Dynamic House Allocations

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Abstract

We consider the problem of assigning individuals (e.g. students) to indivisible goods (schools) when these assignments have to be made repeatedly and when individuals face uncertainty about the intensity of their future preferences. For the setting with two schools and two periods, where all individuals have an ordinal preference for the same school, we show that expected utility under a dynamic mechanism is strictly greater for every individual than under a static mechanism, where a mechanism is called dynamic (static) if its future assignments depend (do not depend) on past reports. We derive conditions under which a simple dynamic mechanism achieves the first-best allocation in the first period and second-best over both periods. Extensions of the model allow, in turn, for individuals who differ with respect to their ordinal preferences, uncertainty about future ordinal preferences, correlated utilities, and the possibility that switching schools is costly.

Keywords: House allocation, matching, mechanism design without transfers, preference intensity.

JEL: C72, C78, D02

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1 Introduction

We consider the problem of assigning individuals (e.g. students) to indivisible goods (schools) when these assignments have to be made repeatedly and when individuals face uncertainty about the intensity of their future preferences.[1] These assumptions are satisfied in many real world situations, such as the allocation of high school students to public schools, college students to campus houses, students to courses at business schools, or scarce resources within an organization in general. Since for example students who are assigned to a course in a given semester typically have to be assigned to a course the following semester, it is clear that these problems are dynamic.[2]

Though this dynamic dimension adds naturally some complications, it also offers potential for substantial efficiency gains. Very much like in the theory of repeated games and the literature on dynamic contracting, future payoffs can serve as incentives in the presence. For example, assume that there are two schools and two periods and that all students agree which school is better. If students who only weakly prefer the good school in period one are credibly promised a sufficiently higher probability of getting into the good school in period two when they apply to the bad school today, the allocation in period one can eventually be improved. As those with the lowest preference for the good school apply to the bad school today, there is either no rationing at the good school in period one, or rationing is more efficient because it occurs among students with a stronger preference for the good school.

Consequently, dynamic mechanisms have the potential of eliciting some cardinal utility information, namely whether individuals care a lot or a little about the good school in period one. This contrasts with a static or myopic mechanism, where in every period all students have a dominant strategy to apply to their preferred school. Moreover, as a consequence of the increased efficiency in the first period, ex ante expected utility under a dynamic mechanism (i.e. expected utility of individuals before they know their valuations) exceeds ex ante expected utility under a static mechanism.

The paper, and the model we adopt, are motivated by the problem of public school choice, where a given number of students have to be assigned to a given

[1] Allocation problems where indivisible goods - "houses" - are assigned to individuals without using monetary transfers have become known as house allocation problems.

[2] Similarly, students who are assigned to a school in a given period will have to be assigned to a school in the next period, and scarce resources within an organization are allocated to members of the organization repeatedly.

number of schools.[3] Because of either political reasons, credit market imperfections, or other practical problems, no transfer payments are allowed for, so that the allocation cannot be relied upon the usual market mechanism.[4] Nonetheless, it is desirable to allow students to choose their most preferred school if this school is available. For example, consider the public school match currently in use in Boston. Each school has a fixed maximum capacity of students it can accommodate, which is determined by the Boston school department. Students are given the opportunity to choose a school in kindergarten, and then in first, sixth and ninth grade. So, clearly, the allocation or matching problem is dynamic. Moreover, it is reasonable to assume that students (and their parents) are uncertain about the intensity of their future preferences for the various schools when taking their decision in the presence. These features are the main ingredients of the model we analyze. We capture the dynamic nature by assuming that students go to school for two periods. There are two schools, or two types of schools, each type with a fixed capacity, and individuals face uncertainty about their future utility when applying to a school in the presence.

Related literature The present paper relates to the literatures on matching theory and mechanism design without monetary transfers. A classic reference for matching is Roth and Sotomayor (1990). Hatfield and Milgrom (2005) generalize some of the key results in matching markets. Static house allocation problems with ordinal preferences have been studied by, among others, Abdulkadiroglu and Sönmez (1998, 1999) and McLennan (2002). Roth, Sönmez, and Ünver (2004) use von-Neumann-Morgenstern utilities to assess the potential welfare gains from an improved matching mechanism for kidney transplants. However, their mechanism does not elicit or use cardinal information. Similarly, Roth, Sönmez, and Ünver (2005b) analyze stochastic exchanges without inducing revelation of cardinal utility information. Course bidding at business

[3] See Abdulkadiroglu and Sönmez (2003), Abdulkadiroglu, Pathak, Roth, and Sönmez (2005) and Abdulkadiroglu, Pathak, and Roth (2005). Though the model is meant to capture the essential features of school assignments, it is also appropriate in other environments, most importantly perhaps, for the problem of queue management. Consider e.g. a large organization, like a firm or a university, and assume that in every period various subunits of the organization (called clients) require the services of a central unit (the server). Assume further that the server's capacity is limited, so that in every period some clients will be rationed. This problem is almost identical to the school assignment problem we study. The main difference is that the continuum assumption is less likely to be appropriate in this setting than in the school assignment problem.

[4] Practical concerns arise, for example, when allocating resources within an organization such as a firm or a university. Though in principle it is possible to create markets and have market mechanisms work inside the organization, there can be quite compelling reasons (such as too much or too little consumption) why this need not be very desirable.

schools in a static setting is studied by Sönmez and Ünver (2005). The application of the mechanism design approach to public school choice was proposed by Abdulkadiroglu and Sönmez (2003),[5] and some of the practical problems and design issues involved are discussed by Abdulkadiroglu, Pathak, Roth, and Sönmez (2005) and Abdulkadiroglu, Pathak, and Roth (2005).[6]

Casella (2003), Abdulkadiroglu (2004) and Jackson and Sonnenschein (2004) have started to study mechanisms without transfers that improve allocations in terms of expected utility, either by linking decisions (Casella, Jackson and Sonnenschein) or by giving individuals "quasi-money" (Abdulkadiroglu). An important difference to Casella's paper is that we look at a problem with a continuum of agents. This allows us to abstract from strategic interactions between agents, whereas she analyzes voting equilibria when there are two or three agents who face a repeated decision problem. A main contrast between our paper and those of Jackson and Sonnenschein and Abdulkadiroglu is the striking simplicity of our mechanism, which nevertheless has very desirable welfare properties. Hortala-Vallve (2004) and Börgers and Postl (2004) consider the possibility of achieving welfare gains by studying more elaborate voting schemes than "one man one vote". Hortala-Vallve proves the impossibility of attaining first-best, while Börgers and Postl show that nonetheless substantial welfare gains, as a matter of fact, almost first-best solutions, are possible.[7]

The remainder of the paper is structured as follows. Section 2 introduces basic concepts and illustrates the main idea of the paper with a simple example. Section 3 analyzes the general two period two school model, where all individuals have identical ordinal preferences, utility for the bad school is normalized to zero and utility draws for the good school are independent over time. In Section 4, we analyze the same model with the modification that students have heterogenous ordinal preferences, i.e. some students prefer one school and some the other. Section 5 introduces uncertainty about ordinal preferences in addition to uncertainty about preference intensity. This setting is natural when a school is not good or bad per se, but rather specialized, say, either in languages

[5]See also Boston Globe (2003).

[6]See also Roth, Sönmez, and Ünver (2005a) and Roth (2002). The former address design problems for kidney exchange, the latter provides a detailed discussion of other applications and issues in economic design.

[7]The basic problem Casella, Jackson and Sonnenschein, Hortala-Vallve, and Börgers and Postl address is a collective decision problem with few (typically two or three) individuals who may differ both with respect to their ordinal preferences as well as the intensities of these preferences. The goal of all four papers is to find mechanism that improve ex ante expected utility.

or in math, and when students are uncertain in period one about the skills they are going to develop during that period. Section 6 extends the model to allow for correlation of utility over time. In Section 7, we study the model when switching from one school to another involves a dead weight loss cost. Section 8 summarizes the main results we obtain for the various models, and Section 9 concludes. A generalization that drops the normalization of utility for the bad school, which is maintained throughout the paper, and an example are deferred to the Appendix.

