# Standard Matching

Marriage $x \in X \equiv M \times W$ one-to-one

Medical $x \in X \equiv D \times H$ many-to-one

hospital: Quota $q_h$

strict pref. on doctors

$\{1,4\} \underset{?}{<} \{2,3\}$ but can pick best from set

DA (man proposing)

DA (doctor proposing)

exists stable match

✓ strategyproof to man (woman)

doctor-optimal stable, etc.

× not SP to hospitals

## With Wages

Job market $x \in X \equiv F \times W \times \underset{\substack{\text{salary} \\ \text{(discrete)}}}{S}$

Workers: $v_i(j, s_{ij})$

$(1, \$100) \succ (1, \$90) \succ (2, \$80) \succ (1, \$80) \succ \ldots$

Firms: $v_j(C_j) - \sum_{i \in C_j} s_{ij}$ quasi-linear

A 5 $(A,1) \succ (AB, 6) \succ (B,4) \succ \ldots$

B 5

AB 8

DA (firm-proposing)

0: Firm j propose to most preferred worker at $s_{ij}^0$
Worker reject/hold.

$t$: $s_j^t = \begin{cases} s_j^{t-1} + 1 & \text{if offer } (j, s_j^{t-1}) \text{ rejected} \\ s_j^{t-1} & \text{otherwise} \end{cases}$

Firm j propose to most preferred workers at $s_{ij}^t$
Workers' reject/hold. (repeat any held from last round)

## Gross substitutes

$$C_j^*(s_j) \in \arg\max_{C_j} \left[ v_j(C_j) - \sum_{i \in C_j} s_{ij} \right]$$

GS: if $i \in C_j^*(s_j)$ then $i$ is some $C_j^*(s_j')$ where $s_j'$ increases to some $i' \neq i$

| | | |
|---|---|---|
| A | 0 | not ok! |
| B | 0 | |
| AB | 4 | |

✓ exists stable match
✓ ~~doctor~~ firm-optimal stable
✓ DSIC to worker if use worker-proposing
× not DSIC to firm

→ Milgrom "Assignment messages & exchanges" paper

Special cases

a) One worker : ascending-price auction
b) N workers (indifferent) : SAA
c) No money : hospital-offer DA

## Generalized Matching

contract $x \in X \equiv D \times H$

allocation $X' \subset X$

doctor : $P_d : x \succ y \succ z \succ \emptyset$ $\quad C_d(X') \in \{X'\} \cup \{\emptyset\}$

strict pref

hospitals : $P_h : \{d_1, d_2\} \succ \{d_1\} \succ \{d_2\} \succ \emptyset$ $\quad C_h(X') \subseteq X' \cup \{\emptyset\}$

strict pref on sets

$$C_D(X') = \bigcup_{d \in D} C_d(X') \qquad C_H(X') = \bigcup_{h \in H} C_h(X')$$

$$R_D(X') = X' - C_D(X') \qquad R_H(X') = X' - C_H(X')$$

Defn Contracts are substitutes if, for every $h$,

$$X' \subset X'' \Rightarrow R_h(X') \subset R_h(X'')$$

ie., every contract rejected from $X'$ is rejected from $X'' \supset X$

Defn Law of aggregate demand if for every $h$,

$$X' \subset X'' \Rightarrow |C_h(X')| \leq |C_h(X'')|$$

eg. $P_h : \{d_3\} \succ \{d_1, d_2\} \succ \{d_1\} \succ \{d_2\}$

Thm

exists stable match
exists d-optimal & h-optimal stable match
doctor proposing DA is strategyproof for doctors
every d, h sign same # contracts in all stable matches

Substitutes $\quad X' \subset X'' \Rightarrow R_h(X') \subset R_h(X'')$

Kelso-Crawford

$X''$ $\quad$ A $\$2, \$3, \ldots$
$\quad\quad$ B $\$2, \$3, \ldots$

$X'$ $\quad$ A $\$3, \ldots$
$\quad\quad$ B $\$2, \ldots$

Can't choose B at $X''$, not at $X'$

Can't reject B at $X'$ & not at $X''$

Example $\quad \{x, z\} \succ \{z\} \succ \{z'\} \succ \{x\}$

$X'' = \{x, z, z'\} \quad R_h(X'') = \{z'\}$

$X' = \{x, z'\} \quad R_h(X') = \{x\}$

Fails.

$\{d_3\} \succ \{d_1, d_2\} \succ \{d_1\} \succ \{d_2\}$

$X'' = \{d_1, d_3\} \quad R_h(X'') = \{d_1\}$

$X' = \{d_1\} \quad R_h(X') = \{\emptyset\}$

OK.

· $X'$ stable if

(a) $C_D(X') = C_H(X') = X'$

(b) no $h$, no $X'' \neq C_h(X')$ s.t. $X'' = C_h(X' \cup X'') \subset C_D(X' \cup X'')$

(a) – individual rational

(b) – no alternative set of contracts preferred by h and D

Theorem 1

If $(X_D, X_H) \subset X^2$ solves

$$X_D = X - R_H(X_H) \qquad \text{(not rejected)}$$

$$X_H = X - R_D(X_D) \qquad \text{(offered )}$$

then $X_D \cap X_H = C_D(X_D) = C_H(X_H)$ is stable.

Why? $X_D \cap X_H = X_D - (X - X_H) = X_D - R_D(X_D) = C_D(X_D)$

$= \quad .. \quad .. = C_H(X_H)$

$$C_H(X') = C_D(X') = X' = X_D \cap X_H \qquad \text{(a)}$$

For (b), notice that additional contracts not in $X_D \cap X_H$ are in $R_H(X_H)$ or $R_D(X_D)$ (or both) and so not preferred.

