Standard Matching

Marriage \( x \in X = M \times W \) one-to-one

Medical \( x \in X = D \times H \) many-to-one

Hospital: quote in
short away or doctors
simply < s, s > but can pick best from set

DA (man propose)

DA (doctor proposes)

exists stable match
✓ strategyproof to man (woman)
doctor optimal stable, etc.
not EP to hospitals

With Wages

Job Market \( x \in X = F \times W \times S \) salary

Worker: \( u_i(j, s_{ij}) \)

\( (1, s_{100}) \succ (1, s_{90}) \succ (2, s_{80}) \succ (1, s_{70}) \succ \ldots \)

Firms: \( u_j(c_j) = \sum_{i \in c_j} s_{ij} \) quasi-linear

\( A \ 5 \) \( (A, 1) \succ (AB, 6) \succ (B, 6) \succ \ldots \)

\( B \ 5 \)

\( AB \ 8 \)
DA (fin. proposi)

0: Firm $j$ propose to most preferred worker at $S_{ij}$. Worker reject/had.

$t: S_j^t = \begin{cases} S_j^{t-1} + 1 & \text{if offer } (j, s_j^{t-1}) \text{ rejected} \\ S_j^{t-1} & \text{otherwise} \end{cases}

From $j$ propose to most preferred worker at $S_j^t$. Worker reject/had. (repeat any held from last round)

\% Gross substitutes

$C^*_j(s_j) = \arg \max_{C_j} \left[ v_j(C_j) - \left[ S_{ij} \right] \right]$

GS: if $i \in C^*_j(s_j)$ then $i$ is some $C^*_j(s_{j'})$ where $s_j'$ increases to some $i' \neq i$

\checkmark exist stable match form stable-optimal stable

\checkmark DSIC to worker: if use weather property

\times not DSIC to firm

Special cases

a) One worker: assuming price anchor

b) $N$ workers (indifferent): SAA

c) No money: hospital offer DA
contract $x \in X \equiv D \times H$

allocated $X' \subset X$

doctors: $P_d: x \ni y \ni z \ni \emptyset \quad C_d(x') \in \{x'\} \cup \{\emptyset\}$

short prof

hospitals: $P_h: \{d_1, d_2\} \ni \{d_1\} \ni \{d_2\} \ni \emptyset \quad C_h(x') \subseteq X' \cup \emptyset$

short prof on nets

$C_d(x') = \bigcup_{d \ni x'} \mathcal{D}_d(x') \quad C_h(x') = \bigcup_{h \ni x'} \mathcal{C}_h(x')$

$R_d(x') = X' - C_d(x') \quad R_h(x') = X' - C_h(x')$

Def: Contracts are substitutes if $\exists x''$ for every $x'$, $X' \subset x'' \implies R_h(x') \subset R_h(x'')$

ie., every contract rejected from $x'$ is rejected from $x'' \supset X$

Def: Law of aggregate demand if for every $x'$, $X' \subset x'' \implies |C_h(x')| \leq |C_h(x'')|$

eg. $P_h: \{d_1, d_3\} \ni \{d_1, d_2\} \ni \{d_1\} \ni \{d_2\}$

Thus there exists stable match
exists $d$-optimal and $h$-optimal stable match

doctor propose $DA$ is equilibrium for doctors

every $d, h$ pair sure to contract in all stable matches
Substitute
\[ x' < x'' \Rightarrow R_u(x') \subseteq R_u(x'') \]

Kelso-Cornwell
\[ x'' \]
\[
A: 52, 53, \ldots \\
B: 52, 53, \ldots
\]
\[ x' \]
\[
A: 53, \ldots \\
B: 52, \ldots
\]
Can't choose B at \( x'' \), not at \( x' \)
Can't reject B at \( x' \), not at \( x'' \)

Example
\[ \{x_1, z\} \cap \{z\} \cap \{z', z'\} \cap \{x\} \]
\[ x'' = \{x_1, z, z', z'\} \quad R_u(x'') = \{z', z'\} \]
\[ x' = \{x_1, z'\} \quad R_u(x') = \{x\} \]
Failed.
\[ \{d_3\} \cap \{d_1, d_2\} \cap \{d_1\} \cap \{d_2\} \]
\[ x'' = \{d_1, d_3\} \quad R_u(x'') = \{d_3\} \]
\[ x' = \{d_1\} \quad R_u(x') = \{\emptyset\} \]
OK.
$X^1$ stable if

(a) $C_D(X^1) = C_H(X^1) = X^1$

(b) no $L$, no $X'' \neq C_{H}(X')$ s.t. $X'' = C_{H}(X'UX'') \subset C_{D}(X'U)$

(a) individual rational

(b) no alternative set of contracts preferred by $H$ and $D$

**Theorem**

If $(X_0, X_H) \subset X^2$ solve

$X_0 = X - R_H(X_H)$ \hspace{1cm} (not rejected)

$X_H = X - R_D(X_0)$ \hspace{1cm} (offered)

Then $X_D \cap X_H = C_D(X_0) = C_H(X_H)$ and is stable.

Why? $X_D \cap X_H = X_D - (X - X_H) = X_D - R_H(X_H) = C_D(X_0)$

\[ C_H(X') = C_D(X') = X' = X_0 \cap X_H \hspace{1cm} (a) \]

For (b), notice that additional contracts are not in $X_D \cap X_H$ or $R_H(X_H)$ (or both) and so not preferred.

