Arrow's Theorem

Alternatives A

Agents $N = \{1, \dots, n\}$

$<_i \in L(A)$ total order on A

$a <_i b$ i prefers a to b

SWF $F: L(A)^n \to L(A)$

SCF $f: L(A)^n \to A$

Properties

(U) unanimity $F(<, <, \dots, <) = <$

(IIA) For every $a, b \in A$, every $<_1 \dots <_n$, $<'_1, \dots, <'_n$,
$a <_i b \Leftrightarrow a <'_i b$ for all i implies $a <_F b \Leftrightarrow a <'_F b$

(ND) non dictatorial; no i s.t. $F(<_1 \dots <_n) = <_i$ for all $<_1 \dots <_n \in L(A)$

Thm If $|A| \geq 3$, every SWF that is (U) and (IIA) is (Dictatorial)

(WP) For every $a, b \in A$, if $a <_i b$ for all i then $a <_F b$

Lemma (U) and (IIA) $\Rightarrow$ (WP)

Proof of Arrow's thm via Geanokoplos' 05 "first proof"

Extremal lemma

If every i puts b at bottom or top of $<_i$ then any (IIA) and (WP) SWF must put b at bottom or top.

Proof By contradiction, suppose $c <_F b$, $b <_F a$. By IIA, unchanged if move c above a for all i. By (WP) need $a <_F c$, contradiction.

Eg.

$$\begin{array}{ccc|c} b & b & a & a \\ c & a & c & b \\ a & c & b & c \\ & & & F \end{array} \longrightarrow \begin{array}{ccc|c} b & b & c & a \\ c & c & a & b \\ a & a & b & c \\ & & & F \end{array}$$

(by IIA) .. but fails (WP)

Extremely Pivotal lemma

For any b, exists $<_1 \dots <_n$ for which an agent i is extremely pivotal and can move b from bottom to top.

Proof

By (WP) By (WP)

Switch b bottom to top one-by-one; eventually get extreme pivot.

Proof of Arrow's thm

	i		F
b	c	c	
c	a	a	
a	b	b	b

α^{I}

		i	F
b	b	c	b
c	c	a	
a	a	b	

α^{II}

			F
b	a	c	a
a	b	a	$\vee$
c	c	b	c

α^{III}

			F
a	b	b	
b	a	c	
	$\vee$		
c	c	a	

α^{IV}

1. Show agent i from extremely pivotal lemma is a dictator for any pair a, c not involving b. Construct α^{III} by moving a above b in α_i, then by IIA have $b <_F a$ from α^{I} and $c <_F b$ from α^{II}. So, $c <_F a$. Construct α^{IV} by changing anything but c, a relation for all agents. So, $c <_F a$ by IIA and now all preferences arbitrary except $a \succ_i c$ for agent i.

2. Show i is dictator for any pair a, b. Consider third outcome, c. Some j is a dictator for any $\{\alpha, \beta\}$ not including c, including $\{a, b\}$. But i controls $\{a, b\}$ from α^{I} to α^{II} and j must be i.

Muller-Satterthwaite Theorem

If $|A| \geq 3$, any SCF that satisfies (U) and (M) is dictatorial.

Agent i always gets top choice

If all agree on top choice, must pick.

$f(\alpha_1 .. \alpha_n) = a$
$\neq a' = f(\alpha_1 .. \alpha_i' \alpha_n)$
then
$a' <_i a$ and
$a <_i' a'$.

Proof via using f to construct SWF F violating Arrow's thm.

(SP) SCF f is (SP) if
$a = f(\alpha_1 .. \alpha_n) \succ_i f(\alpha_1 .. \alpha_i' .. \alpha_n) = a'$
for all $\alpha_1 .. \alpha_n, \alpha_i'$

Lemma Monotonic $\Leftrightarrow$ SP

Gibbard-Satterthwaite thm

If $|A| \geq 3$, any SCF that is (SP) and onto must be (Dictatorial)

Lemma (Onto) and (SP) $\Rightarrow$ (U)

Follows as a corollary of ~~M-S~~ Muller-Satterthwaite

How to escape?

