```
// Upward pass
foreach node i, bottom up do
    m_i ← v̂_i − c(l_i)
    foreach node j ∈ children of i do
        m_i ← m_i + max(m_j, 0)
// Downward pass
S ← ∅
s_root ← m_root
foreach node i, top down do
    if s_i ≥ 0 then
        S ← S ⋃ {i}
        p_i ← max(v̂_i − s_i, 0)
    foreach node j ∈ children of i do
        s_j ← min(s_i, m_j)
```

Figure 10.7 A distributed algorithm for computing the efficient allocation and VCG payments for multicast cost sharing.

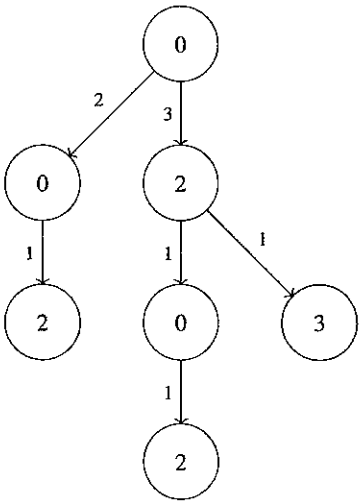
does not connect j. Thus, s_j can also be seen as the maximum amount by which agents in the subtree rooted at s_j could reduce their joint value declaration while remaining connected. Each connected node j is charged $\max(\hat{v}_j - s_j, 0)$, meaning that his surplus is equal to the amount he could have under-reported his value without being disconnected. These payments are illustrated in Figure 10.8d.

10.6.4 *Two-sided matching*

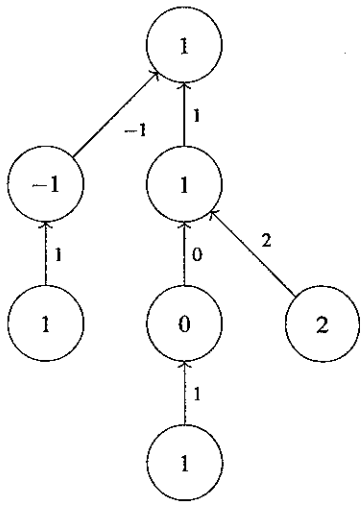
two-sided matching

So far in this chapter we have concentrated on mechanism design in quasilinear settings, meaning that we have assumed that money can be transferred between agents and the mechanism. However, there exist many interesting settings where such transfers are impossible, for example, because of legal restrictions. Examples of such problems include kidney exchanges, college admissions, and the assignment of medical interns to hospitals. *Two-sided matching* is a widely studied model that can be used to describe such domains. Under this model, each agent belongs to one of two groups. Members of each group are matched up, based on their declared preferences over their candidate partners. The mechanism design problem is to induce agents to disclose these preferences in a way that allows a desirable matching to be chosen, despite the restriction that payments cannot be imposed.

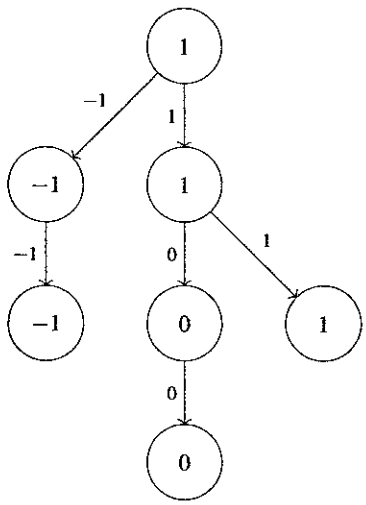
We will use the running example of a cohort of graduate students who must align with thesis advisors. Each student has a preference ordering over advisors (depending on their research interests, personalities, etc.), and likewise each potential advisor has a preference ordering over students. In this setting a social choice function is a decision about which students should be assigned to which advisors, given their preferences; as always, the mechanism design concern is how to implement a desired social choice function.