**Fix Point**

$F: X \times X \rightarrow X \times X$

$F_1(X') = X - R_H(X')$

$F_2(X') = X - R_D(X')$

$F(X_D, X_H) = (F_1(X_H), F_2(F_1(X_H)))$
Partial order

\[(X_0, X_H) \geq (X_0', X_H') \equiv (X_0 \supset X_0') \land (X_H \subseteq X_H')\]

i.e. more rejected \& more offered

\((X, 2)\) is a finite lattice, \((X, \emptyset)\) and \((\emptyset, X)\) are its maximal \& minimal elements.

**Lemma** \(F : X \times X \rightarrow X \times X\) is monotone weakly

\[(X_0, X_H) \geq (X_0', X_H') \Rightarrow F(X_0, X_H) \geq F(X_0', X_H')\]

when substitutions hold.

Then (via Tarski fixed point)

Fix point from non-empty lattice

starting from \((X, 0)\) get to \((X_0, X_H)\), i.e. stronger optimal
Example

\[ X_0(t) = X - R_H(X_H(t-1)) \]  not rejected

\[ X_H(t) = X - R_D(X_0(t)) \]  rejected

\[ P_d: h_1 \vee h_2 \]

\[ P_{d_e}: h_1 \vee h_2 \]

\[ P_{h_1}: \{d_1\} \neq \{d_2\} \neq \emptyset \]

\[ P_{h_2}: \{d_1, d_2\} \neq \{d_1\} \neq \{d_2\} \neq \emptyset \]

\[
\begin{array}{cccc}
\text{can not reject} & \text{not offered} & \text{can. offered} & \text{rejected} \\
X_b(t) & R_0(X_0(t)) & X_H(t) & R_H(X_H(t)) \\
\end{array}
\]

\[ t=1 \]

\[ \times \]

\[ (d_1, h_2) (d_2, h_2) \]

\[ (d_1, h_1) (d_2, h_1) \]

\[ (d_2, h_1) \]

\[ t=2 \]

\[ (d_1, h_1)(d_1, h_2) \]

\[ (d_1, h_2) \]

\[ (d_1, h_1)(d_2, h_2) \]

\[ (d_2, h_1) \]

\[ t=3 \]

\[ \ldots \]

\[ \text{unchanged} \]

Outcome: \[ \{ (d_1, h_1), (d_2, h_2) \} \]

doctor: optimal, stable
Thm 8. Subst \ \text{Law} \ \alpha_{ij}, \ \text{demand} \Rightarrow \text{rural hospitals theorem}

\[ M \ni |M| - |\{W\}| \quad \text{... same set \ \in \ \text{all}} \]
\[ M \ni |M'| = |\{W\}| \]

Thm 10. Subst \ implies \ if \ get \ x \ with \ P_d^* : z_1 > z_2 > \ldots > z_n > x \]
\[ \text{then get \ x \ with \ P_d : x} \]

Proof: Match under \ P_d^* : z_1 > z_2 > \ldots > z_n > x \ still \ \in \ \text{care}\]
under \ P_d : x \ because \ d \ \text{can't block (only change). Since get}
doctor-stable, must get same match.

Thm 11. Subst \ + \ \text{Law} \ \alpha_{ij}, \ \text{demand} \Rightarrow \text{DSIC for doctors}

Consider \ P_d : x \ and \ note \ that \ d \ employed \ at \ all \ stable
matches \ by \ RHT. \ Consider \ P_d^* : y_1 > y_2 > \ldots > y_n > x, \ note
that every outcome with \ d \ unemployed \ still \ blocked \ \Rightarrow \ get
at least \ x \ \in \ \text{doctor-optimal stable outcome}. \ Submitting
P_d : y_1 > y_2 > \ldots > y_n > x \ \Rightarrow \ \text{doesn't change (no new block .j.)}
Examples

Marriage
one-to-one
subst ✓
no contract rejected at x', not at x''
low avg demand ✓

Medical
many-to-one
subst ✓ (response)
don't stop rejecting at x'
low avg demand ✓

Job matching
many to one
QL + subst = avg demand
worker propose DSIC

Proxy anchor
many to one
x ∝ x = B × 2^x × R

bidder propose
anchorer accumulates a test, holds best set
gets anchor. Sethi for anchorer
⇒ stable
DSIC for big-group modular (sub case of subset)

Generalize anchor connection
\[ x \in X = N \times A \]

hidden agenda

strict profit both sides
substitution: don't stop restrict an net increase
ag demand: trivial

\[ \Rightarrow \text{Biddle papers DA \& DSIC} \]

E.g. \((a,1) \succ (a,2) \succ (a,3) \succ \ldots \succ (d,1) \succ \ldots \succ (d,3)\)

Bidders:
- \(p_1: (c,1) \succ (d,1) \succ \emptyset\)
- \(p_2: (d,2) \succ (b,2) \succ (a,2) \succ \emptyset\)
- \(p_3: \emptyset\)

Paper: Bid \((c,1), (a,2), \emptyset\).

#2 wins. Finally pay \((b,2)\)

DA

\[
\begin{array}{c|c|c}
  t=1 & B_1 & B_2 \\
  \hline
  (c,1) & (d,2) \\
  \hline
  t=2 & (c,1) & (b,2) \\
  t=3 & (d,1) & (b,2) \\
\end{array}
\]

\[ \Rightarrow \text{don't need bid-for-know knowledge, get iterative} \]

Prop 1 (R) Prop 2 (PE) Prop 3 (DSIC)