Structured preferences

- Add money / QL
- Single-peaked prefs
- House allocation
  - Two sided matching

Utility theory

C T S D M C

exists $u_i : O \to [0,1]$ s.t. $u_i(o_1) \geq u_i(o_2) \Leftrightarrow o_1 \succsim o_2$

$u_i(\text{lottery}) =$ exp. util.

B. Normal form game (N, A, u)

		p	H
		LW	WL
S	LW	2,1	0,0
N	WL	0,0	1,2

$u_i : A \to \mathbb{R}$

strategic interdependence

$s_i \in \Delta(A_i)$

$U_i(s_1 .. s_n) = \sum_{a \in A_1 .. A_n} u_i(a_1, \ldots, a_n) \prod_j s_j(a_j)$

$2p + 0(1-q) = 0p + 1(1-p) \Leftrightarrow p = 1/3$

$q = 2/3$

$\begin{bmatrix} 2/9 & 4/9 \\ 1/9 & 2/9 \end{bmatrix}$ Payoff $\frac{4}{9} \times \frac{3}{2} = \frac{2}{3}$

Defn.

s^* is NE if $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$ $\forall i, \forall s_i$

Mixed
- randomizing
- belief about play of other
- large + random matching

C. Existence

Convex comb., weights $\sum_0^n \lambda_i = 1$, $\lambda_i \geq 0$ $T_n \subseteq \mathbb{R}^{n+1}$

tile, intersect on entire face

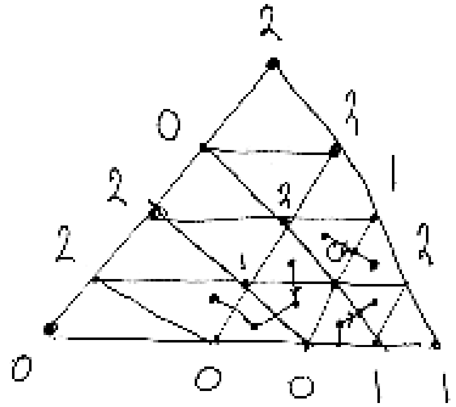
Sperner's Lemma Simplex $T_n = \underbrace{x^0 \cdots x^n}_{\in \mathbb{R}^{n+1}}$, simplicially subdivided with proper labeling L has odd # completely labeled subsimplexes.

$L : V \to \{0, \ldots, n\}$

$L(v) \in \chi(v)$ support

induction on n. Base case $n=0$ trivial.

one subsimplex, one label, ...



Uniquely defined. Either complete or one exit.

Odd # on subsimplex. $\boxed{\text{Walk}}$, either unique + get back or get complete.

Any not reached must pair. Can't go back to edge.

Brouwer Fixpoint Theorem $f \in \mathbb{R}^{m+1}$ $\sum f_i = 1$ $f_i \geq 0$

$f : \Delta_m \to \Delta_m$, continuous. Exists z s.t. $f(z) = z$.

Construct ε-simplicial subdivision

$h(v) \in X(v) \cap \{i : f_i(v) \leq v_i\}$

NB: can't all go larger else $\sum_i v_i = 1$ and $\sum_i f_i(v_i) > 1$.

lim $\varepsilon \to 0$, consider subsequence of centroids, ...
(complete)

~~So~~

$f_i(z) \leq z_i$ for all i

$\sum f_i(z) = 1$ ... $\boxed{f_i(z) = z_i}$ Fixpoint.

$\sum z_i = 1$

E Extend BFP to simplotope $\Delta(S_1) \times \ldots \Delta(S_n)$

Define $f: S \to S$ s.t. fix point $\equiv$ NE via function that puts more prob weight on better response.

D. Regret(a_{-i}, a_i)

MaxRegret $\max_{a_{-i}}$ Regret(a_{-i}, a_i)

MinMaxRegret $\min_{a_i} [\text{——}]$

Safety-level strategies

Tennenholtz, Boutilier, Pass & Halpern

E. Corr. Equil $\supseteq$ NE ... exist

continuum (convex comb.)