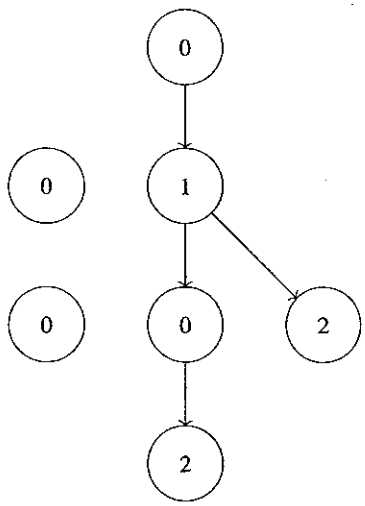
(a) Multicast spanning tree: nodes are labeled with values, edges are labeled with costs.



(b) Upward pass: each node i computes m_i and passes it to its parent.



(c) Downward pass: each node i computes s_j for child j and passes it down.



(d) Final allocation: only connected edges are shown; nodes are labeled with payments.

Figure 10.8 An example run of the algorithm from Figure 10.7.

We now define the setting more formally. Let A be a set of advisors and let S be a set of graduate students. We do not assume that $|A| = |S|$; thus, some students and/or advisors may remain unpaired. We assume that each student can have at most one advisor and each advisor will take at most one new student.[13] Each student i has a preference ordering $\succ_i$ over the advisors, and each advisor j has a preference ordering $\succ_j$ over the students. We write $a \succ_s a'$ to mean that student

13. Many but not all of the results in this section can also be extended to the case where advisors can take multiple students.

s prefers advisor a to advisor a', and $\varnothing \succ_s a$ to mean that s prefers not finding a supervisor to aligning with advisor a. In the latter case we say that advisor a is *unacceptable* to student s. Similarly, we write $s \succ_a s'$ and $\varnothing \succ_a s$. Note that we have assumed that all preferences are strict,[14] but that each agent can identify a set of partners with whom he would prefer not to be matched, effectively expressing a tie among unacceptable partners. In what follows, we adopt the convention that all advisors are female and all students are male. The resulting problem of finding good male–female pairings pays homage to the problem introduced in the two-sided matching literature a half-century ago, so-called *stable marriage*.

unacceptable matching

stable marriage

matching

Definition 10.6.7 (Matching) *A* matching $\mu : A \cup S \to A \cup S \cup \{\varnothing\}$ *is an assignment of advisors to students such that each advisor is assigned to at most one student and vice versa. More formally,* $\mu(s) = a$ *if and only if* $\mu(a) = s$. *Furthermore,* $\forall s \in S$, *either* $\exists a \in A, \mu(s) = a$ *or* $\mu(s) = \varnothing$ *(the student is unpaired), and likewise* $\forall a \in A$, *either* $\exists s \in S, \mu(a) = s$ *or* $\mu(a) = \varnothing$.

Note that it is always possible that some student s has the same match under two different matchings μ and μ', that is $\mu(s) = \mu'(s)$. In this case, s must be indifferent between matchings μ and μ'. A similar argument is true for advisors. Therefore, we use the operator $\succeq$ as well as $\succ$ when describing an agent's preference relation over matchings. More formally, $\mu(s) \succeq_s \mu'(s)$ means that either $\mu(s) \succ_s \mu'(s)$ or $\mu(s) = \mu'(s)$. Similarly, $\mu(a) \succeq_a \mu'(a)$ means that either $\mu(a) \succ_a \mu'(a)$ or $\mu(a) = \mu'(a)$.

Clearly, there are many possible matchings. The key question is which matching should be chosen, given students' and advisors' preference orderings. In other words, what properties does a desirable matching have? We identify two.

individually rational matching

Definition 10.6.8 (Individual rationality) *A matching* μ *is* individually rational *if no agent* i *prefers to remain unmatched than to be matched to* $\mu(i)$.

unblocked matching

Definition 10.6.9 (Unblocked) *A matching* μ *is* unblocked *if there exists no pair* (s, a) *such that* $\mu(s) \neq a$, *but* $a \succ_s \mu(s)$ *and* $s \succ_a \mu(a)$.

Intuitively, a matching is individually rational if no agent is matched with an unacceptable partner; a matching is unblocked if there exists no pair that would prefer to be matched with each other than with their respective partners. Putting these two definitions together, we obtain the concept of a stable matching.

stable matching

Definition 10.6.10 (Stable matching) *A matching* μ *is* stable *if and only if it is individually rational and unblocked.*

It turns out that in the setting we have defined above, no matter how many students and advisors there are and what preferences they have, there always exists at least one stable matching.