2 Example

For the purpose of illustration, we first consider a simple example. There is a continuum of students whose total mass is two. There are two schools, or two types of schools, A and B, each with a capacity to accommodate students of mass one. Students go to school for two periods, $t = 1, 2$. Within a period, a school is an indivisible good, but students can switch school from one period to another at zero costs.

Preferences over schools are as follows. Denote by x_{kt} the instantaneous utility of a student when attending school $k = A, B$ in period t. We assume that $x_{Bt} = 0$ for all t and all students. Instantaneous utility for A, x_{At}, is drawn randomly from the distribution G on $[0, M]$ with $M > 0$ and $G(0) = 0$, where these draws are i.i.d. over time and across students.[8] Because utility of school B is always zero, we write x for the utility of school A for simplicity. Denote by Eu the expected value of x. The realizations of the instantaneous utilities are known by individuals before applying to a school within a given period. Utility is additive over time, and there is no discounting, so that in period one the expected utility of an individual who is sure to attend school A in both periods and whose first period draw is x is $x + Eu$; Figure 1 illustrates the timing. Also, we let $F(x)$ be the measure of students whose utility draw for school A in period one is no larger than x. Because students have mass two in total, we have $F(x) \equiv 2G(x)$.

[8] The assumption that $G(.)$ is a continuous function is made merely for expositional convenience. All the main results would go through if the random variable is of the discrete type, as long as its support includes sufficiently many points; in particular, all the main results go through if the support contains the points (if they exist) x_j with $j = 1, .., 4$ defined in the proof of Proposition 1 below.

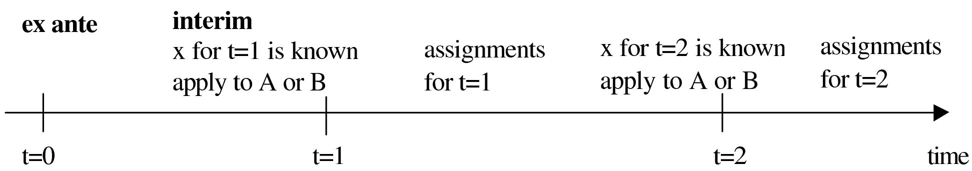

Figure 1: Timing.

2.1 Mechanisms

We investigate the potential of using dynamic mechanisms in a school assignment (or house allocation) problem with two periods and two schools. We restrict attention to mechanisms that are simple in that the number of possible messages is equal to the number of schools. This type of mechanism is particularly apt for practical use since asking (and answering) the question "To which school do you want to apply?" is a very natural thing in a school assignment problem. Though more complicated mechanisms exist that may be used in practical applications as well, the analysis of such mechanisms is beyond the scope of the present paper. There are two motivations for this. First, it simplifies the analysis. Second, as will be shown below, there are reasonable conditions under which even a very simple mechanism achieves the optimal incentive compatible allocation over both periods.

In particular, we consider and compare the following two types of mechanisms.

Static (or Myopic) Mechanism Let each individual report his or her preferences in period one. If there is excess demand for a school, individuals are allocated randomly. The same procedure is repeated in period two.

We contrast the static (or myopic) mechanism with the dynamic mechanism.

Dynamic Mechanism Those students who apply to the less preferred school in period one are given priority for their preferred school in period two. (It is assumed that everybody knows which school is preferred at the time the mechanism is run.) Students who apply to the preferred school in period one have priority for this school in period one. If there is excess demand in any period for some school, students with the same priority are allocated randomly.

2.2 Equilibrium

We now derive equilibrium under both mechanisms.

Equilibrium under the Static Mechanism Let $\mu_2 \in [0,2]$ be the measure of other students an individual believes will apply to A in $t = 2$ (not necessarily the correct measure). Denote by $U_k(x)$ the expected utility of an individual whose utility draw for school A is x and who applies for school k. Under the static mechanism, "apply to A" is a strictly dominant strategy in $t = 2$ since

$$U_A(x) = \min\left\{1, \frac{1}{\mu_2}\right\} x > 0 = U_B(x)$$

for all $x > 0$. Thus, the static mechanism induces a unique equilibrium in $t = 2$. In this equilibrium, all individuals apply to A.

Consequently, the expected utility of an individual in $t = 1$ whose first period utility draw is x and who believes that $\mu_1 \in [0,2]$ others apply to A in $t = 1$ when applying to A (taking equilibrium behavior in $t = 2$ into account) is

$$U_A(x) = \min\left\{1, \frac{1}{\mu_1}\right\} x + \frac{1}{2}Eu > \frac{1}{2}Eu = U_B(x),$$

where Eu is the expected utility for school A and where the strict inequality holds for all $x > 0$. Thus, "apply to A" is a strictly dominant strategy in both periods.

Equilibrium under the Dynamic Mechanism Consider the expected utility of an individual who believes μ others apply to A in period one (and who correctly anticipates that in $t = 2$ all individuals will apply to A) after his period one utility x is realized, i.e. at the *interim stage*:

$$\begin{aligned}
U_A(x) &= \min\left\{1, \frac{1}{\mu}\right\} x + \max\left\{0, 1 - \frac{1}{\mu}\right\} Eu \\
U_B(x) &= \max\left\{0, \frac{1-\mu}{2-\mu}\right\} x + \min\left\{1, \frac{1}{2-\mu}\right\} Eu.
\end{aligned}$$

So, an individual with utility x applies to A if and only if

$$\begin{aligned}
U_A(x) - U_B(x) &= \left(\min\left\{1, \frac{1}{\mu}\right\} - \max\left\{0, \frac{1-\mu}{2-\mu}\right\}\right) x \\
&+ \left(\max\left\{0, 1 - \frac{1}{\mu}\right\} - \min\left\{1, \frac{1}{2-\mu}\right\}\right) Eu \geq 0.
\end{aligned}$$

It is easy to see that regardless of whether $\mu \geq 1$ or $\mu < 1$, the individual will apply to A if and only if $x \geq Eu$. Thus, every individual has a strictly dominant strategy, which is either "apply to A" or "apply to B". Which strategy is dominant depends on the first period utility realization x. Because the equilibrium is in strictly dominant strategies, it is necessarily unique.

2.3 Uniform Distribution

For the purpose of illustration, we consider now an example where G is uniform on $[0, M]$. Thus, $Eu = \frac{M}{2}$ and $F(Eu) = 1$. In $t = 1$, all individuals with $x \leq Eu$ apply to B, and all others apply to A. Note that maximizing social welfare requires that the fifty percent of students with the most intense preference for A are assigned to A, which is exactly what is achieved under the dynamic mechanism. Thus, from a social point of view, the first-best allocation is achieved in $t = 1$.

Quantifying Welfare Gains First, since all students with $x \leq Eu$ apply to B in $t = 1$, there is no rationing (or excess demand for school A) and no demand shortage for school B in period one. Second, because of this and because all high utility students (i.e. all students with $x > Eu$) go to school A in $t = 1$, the period one allocation maximizes social welfare. Overall welfare under the dynamic mechanism is given by

$$W^{DM} = E[x \mid x > M/2] + Eu = \frac{5}{4}M.$$

Since $M > 0$, this is greater than expected welfare under the static mechanism, which is $W^{SM} = M$. The expected welfare gain generated by the dynamic mechanism is thus $\frac{M}{4}$, or twenty-five percent.

3 The Model

We now consider a more general model, in which utility for the good school, A, is drawn from the distribution function G with support $[0, M]$ and where school A can have any capacity $\alpha \in (0, 2)$. Capacity of school B is $2 - \alpha$.[9] As before, we denote by x the instantaneous utility for A (which is private information) and by Eu the expected value of x and we let $F(x) \equiv 2G(x)$ be the mass of students with utility draw no larger than x. The median value of x is denoted

[9] Aside from the fact that capacities of schools may, in general, vary, the exercise of allowing for varying capacities is motivated by the model of Section 4, where students differ with respect to their preferred school.

by m, i.e. $G(m) = \frac{1}{2}$. Notice that $F(m) = 1$. Utility for school B is zero for all individuals and both periods, and there are no costs of switching from one school to the other after period one. All of these assumptions are common knowledge.