$s_i : D_i \to A_i$ $\quad \pi(d_1 \ldots d_n)$ common

$\cancel{u_i(s_i^*(d_i), s_{-i}^*(d_{-i})) \geq u_i(s_i'(d_i), s_{-i}^*(d_{-i}))}$

exp. utility $\mathbb{E}_{d_{-i}} [u_i(s_i^*(d_i), s_{-i}^*(d_{-i})) \mid \pi, d_i] \geq \mathbb{E}_{d_{-i}} [u_i(s_i'(d_i), s_{-i}^*(d_{-i}))]$

$\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$ same signal.

Payoff 1.5

Why

LP solve (even concise)

learning

THPE ε-Nash Selten 8

/

1,1	0,0
$1+\frac{\varepsilon}{2}$, 1	500,500

Problem

refinement.

Must remain BR to each of a fully mixed convergent sequence

s_i BR to s_{-i}^k s^0, s^1 & .. $\lim_{n \to \infty} s^n = s$

	L	R
U	1,1	2,0
D	0,2	2,2

(From Wikipedia)

(U,L) (D,R) NE

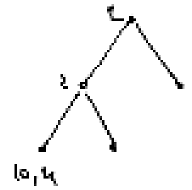
THPE ↳ in fact, player 2 will switch away from R

Extensive form THPE

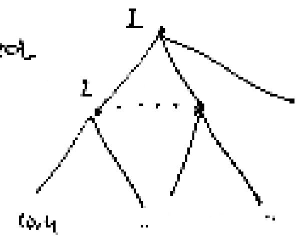
↳ move at all info sets

Extensive form

Perfect



Imperfect

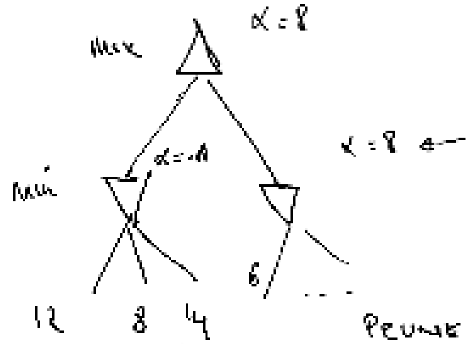


SPNE.
via backwards induction.

SE. fully mixed
$(S,\mu) = \lim_{n\to\infty} (S^n, \mu^n)$

S_i is BR for i given μ
Removes beliefs that are "incredible"....
threats (puts some prob. everywhere.)

α-β pruning
Zero-sum games



$\alpha = 8$ ← what can achieve just above

H. Repeated Games

Why do we see cooperation / g. PD). Punishment!

NE exist that support $r_1 \dots r_n$

(i) ~~feasible~~ enforceable $r_i \geq$ min max i
PUNISH

(ii) feasible available via average of cycling ...

+ Discounting. Perfect. ....

A. Games of Incomplete Information

$(N, X_1, \ldots, X_n, \Theta_1, \ldots, \Theta_n, u_1, \ldots, u_n)$

e.g. auctions

$u_i : X_1 \times \ldots \times X_n \times \Theta_i \to \mathbb{R}$

$s_i : \Theta_i \to X_i$

<u>Dom. strat. equil.</u> $u_i(s_i^*(\theta_i), s_{-i}(\theta_{-i}), \theta_i) \geq u_i(s_i'(\theta_i), s_{-i}(\theta_{-i}), \theta_i)$

$\forall i \; \forall \theta_i \; \forall \theta_{-i} \; \forall s_{-i} \; \forall s_i'$

<u>Ex post Nash equil</u> — " — $s_{-i}^*(\theta_{-i})$ — " — — " — $s_{-i}^*(\theta_{-i})$ — " —