14. Our assumption that preferences are strict is restrictive; some of the results presented in this section no longer hold if it is relaxed.

Step 1: *each student applies to his most preferred advisor.*
repeat
  Step 2: *each advisor keeps her most preferred acceptable application (if any) and rejects the rest (if any).*
  Step 3: *each student who was rejected at the previous step applies to his next acceptable choice.*
until *no student applied in the last step*

Figure 10.9 Deferred acceptance algorithm, student-application version.

Theorem 10.6.11 (Gale and Shapley, 1962) *A stable matching always exists.*

Proof. The proof is obtained by giving a procedure that produces a stable matching given any set of student and advisor preferences. Here, we describe the so-called "student-application" version of the algorithm (Figure 10.9). There is an analogous algorithm in which advisors apply to students. This algorithm must stop in at most a quadratic number of steps, since no student ever applies more than once to any advisor. The outcome is a matching, since at any step each student is paired with at most one advisor and vice versa. The matching is individually rational, since no student or advisor is ever matched to an unacceptable agent.

It only remains to show that the matching is unblocked. Let μ be the matching produced by the algorithm. Assume for contradiction that μ is blocked by some student s and advisor a. Since s prefers a to his own match at μ, a must be acceptable to s, and so he must have applied to her before having applied to his match. Since s is not matched to a in μ, he must have been rejected by her in favor of someone she liked better. Therefore, (s, a) does not block μ, a contradiction. ■

Thus, there always exists at least one stable matching. However, these matchings are not necessarily unique—given a set of student and advisor preferences, there may exist many stable matchings. Let us now consider how different matchings can be compared.

student-optimal matching

Definition 10.6.12 *A stable matching μ is* student optimal *if every student likes it at least as well as any other stable matching; that is, $\forall s \in S$ and for every other stable matching μ', $\mu(s) \succeq_s \mu'(s)$.*

advisor-optimal matching

Along the same lines, we can define *advisor-optimal* matching. Now we can draw the following conclusions about stable matchings.

Theorem 10.6.13 *There exists exactly one student-optimal stable matching and one advisor-optimal stable matching. The matching produced by the student-application version of the deferred application algorithm is the student-optimal stable matching, and the matching produced by the advisor-application version of the deferred application algorithm is the advisor-optimal stable matching.*

Next, it turns out that any stable matching that is better for all the students is worse for all the advisors and vice versa.

Theorem 10.6.14 *If μ and μ' are stable matchings, $\forall s \in S$, $\mu(s) \succeq_s \mu'(s)$ if and only if $\forall a \in A$, $\mu'(a) \succeq_a \mu(a)$.*

achievable match

Say that an advisor a is *achievable* for student s, and vice versa, if there is a stable matching μ that matches a to s. Then we can state one implication of the above theorem: that the student-optimal stable matching is the worst stable matching from each advisor's point of view, and vice versa.

Corollary 10.6.15 *The student-optimal stable matching matches each advisor with her least preferred achievable student, and the advisor-optimal stable matching matches each student with her least preferred achievable advisor.*

Now let us move to the mechanism design question. If agents' preferences are private information, can we find a mechanism that ensures that a stable matching will be achieved? As is common in the matching literature, we restrict our attention to settings in which neither the agents' equilibrium strategies nor the mechanism itself are allowed to depend on the distribution over agents' preferences. Thus, we must rely on either dominant-strategy or *ex post* equilibrium implementation. Unfortunately, it turns out that stable matchings cannot be implemented under either equilibrium concept.

Theorem 10.6.16 *No mechanism implements stable matching in dominant strategies.*

Proof. By the revelation principle, if such a mechanism exists, then there also exists a direct truthful mechanism that selects matchings that are stable with respect to the declared preference orderings. Consider a setting with two students, s_1 and s_2, and two advisors, a_1, and a_2. Imagine that s_1, s_2 and a_1 declare the following preference orderings: $a_1 \widehat{\succ}_{s_1} a_2$, $a_2 \widehat{\succ}_{s_2} a_1$, and $s_2 \widehat{\succ}_{a_1} s_1$. Assume that a_2's true preference ordering is the following: $s_1 \succ_{a_2} s_2$. If a_2 declares the truth, then (1) the setting will have two stable matchings, μ and μ', given by $\mu(s_i) = a_i$ for $i \in \{1, 2\}$, and $\mu'(s_i) = a_j$ for $i, j \in \{1, 2\}$, $j \neq i$, and (2) any stable matching mechanism must choose one of μ or μ'. Suppose the mechanism chooses μ. Observe that if a_2 declares that her only acceptable student is s_1, then μ' is the only stable matching with respect to the stated preferences and the mechanism must select μ'—which a_2 prefers to μ. Similarly, we can show that if the mechanism chooses μ' when the above preference orderings are stated, then in a setting where $a_2 \succ_{s_2} a_1$ is s_2's true preference ordering, s_2 benefits by misreporting his preference ordering. Therefore, declaring the truth is not a dominant strategy for every agent. ■