The assumptions of two schools is maintained throughout the paper. The assumption of homogenous and time invariant ordinal preferences, i.e. that all students have an ordinal preference for school A in both periods, is relaxed in Section 4. Note that time invariant ordinal preferences imply certainty about future ordinal preferences. This assumption will be relaxed in Section 5. The assumption that switching from one school to another is costless is relaxed in Section 7. The normalization of utility for school B is dropped in the Appendix, where we show that the normalization does not affect the results in any qualitatively important way.

We begin with the definition of a cutoff equilibrium.

Definition 1 *$x^* \in (0, M)$ is an (interior) equilibrium cutoff point if given the behavior of all others it is optimal*

- for every student with $x < x^*$ to apply to B,

- for every student with $x \geq x^*$ to apply to A, and

- $\mu = 2 - F(x^*)$.

We are now ready to state one of the main results of this section, which is that under fairly general conditions a cutoff equilibrium under the dynamic mechanism exists.

Proposition 1 *For any $\alpha \in \left(0, \frac{2M}{Eu+M}\right)$ and any G with full support on $[0, M]$ and $G(0) = 0$, there is an interior cutoff equilibrium. A sufficient condition for the existence of an interior cutoff equilibrium for any $\alpha \in (0, 2)$ is $Eu = m$.*

Notice that $\frac{2M}{Eu+M} > 1$ for any non-degenerate distribution G.

Proof: The proof consists of three steps. In step 1, we derive the four cases that have to be distinguished and the necessary conditions for an equilibrium for each case. In step 2, we construct an equilibrium for any $\alpha \leq 1$ and in step 3, we construct an equilibrium for all $\alpha > 1$.

Step 1: Consider Figure 2 to see that as a function of α and μ, which remains to be determined in steps 2 and 3, there are four cases that can occur.

Case 1: Assume $2 - \mu < \alpha < \mu$. In this case, the number of applicants to B in $t = 1$ is smaller than the capacity of A, which in turn is smaller than the number of applicants to A. For this case to occur in a cutoff equilibrium, the

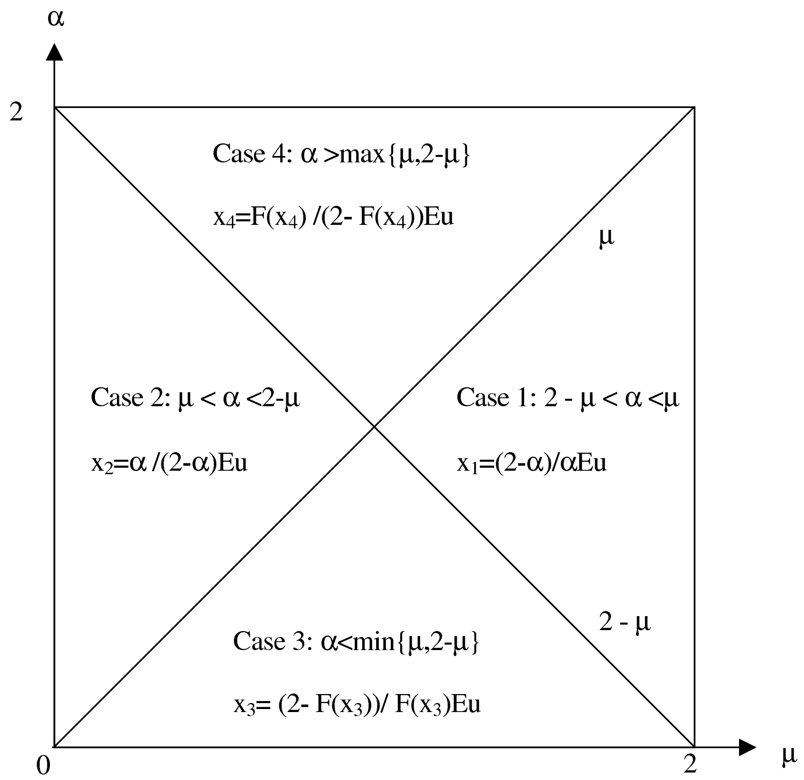

Figure 2: Four cases.

following must hold for some x:

$$\begin{aligned} U_A(x) = \frac{\alpha}{\mu}x + \frac{\alpha-(2-\mu)}{\mu}Eu &= Eu = U_B(x) \\ &\Leftrightarrow \\ x^* &= \frac{2-\alpha}{\alpha}Eu. \end{aligned}$$

Clearly, $\frac{\partial U_A}{\partial x} = \frac{\alpha}{\mu} > 0 = \frac{\partial U_B}{\partial x}$.

Case 2: Assume $\mu < \alpha < 2-\mu$. Then, the capacity of B is smaller than the number of applicants to B in $t = 1$. Consequently, no applicant to A in $t = 1$ will get into A in $t = 2$.

$$\begin{aligned} U_A(x) = x &= \frac{\alpha-\mu}{2-\mu}x + \frac{\alpha}{2-\mu}Eu = U_B(x) \\ &\Leftrightarrow \\ x^* &= \frac{\alpha}{2-\alpha}Eu. \end{aligned}$$

It is easy to see that $\frac{\partial U_A}{\partial x} = 1 > \frac{\alpha-\mu}{2-\mu} = \frac{\partial U_B}{\partial x}$.

Case 3: Assume $\alpha < \min\{\mu, 2-\mu\}$. Note that $\alpha < \mu \Leftrightarrow 2-\alpha > 2-\mu$, i.e. there are less applicants to B in $t = 1$ than seats in B. Moreover, $\alpha < 2-\mu$ implies that there are more applicants to B in $t = 1$ than the capacity of A can accommodate. Consequently, none of those who apply to A in $t = 1$ will get into A in $t = 2$.

$$\begin{aligned} U_A(x) = \frac{\alpha}{\mu}x &= \frac{\alpha}{2-\mu}Eu = U_B(x) \\ &\Leftrightarrow \\ x^* &= \frac{\mu}{2-\mu}Eu. \end{aligned}$$

Obviously, $\frac{\partial U_A}{\partial x} = \frac{\alpha}{\mu} > 0 = \frac{\partial U_B}{\partial x}$.

Case 4: Assume $\alpha > \max\{\mu, 2-\mu\}$. The condition $\alpha > 2-\mu$ implies that the number of applicants to B in $t = 1$ is smaller than the capacity of A.

$$\begin{aligned} U_A(x) = x + \frac{\alpha-(2-\mu)}{\mu}Eu &= \frac{\alpha-\mu}{2-\mu}x + Eu = U_B(x) \\ &\Leftrightarrow \\ x^* &= \frac{2-\mu}{\mu}Eu. \end{aligned}$$

It is easy to check that $\frac{\partial U_A}{\partial x} = 1 > \frac{\alpha-\mu}{2-\mu} = \frac{\partial U_B}{\partial x}$.

These conditions and cases can be summarized as follows:

$$\begin{aligned} x_1 &= \frac{2-\alpha}{\alpha}Eu \quad 2-\mu \leq \alpha \leq \mu \quad \text{(Case 1)} \\ x_2 &= \frac{\alpha}{2-\alpha}Eu \quad \mu \leq \alpha \leq 2-\mu \quad \text{(Case 2)} \\ x_3 &= \frac{\mu}{2-\mu}Eu \quad \alpha \leq \min\{\mu, 2-\mu\} \quad \text{(Case 3)} \\ x_4 &= \frac{2-\mu}{\mu}Eu \quad \alpha \geq \max\{\mu, 2-\mu\} \quad \text{(Case 4)}, \end{aligned}$$

where we have dropped the "star" and used subscripts to indicate the respective cases. Second, replace μ by $2 - F(x_i)$ for $i = 1, .., 4$ to get:

$$\begin{aligned} x_1 &= \frac{2-\alpha}{\alpha}Eu && F(x_1) \leq \alpha \leq 2 - F(x_1) \\ x_2 &= \frac{\alpha}{2-\alpha}Eu && 2 - F(x_2) \leq \alpha \leq F(x_2) \\ x_3 &= \frac{2-F(x_3)}{F(x_3)}Eu && \alpha \leq \min\{F(x_3), 2 - F(x_3)\} \\ x_4 &= \frac{F(x_4)}{2-F(x_4)}Eu && \alpha \geq \max\{F(x_4), 2 - F(x_4)\}. \end{aligned}$$

Note that $\frac{\partial x_1}{\partial \alpha} < 0$ and $\frac{\partial x_2}{\partial \alpha} > 0$.