$\forall i \; \forall \theta_i \; \forall \theta_{-i} \; \forall s_i'$

Bayesian games

↳ common prior $Pr(\Theta) \in [0,1]$ on joint types

<u>Bayes Nash equil</u> $\sum_{\theta_{-i} \in \Theta_{-i}} Pr(\theta_{-i} | \theta_i) \, u_i(s_i^*(\theta_i), s_{-i}^*(\theta_{-i}), \theta_i) \geq$

$\sum_{\theta_{-i} \in \Theta_{-i}} Pr(\theta_{-i} | \theta_i) \, u_i(s_i'(\theta_i), s_{-i}^*(\theta_{-i}), \theta_i)$

$\forall i \; \forall \theta_i \; \forall s_i'$

<u>Common prior</u>

Aumann People with same info have "no rational basis" for different priors

↳ if due to info processing then model directly

Posteriors <u>may</u> differ through info. acquisition

Morris: methodological defense

↳ but judiciously relax

Becomes important in MAS because need to make inferences on θ_i

$\boxed{B.}$ Escaping Gibbard-Satterthwaite

$f : L^n \to A$

DSIC, onto, $|A| \geq 3$, then dictatorial

Quasi-linear

$v_i : A \times \Theta_i \to \mathbb{R}$

$u_i = v_i(a, \theta_i) - m$

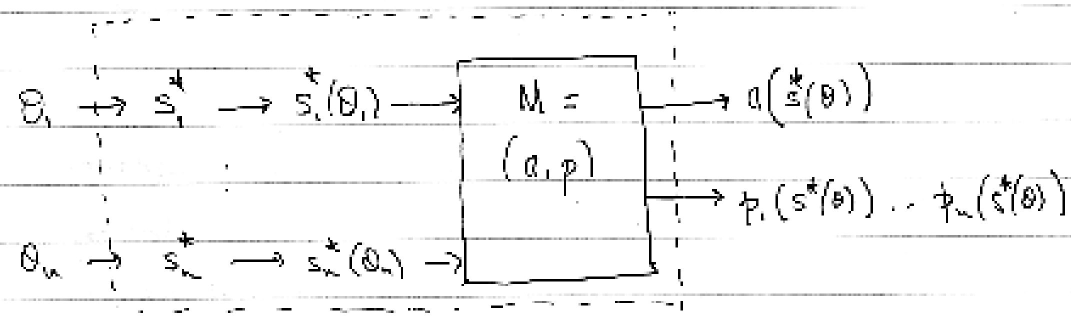
Mechanism $(X, a, p_1 \dots p_n)$ — payment rule

$\hookrightarrow p_i : X_1 \times \dots \times X_n \to \mathbb{R}$

$X_1 \dots X_n$

$a : X_1 \times \dots \times X_n \to A$

outcome rule



$M = (a, p)$ <u>implements</u> $f : \Theta_1 \times \dots \times \Theta_n \to A$ in DSE if $f(\theta) = a(s^*(\theta))$ $\forall \theta$, and s^* is a DSE.

$\hookrightarrow$ one equilibrium sufficient (!)

<u>Revelation principle</u> If exist M that <u>implements</u> f in DSE then exists a direct, DSIC M' that implements f.

<u>Direct Mechanism</u> (Θ, f, p)

$\hookrightarrow p_i : \Theta_1 \times \dots \times \Theta_n \to \mathbb{R}$

$\Theta_1 \times \dots \Theta_n$

$f : \Theta_1 \times \dots \Theta_n \to A$

C. Strategyproof mechanisms: VCG

↳ Role of indirect

↳ SP for free (?)