Furthermore, it does not help to move to the *ex post* equilibrium concept, as can be proved along the same lines as Theorem 10.6.16.

Theorem 10.6.17 *No mechanism implements stable matching in* ex post *equilibrium.*

All is not hopeless, however—it turns out that we can obtain a positive mechanism design result for stable two-sided matching. The key is to relax our assumption that *all* agents are strategic. In our setting we will assume that advisors can

be compelled to behave honestly. Under this assumption, it is enough to prove the following result.

Theorem 10.6.18 *Under the direct mechanism associated with the student-application version of the deferred acceptance algorithm, it is a dominant strategy for each student to declare his true preferences.*

Proof. This proof proceeds by contradiction. Suppose that the claim is not true and, without loss of generality, say that it is not a dominant strategy for student s_1 to state his true preference ordering. Then, there is a preference profile $[\widehat{\succ}] = (\succ_{s_1}, \widehat{\succ}_{s_2}, \ldots, \widehat{\succ}_{s_{|S|}}, \widehat{\succ}_{a_1}, \ldots, \widehat{\succ}_{a_{|A|}})$ such that s_1 benefits from reporting $\succ'_{s_1} \neq \succ_{s_1}$. Let μ be the stable matching obtained by applying the student application version of the deferred acceptance algorithm to $[\widehat{\succ}]$. By Theorem 10.6.13, μ is student optimal with respect to $[\widehat{\succ}]$. Let μ' be the stable matching obtained by applying the same algorithm to $[\widehat{\succ}'] = (\succ'_{s_1}, \widehat{\succ}_{s_2}, \ldots, \widehat{\succ}_{s_{|S|}}, \widehat{\succ}_{a_1}, \ldots, \widehat{\succ}_{a_{|A|}})$. Note that except for s_1, all the other students and advisors declare the same preference ordering under $[\widehat{\succ}]$ and $[\widehat{\succ}']$.

Let $R = \{s_1\} \cup \{s : \mu'(s) \widehat{\succ}_s \mu(s)\}$ denote the set of students who strictly prefer μ' to μ (with respect to their declared preferences $[\widehat{\succ}]$). Note that we have included s_1 in R because, by assumption, $\mu'(s_1) \succ_{s_1} \mu(s_1)$. Let $T = \{a : \mu'(a) \in R\}$ denote the set of advisors who are matched with some student from R under μ'. In what follows we first show (Part 1) that any advisor $a \in T$ is matched with an (always different) student from R under μ; that is, $\{a : \mu'(a) \in R\} = \{a : \mu(a) \in R\} = T$. Then (Part 2) we show that there exist some $a_\ell \in T$ and $s_r \notin R$ such that (s_r, a_ℓ) blocks μ' at $[\widehat{\succ}']$ and therefore μ' is not stable with respect to $[\widehat{\succ}']$. This contradicts our assumption that μ' is a stable matching with respect to $[\widehat{\succ}']$.

Part 1: For any $s \in R$, let $a = \mu'(s)$. Stability of μ with respect to $[\widehat{\succ}]$ requires that advisor a be matched to some student under μ (rather than being unpaired), as otherwise (s, a) would block μ at $[\widehat{\succ}]$; let $s' = \mu(a)$. If $s' = s_1$, then since s_1 prefers his match under μ' to his match under μ, $s' \in R$. Otherwise, since (with respect to his preferences declared in $[\widehat{\succ}]$) s strictly prefers $\mu'(s)$ to $\mu(s)$, stability of μ with respect to $[\widehat{\succ}]$ implies that $s' \widehat{\succ}_a s$. Since we defined $s = \mu'(a)$, thus $s' \widehat{\succ}_a \mu'(a)$. Then, stability of μ' with respect to $[\widehat{\succ}']$ implies that $\mu'(s') \widehat{\succ}_{s'} a$. Since we defined $a = \mu(s')$, thus $\mu'(s') \widehat{\succ}_{s'} \mu(s')$ and therefore $s' \in R$. As a result, we can write $T = \{a : \mu'(a) \in R\} = \{a : \mu(a) \in R\}$.