Step 2: Another necessary condition for some x^* and a given μ to constitute an equilibrium is that x^* and μ must be consistent, i.e. it must be that $\mu = 2 - F(x^*)$.

So as to show that for $\alpha \in \left(0, \frac{2M}{M+Eu}\right)$ there is always an equilibrium with an interior cutoff point, it is useful to distinguish the cases with $\alpha \leq 1$ and $\alpha > 1$. We begin with the former and show that such a pair x^* and $\mu = 2 - F(x^*)$ always exists, which then proves existence.

For $\alpha \leq 1$, the strategy of the proof is to start with an x close to the median such that the restriction $\alpha \leq \min\{F(x), 2-F(x)\}$ of a case 3 equilibrium is met. If this is not an equilibrium (i.e. if for no such x, $x = x_3$ holds), then we can either decrease x until we have an x that satisfies all of the restrictions of a case 1 equilibrium, or we can increase x until we have a case 2 equilibrium. For $\alpha > 1$, the strategy of the proof is completely analogous, except that we start with an x close to the median that satisfies the restrictions $\alpha \geq \max\{F(x), 2 - F(x)\}$ of a case 4 equilibrium.

Recall that m denotes the utility of the median, i.e. $F(m) = 1$, and consider case 3. Clearly, for $\alpha \leq 1$ there is always an x such that $\alpha \leq \min\{F(x), 2 - F(x)\}$, since we can always choose $x = m$. If in addition, $x = \frac{2-F(x)}{F(x)}Eu$ holds, we have an equilibrium. So, assume either $x > \frac{2-F(x)}{F(x)}Eu$ or $x < \frac{2-F(x)}{F(x)}Eu$ for all x for which the restriction $\alpha \leq \min\{F(x_3), 2 - F(x_3)\}$ holds. We consider the first case first. Start with an x such that $\alpha \leq \min\{F(x), 2 - F(x)\}$ holds and decrease x until $x = \tilde{x}$ with $\tilde{x} > \frac{2-F(\tilde{x})}{F(\tilde{x})}Eu$ and $\alpha = F(\tilde{x}) < 2 - F(\tilde{x})$. The last inequality holds because initially $2 - F(x) \geq \alpha$ holds and as x decreases to $\tilde{x}$, $2 - F(\tilde{x}) > \alpha$ follows. Note that $\tilde{x} > \frac{2-\alpha}{\alpha}Eu$. Now decrease x further until $\hat{x} = \frac{2-\alpha}{\alpha}Eu$. As $\hat{x} < \tilde{x}$ implies $F(\hat{x}) < \alpha < 2 - F(\hat{x})$, it follows that we have an equilibrium of the form described in case 1.

Notice that $x_1 = \frac{2-\alpha}{\alpha}Eu > 0$ for any $\alpha < 1$. The only concern for the existence of an interior equilibrium is thus that $x_1 > M$. However, since x_1 is only needed to prove equilibrium existence when x_3 is not an equilibrium cutoff point and because in this case x_1 is known to be smaller than some $x < M$, we know that $x_1 < M$ holds.

So as to complete the case where $\alpha \leq 1$, assume now that for any x that satisfies the restriction $\alpha \leq \min\{F(x), 2 - F(x)\}$, $x < \frac{2-F(x)}{F(x)}Eu$ holds. Start with any such x and increase it until x equals $\tilde{x}$, which is such that $\tilde{x} < \frac{2-F(\tilde{x})}{F(\tilde{x})}Eu$ and $\alpha = 2 - F(\tilde{x})$. Note that $2 - F(\tilde{x}) = \alpha < F(\tilde{x})$, since initially both $F(x)$ and $2 - F(x)$ were larger than α and since the term $2 - F(x)$ decreases when x increases. Note also that $\tilde{x} < \frac{\alpha}{2-\alpha}Eu$. So, increase x further until $\hat{x} = \frac{\alpha}{2-\alpha}Eu$. This is clearly an equilibrium of the type analyzed in case 2, with $2 - F(\hat{x}) < \alpha < F(\hat{x})$.

Notice that $x_2 = \frac{\alpha}{2-\alpha}Eu \leq Eu$ for any $\alpha \leq 1$ and that x_2 is positive. Thus,

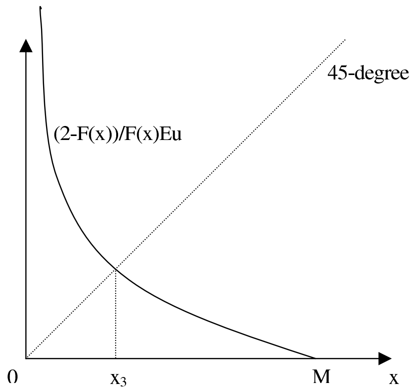

Figure 3: A fix point x_3 always exists.

whenever x_2 is an equilibrium cutoff point for $\alpha \leq 1$, it is an interior cutoff point.

Step 3: The reasoning for the case with $\alpha > 1$ is almost completely analogous. Consider an x such that $\alpha > \max\{F(x), 2 - F(x)\}$. Because $\alpha > 1$, we know that such x's always exist if we choose them close enough to m. If in addition for one such x, $x = \frac{F(x)}{2-F(x)}Eu$ holds, we have an equilibrium of the case 4 type. So, assume that no such x exists, i.e. whenever $\alpha > \max\{F(x), 2-F(x)\}$ is satisfied, we either have $x > \frac{F(x)}{2-F(x)}Eu$ or $x < \frac{F(x)}{2-F(x)}Eu$. Consider first the case where $x > \frac{F(x)}{2-F(x)}Eu$ and the restriction $\alpha \geq \max\{F(x_4), 2-F(x_4)\}$ is satisfied. As we decrease x, $F(x)$ decreases and $2 - F(x)$ increases. Since initially $\max\{F(x), 2-F(x)\} < \alpha$, the constraint that will become binding for some sufficiently small $\tilde{x}$ is $\alpha = 2-F(\tilde{x}) > F(\tilde{x})$. Assume that $\tilde{x} > \frac{F(\tilde{x})}{2-F(\tilde{x})}Eu = \frac{2-\alpha}{\alpha}Eu$. Clearly, as we decrease x further until $\hat{x} = \frac{2-\alpha}{\alpha}Eu$, $F(\hat{x}) < \alpha < 2 - F(\tilde{x})$ holds, and we have an equilibrium of the case 1 type. Because we reach x_1 by decreasing x, starting from some $x < M$, we know that x_1 is an interior cutoff point.

Finally, consider the case where for any x such that $\alpha > \max\{F(x), 2-F(x)\}$ holds, we have $x < \frac{F(x)}{2-F(x)}Eu$. Increase x until $\alpha = F(\tilde{x}) > 2 - F(\tilde{x})$. Note that $\tilde{x} < \frac{F(\tilde{x})}{2-F(\tilde{x})}Eu = \frac{\alpha}{2-\alpha}Eu$. Increase x further up to $\hat{x} = \frac{\alpha}{2-\alpha}Eu$. Since $F(\hat{x}) > \alpha > 2-F(\hat{x})$, we have an equilibrium of the type considered under case 2, provided $x_2 < M$, which is implied by the assumption $\alpha < \frac{2M}{M+Eu}$.

That $Eu = m$ is a sufficient condition for an interior cutoff equilibrium to exist for any $\alpha \in (0,2)$ follows directly by plugging $x_3 = m$ and $x_4 = m$ into the conditions $x_3 = \frac{2-F(x_3)}{F(x_3)}Eu$ and $x_4 = \frac{F(x_4)}{2-F(x_4)}Eu$ and the corresponding restrictions $\alpha \leq \min\{F(x_3), 2-F(x_3)\}$ and $\alpha \geq \max\{F(x_4), 2-F(x_4)\}$, which yields $x_3 = x_4 = Eu$ and $\min\{F(x_3), 2-F(x_3)\} = \max\{F(x_4), 2-F(x_4)\} = 1$.