Fix Point

$F: X \times X \to X \times X$

$F_1(X') = X - R_H(X')$

$F_2(X') = X - R_D(X')$

$F(X_D, X_H) = (F_1(X_H), F_2(F_1(X_H)))$

Fixpoint

$X_D = F_1(X_H) = X - R_H(X_H)$

$X_H = F_2(F_1(X_H)) = X - R_D(X_D)$

Partial order

$$(X_D, X_H) \geq (X_D', X_H') \equiv (X_D \supset X_D') \wedge (X_H \subset X_{H'})$$

i.e. more rejected + more offered

$(X, \geq)$ is a finite lattice. $(X, \emptyset)$ and $(\emptyset, X)$ are its maximal + minimal elements.

<u>Lemma</u> $F: X \times X \to X \times X$ is weakly monotone increasing

$$(X_D, X_H) \geq (X_D', X_H') \Rightarrow F(X_D, X_H) \geq F(X_D', X_H')$$

when substitutes holds.

<u>Thm</u> (via Tarski fixed point)

Fix points form non-empty lattice

starting from $(X, 0)$ get to $(\bar{X}_D, \bar{X}_H)$, i.e. doctor optimal

Ex

## Example

$X_D(t) = X - R_H(X_H(t-1))$ not rejected

$X_H(t) = X - R_D(X_D(t))$ offered

$P_{d_1} : h_1 \succ h_2$

$P_{d_2} : h_1 \succ h_2$

$P_{h_1} : \{d_1\} \succ \{d_2\} \succ \emptyset$

$P_{h_2} : \{d_1, d_2\} \succ \{d_1\} \succ \{d_2\} \succ \emptyset$

| | cum not rejected | not offered | cum. offered | rejected |
|---|---|---|---|---|
| | $X_D(t)$ | $R_D(X_D(t))$ | $X_H(t)$ | $R_H(X_H(t))$ |
| $t=1$ | $X$ | $(d_1,h_2)$ $(d_2,h_2)$ | $(d_1,h_1)$ $(d_2,h_1)$ | $(d_2,h_1)$ |
| $t=2$ | $(d_1,h_1)(d_1,h_2)$ $(d_2,h_2)$ | $(d_1,h_2)$ | $(d_1,h_1)$ $(d_2,h_2)$ $(d_2,h_1)$ | $(d_2,h_1)$ |
| $t=3$ | .. | unchanged | | |

Outcome: $\{(d_1,h_1), (d_2,h_2)\}$

doctor-optimal, stable

Thm 8 Subst + Law agg. demand $\Rightarrow$ rural hospitals theorem

$$\begin{array}{lcl} \mu_M & |M| = |W| & \text{... same set in all} \\ & \vee\!\!\mid \quad \wedge\!\!\mid & \\ \mu_W & |M'| = |W'| & \end{array}$$

Thm 10 Subst $\Rightarrow$ if get $x$ with $P_d^*: z_1 > z_2 > .. > z_n > x$
then get $x$ with $P_d': x$

Proof Match under $P_d^*: z_1 > z_2 > .. > z_n > x$ still in core under $P_d': x$ because $d$ can't block (only change). Since get doctor-stable, must get same match.

Thm 11 Subst + law agg demand $\Rightarrow$ DSIC for doctors

Consider $P_d': x$ and note that $d$ employed at all stable matches by RHT. Consider $P_d^*: y_1 > y_2 > .. > y_n > x$, note that every outcome with $d$ unemployed still blocked $\Rightarrow$ get at least $x$ in doctor-optimal stable outcome. Submitting $P_d'': y_1 \; .. \; y_n \; x \; z_1 \; .. \; z_m$ doesn't change (no new blocking.)

Examples

Marriage one-to-one
subst ✓ no contract rejected at $X'$ not at $X''$
law agg demand ✓

Medical many-to-one
subst ✓ (responsive) don't stop rejecting as $X'$ increases
law agg demand ✓
doctor-proposing DSIC

Job matching many to one
QL + subst ⇒ agg demand
worker proposing DSIC

[Proxy auction many to one
$x \in X = B \times 2^G \times \mathbb{R}$
bidder-propose
~~hospital~~ auctioneer accumulates + tent. holds best set
gets monot. better for auctioneer
⇒ stable
DSIC for buyer-out modular (sub case of subst.)]

Generalize auction connection

## Qual. Vickrey Auction

one-to-one

$x \in X = N \times A$
bidder alternative

strict pref both side
substitutes : don't stop rejecting as set increases
agg demand : trivial

$\Rightarrow$ Bidder-propose DA is DSIC

E.g. $(a,1) \succ_A (a,2) \succ_A (a,3) \succ_A \cdots \succ_A (d,1) \succ_A \cdots \succ_A (d,3)$

Bidders: $P_{b_1} : (c,1) \succ (d,1) \succ \emptyset$
$P_{b_2} : (d,2) \succ (b,2) \succ (a,2) \succ \emptyset$
$P_{b_3} : \emptyset$

Paper: Bid $(c,1)$ $(a,2)$ $\emptyset$.
#2 wins. Finally pay $(b,2)$

DA

| | $B_1$ | $B_2$ |
|---|---|---|
| $t=1$ | $\boxed{(c,1)}$ | $(d,2)$ |
| $t=2$ | $(c,1)$ | $\boxed{(b,2)}$ |
| $t=3$ | $(d,1)$ | $\boxed{(b,2)}$ |

continuum
equipeaked
wrt utility $_{\text{centre}}$ a $\Omega_i$
$\Downarrow$
law agg.
subst. ..

Same outcome.

$\Rightarrow$ don't need bid-taker knowledge, get iterative
Prop 1 (IR) Prop 2 (PE) Prop 3 (DSIC)