Part 2: Since every student $s \in R$ prefers $\mu'(s)$ to $\mu(s)$, stability of μ with respect to $[\widehat{\succ}]$ implies that $\forall a \in T$, $\mu(a) \widehat{\succ}_a \mu'(a)$. Therefore, during the execution of the student-application algorithm on $[\widehat{\succ}]$, each student $s \in R$ will apply to $\mu'(s)$ and will get rejected by $\mu'(s)$ at some iteration. In other words, each $a \in T$ rejects $\mu'(a) \in R$ at some iteration. Let s_ℓ be (weakly) the last student in R who applies to an advisor during the execution of the student-application algorithm. This application is sent to $\mu(s_\ell) \in T$; let $\mu(s_\ell) = a_\ell$. By construction, a_ℓ must have rejected $\mu'(a_\ell)$ at some strictly earlier iteration of the algorithm. Thus, when s_ℓ applies to a_ℓ, a_ℓ must reject an application from some $s_r \notin R$ such that $s_r \widehat{\succ}_{a_\ell} \mu'(a_\ell)$ (fact 1). Note that $s_r \neq s_1$, since

$s_r \notin R$ and $s_1 \in R$. Since s_r applies to a_ℓ before he finally gets matched to $\mu(s_r)$, we have that $a_\ell \widehat{\succ}_{s_r} \mu(s_r)$. Furthermore, since $s_r \notin R$, we also have that $\mu(s_r) \widehat{\succeq}_{s_r} \mu'(s_r)$. Therefore $a_\ell \widehat{\succ}_{s_r} \mu'(s_r)$ (fact 2). Thus, from (fact 1) and (fact 2), (s_r, a_ℓ) blocks μ' at $[\widehat{\succ}']$ and μ' is not stable with respect to $[\widehat{\succ}']$, yielding our contradiction. ■

Of course, it is similarly possible to achieve a direct mechanism under which truth telling is a dominant strategy for advisors by using the advisor-application version of the deferred acceptance algorithm.

10.7 Constrained mechanism design

So far we have assumed that the mechanism designer is free to design any mechanism, but this assumption is violated in many applications—the ones discussed in this section, and many others. In particular, often one starts with given strategy spaces for each of the agents, with limited or no ability to change those. Examples abound:

- A city official who wishes to improve the traffic flow in the city cannot redesign cars or build new roads;
- A UN mediator who wishes to incent two countries fighting over a scarce resource to cease hostilities cannot change their military capabilities or the amount or value of the resource;
- A computer network operator who wishes to route traffic a certain way cannot change the network topology or the underlying routing algorithm.

Many other examples exist, and in fact such constraints can be thought of as the norm rather than the exception. How can such would-be mechanism designers intervene to influence the course of events?

social law

In Chapter 2 we already encountered this problem. Specifically, in Section 2.4 we saw how imposing *social laws*—that is, restricting the options available to agents—can be beneficial to all agents. Social laws played an important coordinating role (as in "drive on the right side of the road") and, furthermore, in some cases prevented the narrow self interests of the agents from hurting them (e.g., allowing cooperation in the Prisoners' Dilemma game). However, in that discussion we made the important assumption that once a social law was imposed (or agreed upon, depending on the interpretation), the agents could be assumed to follow it.

Here we relax this assumption, and we do so in three ways. First, we view the players as having the option of entering into a contract among themselves. Once they do—and only then—the center can impose arbitrary fines on law breakers, if he is aware of such deviations. The question in this case is which contracts the agents can be expected to enter, and how the work of the center can be minimized or even eliminated. Second, we consider the case in which the center can simply bribe the players to play a certain way (or, in more neutral language, offer positive incentives for certain actions). The question in this case is how the center can bias the outcome toward the desired one while minimizing his cost. Finally, we