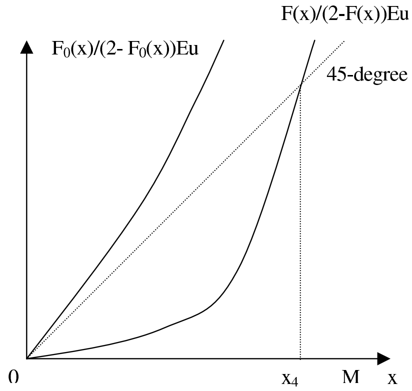

Figure 4: A fix point x_4 may fail to exist.

Thus, for $\alpha \leq 1$, case 3 is an equilibrium, and otherwise case 4 is an equilibrium. ■

Notice that an interior fix point $x_3 = \frac{2-F(x_3)}{F(x_3)} Eu$ always exists, as illustrated in Figure 3. An interior fix point $x_4 = \frac{F(x_4)}{2-F(x_4)} Eu$, on the other hand, need not exist because $F(.)$ may be such that $\frac{F(x)}{2-F(x)} Eu > x$ for all $x > 0$; see Figure 4. This happens, for example, with $G(x) = x^{\frac{1}{2}}$ for $0 \leq x \leq 1$, then $Eu = \frac{1}{3}$ and $\frac{F(x)}{2-F(x)} Eu = \frac{G(x)}{1-G(x)} Eu > x$ for all $x \in [0, 1]$.[10]

Not only the restriction $Eu = m$, but also the restriction $\alpha < \frac{2M}{M+Eu}$ in the first part of the proposition is only sufficient. In Appendix B, we briefly illustrate this 'sufficiency without necessity' with a simple example. We show also that the restrictions have grip by providing an example where an equilibrium does not exist when both restrictions are violated.

It is also worth mention that the possibility that a cutoff equilibrium does not exist depends on the normalization of utility for school B in the following sense. As shown in Appendix A, if utilities for school B and A, x_B and x_A, are drawn independently from the distributions $G[0, M]$ and $G[M, 2M]$, respectively, then a case 4 equilibrium exists for any $\alpha \geq 1$. Thus, without the normalization, the restriction on G that $Eu = m$ can be dispensed with.[11]

3.1 Welfare Properties

Proposition 1 asserts the existence of a cutoff equilibrium under fairly general conditions. We are now going to discuss the welfare properties of these cutoff equilibria.

[10] One of the reasons why a fix point x_4 does not exist in this case is that at $x = 0$, the derivative of $\frac{G(x)}{1-G(x)} Eu$ is infinite.

[11] Of course, some symmetry is contained in the assumption that x_B and x_A are drawn from the same distribution G with disjoint supports.

Proposition 2 *In any cutoff equilibrium, every individual expects a higher utility at the interim stage in $t = 1$ under the dynamic mechanism than under the static mechanism.*

Proof: Denote by $U(x)$ the expected interim utility of an individual under a static mechanism, where everybody always applies to A. That is, $U(x) \equiv \frac{\alpha}{2}x + \frac{\alpha}{2}Eu$. Given μ, the utility of applying to A under the dynamic mechanism is

$$U_A(x) = \min\left\{1, \frac{\alpha}{\mu}\right\} x + \max\left\{0, \frac{\alpha - (2-\mu)}{\mu}\right\} Eu$$

and the utility of applying to B is

$$U_B(x) = \max\left\{0, \frac{\alpha - \mu}{2-\mu}\right\} x + \min\left\{1, \frac{\alpha}{2-\mu}\right\} Eu.$$

Case 1: If $2 - \mu < \alpha < \mu$, then $U_A(x) = \frac{\alpha}{\mu}x + \frac{\alpha-(2-\mu)}{\mu}Eu \geq \frac{\alpha}{2}(x + Eu) = U(x) \Leftrightarrow x \geq \frac{2-\alpha}{\alpha}Eu$. On the other hand, $U_B(x) = Eu > \frac{\alpha}{2}(x + Eu) = U(x) \Leftrightarrow x < \frac{2-\alpha}{\alpha}Eu$.

Case 2: If $\mu < \alpha < 2 - \mu$, then $U_A(x) = x \geq \frac{\alpha}{2}(x + Eu) = U(x) \Leftrightarrow x \geq \frac{\alpha}{2-\alpha}Eu$. For $U_B(x) > U(x)$, we need $\frac{\alpha-\mu}{2-\mu}x + \frac{\alpha}{2-\mu}Eu > \frac{\alpha}{2}(x+Eu) \Leftrightarrow x < \frac{\alpha}{2-\alpha}Eu$.

Case 3: If $\alpha < \min\{\mu, 2-\mu\}$, then $U_A(x) = \frac{\alpha}{\mu}x \geq \frac{\alpha}{2}(x + Eu) = U(x) \Leftrightarrow x \geq \frac{\mu}{2-\mu}Eu$. For $U_B(x) > U(x)$, we need $\frac{\alpha}{2-\mu}Eu > \frac{\alpha}{2}(x + Eu) \Leftrightarrow x < \frac{\mu}{2-\mu}Eu$.

Case 4: If $\alpha > \max\{\mu, 2-\mu\}$, then $U_A(x) = x + \frac{\alpha-(2-\mu)}{\mu}Eu \geq \frac{\alpha}{2}(x + Eu) = U(x) \Leftrightarrow x \geq \frac{2-\mu}{\mu}Eu$. For $U_B(x) > U(x)$, we need $U_B(x) = \frac{\alpha-\mu}{2-\mu}x + Eu > \frac{\alpha}{2}(x + Eu) = U(x) \Leftrightarrow x < \frac{2-\mu}{\mu}Eu$. ■

For the static mechanism, it is immaterial whether it is a one period or a two period problem. At the interim stage in $t = 1$, the expected utility of an individual with utility draw x is $U = \frac{\alpha}{2}x + \frac{\alpha}{2}Eu$. By Proposition 2, one can ask individuals at the interim stage whether they prefer the dynamic or the static mechanism. Under our assumptions, they will all prefer the dynamic mechanism. Moreover, because interim expected utility under the dynamic mechanism exceeds interim expected utility under a static mechanism for every individual, we have also shown:

Corollary 1 *Ex ante expected utility under the dynamic mechanism is larger than under a static mechanism.*

Overall Second-best Welfare Next we show that whenever the first-best allocation is achieved in $t = 1$ under a dynamic mechanism, then this mechanism achieves the second-best allocation over both periods, i.e. implements the optimal incentive compatible allocation overall.

Lemma 1 *In the two-period game where all students prefer A to B, the optimal incentive compatible second period allocation is a random allocation.*

Proof In $t = 2$, the game reduces to a static game. Thus, all individuals have a dominant strategy to apply to A (or to report the highest utility if asked to report utilities). Consequently, based on period two reports, the allocation can only be random. On the other hand, because of the i.i.d. assumption, period one reports cannot be informative about period two utilities. Thus, there is no way to improve upon a random allocation in period two. ■

For example, the dynamic mechanism establishes first-best in $t = 1$, and second-best in $t = 2$ when both schools have equal capacities and when $Eu = m$. Consequently, no other mechanism can do better. Thus:

Proposition 3 *If the dynamic mechanism establishes first-best in period one, it establishes second-best overall.*

3.2 Unique Equilibrium vs. Multiple Equilibria

The following proposition characterizes completely the sets of equilibria under the dynamic mechanism.

Proposition 4 *Two cases occur:*

- *For $\alpha \leq 1$, there is a unique equilibrium under the dynamic mechanism. This equilibrium is a cutoff equilibrium.*
- *For $\alpha > 1$, all equilibria under the dynamic mechanism are either cutoff equilibria, or all students apply to B.*

Proof: First, we show that for any $\alpha \in (0, 2)$, there is no equilibrium where all $x \in [\underline{x}, \overline{x}]$ do the same (e.g. apply to A), while some $x' < \underline{x}$ and some $x'' > \overline{x}$ do the converse (i.e. apply to B), where $0 < \underline{x} \leq \overline{x} < M$. Therefore, all equilibria will be either cutoff or such that all individuals do the same. Second, we show that for any $\alpha \in (0, 2)$ there is no equilibrium where all apply to A. Moreover, for $\alpha \leq 1$, there is no equilibrium where all apply to B. For $\alpha > 1$, on the other hand, it cannot be ruled out in general that all apply to B. Third, we deal with the question of multiple cutoff equilibria. This multiplicity may occur for $\alpha > 1$, but not for $\alpha \leq 1$.