Groves

$$f(\theta) \in \arg\max_a \sum_i v_i(a, \theta_i)$$

$$p_i(\theta) = h_i(\theta_{-i}) - \sum_{j \neq i} v_j(a^*, \theta_j) \underset{(VCG)}{=} \sum_{j \neq i} v_j(a^{-i}, \theta_j) - \sum_{j \neq i} v_j(a^*, \theta_j)$$

$a^* = f(\theta)$ $\quad$ $a^{-i} = f(\theta_{-i})$

DSIC: Fix $\hat{\theta}_{-i}$. Utility $v_i(a, \theta_i) + \sum_{j \neq i} v_j(a, \hat{\theta}_j) - h_i(\theta_{-i})$

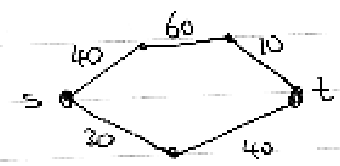
$$a \in \arg\max_a v_i(a, \hat{\theta}_i) + \sum_{j \neq i} v_j(a, \hat{\theta}_j)$$

Examples

(i) Single item auction $\{10, 6, 3\}$

(ii) two units for sale

(iii) ~~Graph~~ Lowest-cost routing



$$p_e = (W - d_{-e}) - (W - d^* + c_e) = d^* - c_e - d_e$$
$$= (70 - 30) - 110$$
$$= -70$$

(iv) (AB, \$2)
(A, \$2)
(B, \$2)

Utility: $\sum_j v_j(a^*, \theta_j) - \sum_{j \neq i} v_j(a^{-i}, \theta_j) \geq 0$ $\quad$ in social choice, allocation ...

Revenue: $\sum_{j \neq i} v_j(a^{-i}, \theta_j) - \sum_{j \neq i} v_j(a^*, \theta_j) \geq 0$ $\quad$ unless we externality

Buy \$10 $\quad$ $5 - 0 = 5$

Sell \$5 $\quad$ $0 - 10 = -10$

Generalizes to $\sum_i \alpha_i v_i(a) + \beta(a)$

↳ Roberts: If $|A| \geq 3$, f is onto A, $\Theta_i = \mathbb{R}^A$ for all i, then only DSIC mechanisms are Groves.

$\boxed{D}$ Strategyproof: Characterization

~~Thm~~ $M = (f, p)$ is DSIC iff (for every i):

$\boxed{i}$ For every θ_{-i}, exists $p_a \in \mathbb{R}$ s.t. for all θ_i with $f(\theta_i, \theta_{-i}) = a$, then $p_i(\theta_i, \theta_{-i}) = p_a$

$\boxed{ii}$ For every θ_{-i}, every θ_i, $f(\theta_i, \theta_{-i}) \in \arg\max_{\substack{a_i \\ \in \text{ange}}} \left(v_i(a, \theta_i) - p_a \right)$

($\Leftarrow$) Easy

($\Rightarrow$) $\boxed{i}$ Get better price if not true — $\boxed{ii}$ Get better outcome if not true

E.g. GSP $\{10, 6, 3\}$

10	Pay 6	
6	Pay 3	
3	Lose	

Satisfies $\boxed{i}$ but not $\boxed{ii}$ if value 2nd slot ≈ 1st slot

$\boxed{E}$ Monotonicity

WMON $f(\theta_i, \theta_{-i}) = a \neq b = f(\theta_i', \theta_{-i})$

$$\Rightarrow v_i(b, \theta_i') - v_i(a, \theta_i') \geq v_i(b, \theta_i) - v_i(a, \theta_i)$$

Necessary

$v_i(a, \theta_i) - p_a \geq v_i(b, \theta_i) - p_b$ $\quad f(\theta_i) = a$

$v_i(b, \theta_i') - p_b \geq v_i(a, \theta_i') - p_a$ $\quad f(\theta_i') = b$

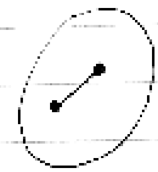
$\Leftrightarrow v_i(a) + v_i(b, \theta_i') \geq v_i(b, \theta_i) + v_i(a, \theta_i')$

