-(d). Drop us a note if you think this includes you! You can also skip Q1 by petition.

*Total points: 145*

1. **Social choice theory.**

   (a) (10 pts) Unanimity (U) and IIA (independence of irrelevant alternatives) are defined on p.212 of the AGT book. A social welfare function $F$ is weak Pareto efficient (WP) if for any alternatives $a, b \in A$ such that $a \prec_i b$ for all $i \in I$ then $a \prec b$ for $\prec = F(\prec_1, \ldots, \prec_n)$. Prove that (U) and (IIA) $\Rightarrow$ (WP).

   (b) (25 pts) For $|A| = 2$, show how to use the *majority* (or plurality) voting rule to obtain a *social welfare function*, i.e. a function $F : L(A)^n \rightarrow L(A)$, that is (U), (IIA) and non-dictatorial. Argue that it has these properties. Construct an example to show that the natural extension of the idea to $|A| = 3$ fails one of these properties.

   (c) (5 pts) The monotonicity (MON) and dominant-strategy incentive-compatibility, or strategyproofness (SP) of a social choice function is defined on p.214 of the AGT book. Some find the proof of Prop. 9.6 a bit mysterious. Provide your own proof, via steps $(MON) \Rightarrow (SP)$ and with $(SP) \Rightarrow (MON)$ established via the contrapositive $\neg(MON) \Rightarrow \neg(SP)$.

   (d) (10 pts) Prove that a social choice function that is *onto* and MON satisfies *unanimity*, defined for a SCF $f$ as "if every agent $i$ has $a$ as its most preferred alternative then $f(\prec_1, \ldots, \prec_n) = a$."

   (e) (15 pts) See p.245-247 of AGT, in regards to single-peaked preferences. Give an example of a social choice function for each of the following cases that is not a median rule (*no proofs necessary*):

i. (SP) and onto
ii. (SP) and anonymous
iii. onto and anonymous

(f) (25 pts) Prove that the generalized median selection mechanism (Thm 10.2) in the AGT book is SP, onto, and anonymous. Is it dictatorial? Is it Pareto optimal for strict preferences, however the phantom peaks are placed?

(g) (5 pts) Explain how to position the additional $n-1$ "phantom peaks" of the median selection mechanism to implement the $k$th order statistic of the peaks announced by the agents (i.e., the alternative at the $k$th highest peak.)

(h) (10 pts) Suppose $|A| \geq 3$, and consider preference domain $\prec_i \in L(A)$ that includes all possible strict orders. Show that this SCF,

$$f(\prec_1, \ldots, \prec_n) = \begin{cases} a & \text{, if for all } i \text{ we have } a \succ_i b \text{ for all } b \neq a \\ a^* & \text{, otherwise.} \end{cases}$$

for some $a^* \in A$, is not SP.

2. **Strategic-form games.**

(a) (10 pts) In the following strategic-form game, what strategies survive iterated elimination of strictly-dominated strategies? What are the two pure strategy Nash equilibrium? Find a non-trivial ( i.e., with support $> 1$) mixed-strategy NE.

| | L | C | R |
|---|---|---|---|
| T | 2,0 | 1,1 | 4,2 |
| M | 3,4 | 1,2 | 2,3 |
| B | 1,3 | 0,2 | 3,0 |

(b) (10 pts) Two agents are bargaining over how to split a dollar. Each simultaneously names the share it would like to have, $s_1$ and $s_2$, where $0 \leq s_1, s_2 \leq 1$. If $s_1 + s_2 \leq 1$, then the agents receive the shares they named; if $s_1 + s_2 > 1$, then the agents receive zero. What are the pure strategy Nash eq.?

(c) (5 pts) Show that there are no (non-trivial) mixed-strategy Nash eq. (i.e. with support greater than one) in the Prisoners' Dilemma game.

| | C | D |
|---|---|---|
| C | 1,1 | -1,2 |
| D | 2,-1 | 0,0 |

(d) (15 pts) *Battle of the Sexes*. Pat and Chris must choose to go for dinner or go to the movies. Both players would rather spend the evening together than apart, but Pat would rather the go for dinner, and Chris would rather they go to the movies.

| | | Chris | |
|---|---|---|---|
| | | Dinner | Movie |
| Pat | Dinner | 2,1 | 0,0 |
| | Movie | 0,0 | 1,2 |

Find two pure Nash equilibria of this game. Let $(q, 1-q)$ be the mixed strategy in which Pat plays Dinner with prob. $q$, and let $(r, 1-r)$ be the mixed strategy in which Chris plays Dinner with prob. $r$. By first determining the best-response correspondences $q^*(r)$ and $r^*(q)$, or otherwise, find the mixed-strategy Nash equilibrium.

(e) Iterated elimination of strictly dominated strategies.

i. (10 pts) Prove that if strategies, $s^* = (s_1^*, \ldots, s_n^*)$, are a Nash eq. in a strategic-form game $G =< N, (S_i), (u_i) >$, then they survive iterated elimination of strictly dominated strategies. (*Hint: proceed by contradiction.*)

ii. (10 pts) Prove that if the process of iterated elimination of strictly dominated strategies in game $G =< N, (S_i), (u_i) >$ results in a *unique* strategy profile, $s^* = (s_1^*, \ldots, s_n^*)$, that this is a Nash eq. of the game. (*Hint: proceed by contradiction.*)

*Good luck!*