Step 1: Denote by $P_t(k)$ the probability of getting into school A in period t when applying to school k in period one, $k = A, B$ and $t = 1, 2$. Assume first that all $x \in [\underline{x}, \overline{x}]$ apply to A. Then, we must have

$$\begin{aligned} U_A(x) = P_1(A)x + P_2(A)Eu &\geq P_1(B)x + P_2(B)Eu = U_B(x) \\ &\Leftrightarrow \\ [P_1(A) - P_1(B)]\, x &\geq [P_2(B) - P_2(A)]\, Eu. \end{aligned}$$

Because it follows from our assumptions that at least $F(\underline{x}) > 0$ individuals have priority over applicants to A in period two, $P_2(B) > P_2(A)$ follows. Therefore, the right-hand side is positive, and consequently, the left-hand side must be positive, too, implying $P_1(A) > P_1(B)$. Therefore, $\frac{\partial U_A}{\partial x} > \frac{\partial U_B}{\partial x} > 0$. Thus, the stipulated behavior cannot be an equilibrium.

If all $x \in [\underline{x}, \overline{x}]$ apply to B, the above inequality must be reversed. But then, all $x < \underline{x}$ strictly prefer to apply to B, too, thus yielding the desired contradiction.

Step 2: Assume all apply to A, i.e. $\mu^* = 2$. But then, for any $\alpha \in (0, 2)$,

$$U_B(x) = Eu > \frac{\alpha}{2}(x + Eu) = U_A(x)$$

for x sufficiently close to zero.

If $\alpha \leq 1$, then there is no equilibrium where all apply to B because

$$U_A(x) = x > \frac{\alpha}{2}(x + Eu) = U_B(x)$$

for $x > \frac{\alpha}{2-\alpha}Eu$, which for $\alpha \leq 1$ holds for any $x > Eu$, and thus in particular for $x = M$. On the other hand, if $\alpha > 1$, then $\frac{\alpha}{2-\alpha}Eu > M$ may be the case. Thus, for $\alpha > 1$ all applying to B can be an equilibrium.

Step 3: From the previous steps, we know that the only equilibria are cutoff for $\alpha \leq 1$. We are now going to show that there is a unique cutoff equilibrium in this case by showing that the conditions of cases 1 trough 4 are incompatible.

Note first that for $\alpha = 1$, all four cases are equivalent, provided their conditions are consistent with equilibrium. That is, $x_1 = x_2 = Eu$. So if case 1 and 2 are both consistent with equilibrium, it must be that $F(x_1) = F(x_2) = 1$. Similarly, if case 3 or case 4 is consistent with equilibrium, the constraints for each of these two cases must hold with equality, implying $F(x_3) = F(x_4) = 1$.

Consider therefore the case with $\alpha < 1$. Trivially, there is no case 4 equilibrium for $\alpha < 1$, so cases 1, 2 and 3 remain to be checked.

Next note that $x_1 = \frac{2-\alpha}{\alpha}Eu > \frac{\alpha}{2-\alpha}Eu = x_2$ implies $F(x_1) > F(x_2)$, which in turn implies $2 - F(x_1) < 2 - F(x_2)$. Because case 1 requires $\alpha < 2 - F(x_1)$ while case 2 requires $2 - F(x_2) < \alpha$, it follows that the two cases are mutually exclusive. Thus, we are left to check consistency of cases 1 and 3 and cases 2 and 3.

So as to see that cases 1 and 3 are incompatible, note first that $\alpha < F(x_3)$ implies $2-\alpha > 2-F(x_3)$, which in turn implies $x_1 = \frac{2-\alpha}{\alpha}Eu > \frac{2-F(x_3)}{F(x_3)}Eu = x_3$ and hence $F(x_1) > F(x_3)$. But $\alpha \leq F(x_3) < F(x_1) \leq \alpha$ is a contradiction. Hence, the two cases are mutually exclusive. In order to see that cases 2 and 3 are incompatible, observe that $\alpha < 2 - F(x_3)$ implies $2 - \alpha > F(x_3)$, which in turn implies $x_3 = \frac{2-F(x_3)}{F(x_3)}Eu > \frac{\alpha}{2-\alpha}Eu = x_2$. Hence, $F(x_3) > F(x_2) \Leftrightarrow 2 - F(x_3) < 2 - F(x_2)$. But for case 3, $\alpha \leq 2 - F(x_3)$ must hold, whereas for case 2, $2 - F(x_2) \leq \alpha$ has to be satisfied, which with $2 - F(x_3) < 2 - F(x_2)$ yields the desired contradiction.

Finally, turn to the case $\alpha > 1$. Clearly, a case 3 equilibrium cannot occur now. However, case 1 and case 2 are now not mutually exclusive since for $\alpha > 1$, $x_1 < x_2$ follows, implying $F(x_1) < F(x_2) \Leftrightarrow 2 - F(x_1) > 2 - F(x_2)$. So, depending on $F(.)$ and α, $F(x_1) < \alpha < 2 - F(x_1)$ and $2 - F(x_2) < \alpha < F(x_2)$ can both hold. Moreover, either case can be consistent with case 4, depending again on $F(.)$ and α. ■

So as to develop some intuition and gain a better understanding of the welfare implications of different cutoffs, an example with multiple cutoff equilibria is instructive.

Welfare Across the Different Equilibria We now consider an example where utility is drawn from the uniform distribution $G(x) = \frac{x}{M}$ on [0,M], so that $F(x) \equiv 2G(x) = \frac{2x}{M}$ and $Eu = m = \frac{M}{2}$. For the four cases we have:

- *Case 1*: $x_1 = \frac{2-\alpha}{\alpha}\frac{M}{2}$ and $F(x_1) = \frac{2-\alpha}{\alpha}$.
- *Case 2*: $x_2 = \frac{\alpha}{2-\alpha}\frac{M}{2}$ and $F(x_2) = \frac{\alpha}{2-\alpha}$.
- *Case 3*: $x_3 = \frac{M}{2}$ because $Eu = m$, so that $F(x_3) = 1$.
- *Case 4*: $x_4 = \frac{M}{2}$ because $Eu = m$, so that $F(x_4) = 2 - F(x_4) = 1$.

For $\alpha < 1$, case 3 is an equilibrium, and from Proposition 4 we know that it is the unique equilibrium. For $\alpha > 1$, case 4 is clearly an equilibrium. Case 1 is an equilibrium whenever $F(x_1) < \alpha < 2 - F(x_1) \Leftrightarrow \frac{2-\alpha}{\alpha} < \alpha < \frac{3\alpha-2}{\alpha}$, which holds for all $\alpha \in (1, 2)$. A necessary condition for a case 2 equilibrium is $2 - F(x_2) < \alpha < F(x_2) \Leftrightarrow \frac{4-3\alpha}{2-\alpha} < \alpha < \frac{\alpha}{2-\alpha}$, which holds for all $\alpha \in (1, 2)$. However, for case 2 to be an interior cutoff, it must also be the case that $x_2 < M$, which requires $\alpha < \frac{4}{3}$. Thus, for $\alpha \in (1, \frac{4}{3})$, there are three cutoff equilibria.

Moreover, for $\frac{\alpha}{2-\alpha}Eu > M \Leftrightarrow \alpha > \frac{4}{3}$, there is also an equilibrium where all apply to B.