$\boxed{\text{WMON} + \text{Convex } \Theta_i \Rightarrow \text{DSIC}}$

N.B. Payment via graph-theoretic argument

N.B. $\Theta_i = \mathbb{R}^B$ WMON ≡ Affine max

$\Theta_i \subseteq \mathbb{R}^A$ ↓ Convex



CA's $v_i(S,\theta_i) \ge v_i(T,\theta_i)$, $S \supseteq T$

$v_i(\emptyset,\theta_i) = 0$

$\Theta_i \equiv 2^G$ numbers

convex space

But if structure, e.g. "double-minded" then not convex

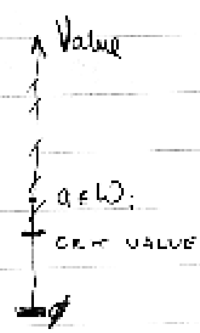
(AB, \$10) (CD, \$20) θ_i

(AC, \$5) (BD, \$15) θ_i'

(AB \$5 AC \$2.50 CD \$10 BD \$7.50)

F Single (1.5) parameter settings

$$v_i(a,\theta_i) = \begin{cases} w_i & \text{if } a \in T_i \\ 0 & \text{otherwise} \end{cases}$$



MON ⇔ DSIC

charge crit. value

Single-minded CA's

$$v_i(S,\theta_i) = \begin{cases} w_i & \text{if } S \supseteq S_i \\ 0 & \text{otherwise} \end{cases}$$

DSIC ⇔ MON in (w_i, S_i)

Charge min w_i' s.t. win at S_i

type: (w_i, S_i)

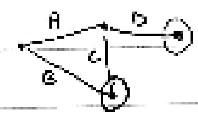
$w_i \in \mathbb{R}$ $S_i \subseteq G$

Win (AB, \$7) → (A, \$7) ?

→ (ABC, \$7) ?

NP-hard via reduction from INDEPENDENT SET

Given undirected graph
Is there a set of ~~…~~ vertices size $\geq k$ that share no edge?



Reduce

(AB, \$1)
(BC, \$1)
(ACD, \$1)
(D, \$1)

Is there ~~a set of bids with value $\geq$~~ \$K
an allocation of value

in NP?
NP-hard ; decision problem is NP-complete.

$\boxed{\text{APPROX}}$ c-approx $\frac{ALG(v)}{OPT(v)} \geq \frac{1}{c}$ for all v

E.g., sort + greedy allocate

(10, A)
(19, AB)
(8, B)

Pay \$11
(22, AB) win

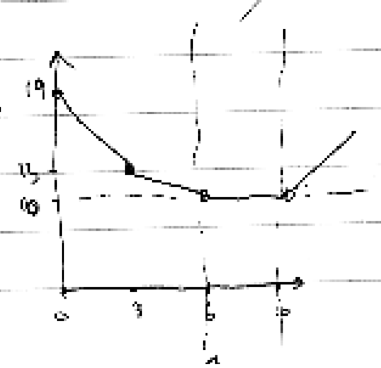
Now ...
charge critical value

In fact, rank by $\frac{v_i}{\sqrt{|S_i|}}$ get $\sqrt{m}$-approx (best possible)

$\boxed{\text{LP/DUAL}}$ $\boxed{D^* \geq V^*}$

Walrasian equilibrium

$\{10, 6, 3\}$



max $\sum x_i v_i$
s.t. $x_i \leq 1$ $\quad \pi_i$
$\sum x_i \leq 1$ $\quad p$
$x_i \geq 0$

min $p + \sum \pi_i$
s.t. $p + \pi_i \geq v_i \quad \forall i$
$p, \pi_i \geq 0$
min $(p, v_i, 0)$

Languages

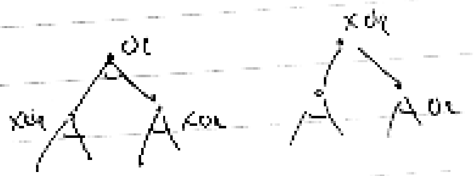
$2^{|G|}$ bundles!

Need concise + expressive

OR (additive?)

XOR (unit demand)

OR/XOR



OR* (ABd, \$10) (CDd, \$15) (A, \$5)

XOR

WDP via OR*

Feed into concise MIPs.

... some operations on Q_i