Clearly, under the equilibrium where all apply to B welfare is lowest because this equilibrium induces just a random assignment in both periods. More interesting is a comparison of welfare across the different cutoff equilibria. Welfare is not monotonously increasing in the cutoff x^*. The reason is that though a higher x^* implies a greater efficiency among those who apply to A, a higher x^* eventually also means more inefficiency among those who apply to B and who eventually are assigned to A. Having fewer applicants for A (or equivalently, having a higher cutoff) is only desirable if there is excess demand for A. If, in equilibrium, the number of applicants is already reduced below A's capacity, then it seems desirable to have a lower cutoff, implying more applicants to A, because the random assignment among those who apply to B is a source of inefficiency as well.

Denote by $g(x)$ the density of $G(x)$. For the uniform distribution we have $g(x) = \frac{1}{M}$. We focus on the case with $\alpha > 1$, for otherwise, there is a unique equilibrium. Since expected welfare in period two is the same for all allocations, we only report period one welfare. We compute the ex ante expected utility of an individual and denote by W_i the first-period welfare in case j with $j = 1, 2, 4$.

- $W_1 = \frac{\alpha}{2-F(x_1)} \int_{x_1}^{M} xg(x)dx = \frac{2+\alpha}{8}M$
- $W_2 = \int_{x_2}^{M} xg(x)dx + \frac{\alpha-(2-F(x_2))}{F(x_2)} \int_{0}^{x_2} xg(x)dx = \frac{4-\alpha}{8}M$
- $W_4 = \int_{\frac{M}{2}}^{M} xg(x)dx + (\alpha - 1) \int_{0}^{\frac{M}{2}} xg(x)dx = \frac{2+\alpha}{8}M.$

Note that $W_1(\alpha) = W_4(\alpha)$. Moreover, for $\alpha = 1$, welfare is the same across all equilibria, but it increase in α for case 1 and 4 while it decreases in α in case 2. Thus, for $\alpha \in (1, \frac{4}{3})$,

$$W_1 = W_4 > W_2.$$

3.3 Many Periods

We conclude this section with a very brief discussion of what happens if there are more than two periods. Denote by $W^{DM(2)}$ the overall welfare achieved in the equilibrium under the dynamic mechanism in the two period model, under the assumption that such an equilibrium exists. Without much loss, assume that the number of periods T is even[12] and denote by $W^{DM(T)}$ equilibrium welfare under a dynamic mechanism for the T period model.

[12] If not, the following statements are true for the model confined to the first $\tilde{T} \equiv T - 1$ periods.

Proposition 5 *Under the above assumptions, there is a dynamic mechanism that induces a cutoff equilibrium such that*

$$W^{DM(T)} \geq \frac{T}{2} W^{DM(2)}$$

holds.

Proof: The existence of a dynamic mechanism that induces a cutoff equilibrium follows immediately from the existence of such an equilibrium under a dynamic mechanism for the two period model: Repeat the two period mechanism $T/2$ times. The resulting equilibrium welfare will be $W^{DM(T)} = \frac{T}{2} W^{DM(2)}$. However, in general the longer time horizon may allow for even better mechanisms. Therefore, $W^{DM(T)} \geq \frac{T}{2} W^{DM(2)}$ holds. ■

4 Heterogeneity in Ordinal Preferences

We now consider four extensions of the basic model. First, we study the model under the assumption that individuals are heterogenous with respect to ordinal preferences. Second, we analyze a simple model when students are uncertain about which school they will prefer in the future. Third, we look at the case when utilities are correlated across time. Lastly, we analyze what happens when there is some inertia in the sense that students dislike switching from one school to another.

4.1 Mechanisms when Ordinal Preferences are not Known

So far, we have assumed that all students prefer school A to B, though the intensity with which they do so may differ across time. We now replace this assumption by the alternative that some students have an ordinal time-invariant preference for school A and others an ordinal and time-invariant preference for school B. These preferences are private information, so that even the fraction of students preferring A to B is not known. These assumptions are natural when a school is not good or bad per se, but rather specialized, say, either in languages or in math, and when it is not known how many students prefer to specialize in math or languages, respectively.

There are two schools A and B with capacities α and $2 - \alpha$, respectively, and two periods. Let x_{kt} be distributed according to $G(.)$ with support $[0, M]$ and let $x_{-kt} = 0$ for all students, where $-k$ means "not k", $k = A, B$ and $t = 1, 2$. We first describe an augmented dynamic mechanism that allows to infer the true preferences before individuals are asked to apply to the good or bad school in period one.[13] Second, we derive equilibrium behavior under this mechanism.

Dynamic Two Phase Mechanism

Phase 1: Ask all individuals i to report their ordinal preferences $\succ_i$. If i's preferred school is available (i.e. if there is no excess demand for this school), i is assigned to this school for both periods. Individuals who prefer a school with excess demand enter phase 2.

Denote the true number of students preferring A to B by a. Without loss, assume that A is the school with excess demand, i.e. $a > \alpha$.

Phase 2: The mechanism announces the number of individuals who reported that they preferred the school with excess demand. Then the dynamic mechanism of Section 3 is applied, i.e. if a student now applies to the less preferred school, he'll have priority for the preferred school in $t = 2$.

4.2 Equilibrium under the Two Phase Mechanism

Lemma 2 *Truth telling in phase 1 is a strictly dominant strategy.*

Proof: Consider an individual i with $A \succ_i B$ and let $\tilde{a}$ be the measure of other individuals who report that they prefer A to B. If $\tilde{a} < \alpha$, then by telling the truth he gets his first-best (i.e. into A in both periods, which is obviously better than lying). If $\tilde{a} \geq \alpha$, there is too little demand for B. Thus, by lying he gets into B in both periods, which is the worst that can happen to him. Thus, truth telling is strictly dominant. ■

The lemma implies that $\tilde{a} = a$ in equilibrium. Consequently, free capacity of school B is $a - \alpha > 0$. Without loss, we can normalize $\hat{a} = 2$ and $\hat{\alpha} = \frac{2}{a}\alpha$. Therefore, we have shown:

Proposition 6 *The game in phase 2 is equivalent to the one studied in Section 3, where capacity of school A is $\hat{\alpha}$.*

5 Uncertainty about Ordinal Preferences

This section contains a generalization to the case where there is uncertainty both about ordinal and cardinal future preferences (as opposed to only the cardinal

[13] Put differently, under this mechanism ordinal and cardinal preference statements are made in separate steps. The motivation is the same as the one of Sönmez and Ünver (2005), who observe that order and intensity of preferences cannot be both inferred when observing only a single variable ("bids").

component). This model is appropriate when each school is specialized in, say, languages or math, and when students are uncertain in period one about the skills they are going to develop. It is shown for the example of a uniform distribution that previous results basically carry over.

5.1 Assumptions

There are two periods, and two schools A and B with equal capacities (i.e. $\alpha = 1$). Total mass of students is two, and instantaneous utilities for school A and B are drawn independently from the distributions $G_A(x)$ and $G_B(x)$ with support $[\underline{\mathcal{M}}, \overline{\mathcal{M}}]$. These draws are i.i.d. across students and time, and as before, there is no discounting. We assume that school A is the good school in the sense that $EU_A > EU_B$.

There are now students who prefer school B to A in one period and A to B in the other period. Denote by γ the probability that a student prefers B to A. That is,

$$\gamma \equiv \Pr(x_B > x_A) = \int_{\underline{\mathcal{M}}}^{\overline{\mathcal{M}}} \int_{\underline{\mathcal{M}}}^{y} dG_A(x) dG_B(y) = \int_{\underline{\mathcal{M}}}^{\overline{\mathcal{M}}} G_A(y) dG_B(y)$$

and $\Pr(x_B \leq x_A) = (1-\gamma)$. Consequently, some students who applied to B in $t = 1$ will prefer school B to A in period two. Therefore, the dynamic mechanism is as follows:

Dynamic Mechanism: If you apply to B in $t = 1$, you are in B, provided there is enough capacity. In $t = 2$, everybody can again apply to A or B. Those who applied to B in $t = 1$ have priority over all those who applied to A, whatever school they apply to in $t = 2$.

Before we can proceed with the equilibrium analysis, some further definitions are needed. The expected utility of A of a student who wants to go to A, i.e. conditional on $x_A \geq x_B$, is

$$EU_A \equiv E[x_A \mid x_A \geq x_B] = \int_{\underline{\mathcal{M}}}^{\overline{\mathcal{M}}} \int_{\underline{\mathcal{M}}}^{x_A} x_A \frac{dG_B(x_B) dG_A(x_A)}{1-\gamma},$$

where $\frac{1}{1-\gamma}$ is the conditioning factor. Next, denote by $\tilde{EU}_B$ the expected utility of school B, conditional on $x_A > x_B$, which is given by

$$\tilde{EU}_B \equiv E[x_B \mid x_A \geq x_B] = \int_{\underline{\mathcal{M}}}^{\overline{\mathcal{M}}} \int_{x_B}^{\overline{\mathcal{M}}} x_B \frac{dG_A(x_A) dG_B(x_B)}{1-\gamma}.$$

This will be of importance because an individual who would like to go to school A may be assigned to B despite $x_A > x_B$, in which case his expected utility is $\tilde{EU}_B$.

5.2 Equilibrium for the Uniform Distribution

Assume G_A and G_B are uniform on $[0, 2M]$ and $[0, M]$, so that their densities are $g_A = \frac{1}{2M}$ for $0 \leq x \leq 2M$ and $g_B = \frac{1}{2M}$ for $0 \leq x \leq M$, respectively. The unconditional expected utilities then are M and $\frac{M}{2}$. The probability of having a preference for school B is then easily seen to be

$$\gamma = \int_0^M \int_0^y \frac{1}{2M^2} dx dy = \frac{1}{4}.$$

The joint density of x_A and x_B is $\frac{1}{2M^2}$ and the probability of $x_B \geq x_A$ is $\gamma = \frac{1}{4}$. Hence, the joint conditional density of x_A and x_B, conditional on $x_A \geq x_B$, is $g(x_A, x_B \mid x_A \geq x_B) = \frac{2}{3M^2}$. Therefore, the relevant conditional expectations are

$$EU_A = E[x_A \mid x_A \geq x_B] = \int_0^M \int_{x_B}^{2M} \frac{x_A}{2M^2} \frac{1}{1-\gamma} dx_A dx_B = \frac{11}{9} M$$

and

$$\tilde{EU}_B = E[x_B \mid x_A \geq x_B] = \int_0^M \int_{x_B}^{2M} \frac{x_B}{2M^2} \frac{1}{1-\gamma} dx_A dx_B = \frac{4}{9} M.$$

Proposition 7 *The game has an equilibrium in which individuals apply to A in $t = 1$ if and only if $\mathbf{x} \equiv (x_A, x_B)$ is such that*

$$x_A \geq x_B + (1 - 2\gamma)[EU_A - \tilde{EU}_B] = x_B + \frac{7}{18} M.$$

In this equilibrium, $\mu^* = \frac{10}{9}$.

Proof: Notice first that $[EU_A - \tilde{EU}_B] = \frac{7}{9} M$ and recall $\gamma = \frac{1}{4}$. Since all individuals in the area above $x_A \geq \frac{7}{18} + x_B$ apply to A, it is easily seen that their measure is $\frac{10}{9}$. So, assume this is an equilibrium. Then, the following holds: $\mu^* > 1 > (2 - \mu^*)$ and $1 > (2 - \mu^*)(1 - \gamma)$, so all who applied to B in $t = 1$ will get into A if they want in $t = 2$. On the other hand, for $\gamma < \frac{1}{2}$ the residual capacity of A in $t = 2$, $1 - (2 - \mu^*)(1 - \gamma)$, does not permit to accommodate all those who applied to A in $t = 1$ and who want to get into A in $t = 2$ since $\gamma < \frac{1}{2} \Leftrightarrow 1 - (2 - \mu^*)(1 - \gamma) < \mu^*(1 - \gamma)$. The cutoff equilibrium condition thus reads

$$\begin{aligned} U_A(\mathbf{x}) &= \frac{1}{\mu^*} x_A + \left(1 - \frac{1}{\mu^*}\right) x_B + (1 - \gamma) \\ &\times \left[\frac{1 - (2 - \mu^*)(1 - \gamma)}{\mu^*(1 - \gamma)} EU_A + \left(1 - \frac{1 - (2 - \mu^*)(1 - \gamma)}{\mu^*(1 - \gamma)}\right) \tilde{EU}_B\right] + \gamma EU_B \\ &\geq x_B + (1 - \gamma) EU_A + \gamma EU_B = U_B(\mathbf{x}). \end{aligned}$$

Re-arranging and simplifying yields the condition in the proposition. ■

5.3 Welfare Comparisons

Static mechanism Again, we restrict attention to period one welfare. Under a static mechanism, all students with a preference for school A apply to this school in $t = 1$. Since total mass of these students is $\frac{3}{2}$ and since $\alpha = 1$, one third of them has to be rationed and sent to school B. On the other hand, since $\gamma < \frac{1}{2}$, all students with a preference for B get into B for sure. Hence, so as to derive welfare under a static mechanism, we first need to compute additionally the expected utility of getting into B conditional on $x_B \geq x_A$, denoted as $E[x_B \mid x_B \geq x_A]$. The joint conditional density of x_A and x_B, conditional on $x_B \geq x_A$, is $g(x_A, x_B \mid x_B \geq x_A) = \frac{2}{M^2}$. Consequently,

$$E[x_B \mid x_B \geq x_A] \quad = \quad \int_0^M \int_0^{x_B} x_B \frac{2}{M^2} dx_A dx_B = \frac{2}{3}M.$$

Thus, expected welfare under a static mechanism, W^{SM}, is

$$\begin{aligned} W^{SM} &= (1-\gamma)\left[\frac{2}{3}E[x_A \mid x_A \geq x_B] + \frac{1}{3}E[x_B \mid x_A \geq x_B]\right] + \gamma E[x_B \mid x_B \geq x_A] \\ &= \frac{13}{18}M = 0.72M. \end{aligned}$$

Next, we consider the equilibrium welfare under the dynamic mechanism.

Dynamic Mechanism Under the dynamic mechanism, only those individuals with $x_A \geq x_B + (1-2\gamma)\left[EU_A - \tilde{EU_B}\right] = x_B + \frac{7}{18}M$ apply to school A. The conditional expectations relevant for welfare are therefore $E[x_B \mid x_A \leq x_B + \frac{7}{18}M]$, $E[x_A \mid x_A \geq x_B + \frac{7}{18}M]$ and $E[x_B \mid x_A \geq x_B + \frac{7}{18}M]$. The probability of $x_A \geq x_B + \frac{7}{18}M$ is $\frac{5}{9}$. Thus, $g\left(x_A, x_B \mid x_A \geq x_B + \frac{7}{18}M\right) = \frac{9}{10M^2}$, and $g\left(x_A, x_B \mid x_A \leq x_B + \frac{7}{18}M\right) = \frac{9}{8M^2}$. Hence

$$\begin{aligned} E[x_B \mid x_A \leq x_B + \frac{7}{18}M] &= \int_0^M \int_0^{x_B} x_B \frac{9}{8M^2} dx_A dx_B = \frac{19}{32}M \\ E[x_A \mid x_A \geq x_B + \frac{7}{18}M] &= \int_0^M \int_{x_B}^{2M} x_A \frac{9}{10M^2} dx_A dx_B = \frac{1013}{720}M \\ E[x_B \mid x_A \geq x_B + \frac{7}{18}M] &= \int_0^M \int_{x_B}^{2M} x_B \frac{9}{10M^2} dx_A dx_B = \frac{17}{40}M. \end{aligned}$$

Since the measure of students applying to A is $\frac{10}{9}$, which exceeds school A's capacity, one tenth of them will be rationed and sent to school B. Thus, expected welfare under the dynamic mechanism, W^{DM}, is given as

$$\begin{aligned} W^{DM} &= \frac{5}{9}\left[\frac{9}{10}E[x_A \mid x_A \geq x_B + \frac{7}{18}M] + \frac{1}{10}E[x_B \mid x_A \geq x_B + \frac{7}{18}M]\right] \\ &+ \frac{4}{9}E[x_B \mid x_B \geq x_B + \frac{7}{18}M] = 0.99M. \end{aligned}$$

Compared to the welfare under a static mechanism, this seems like a substantial improvement. Now let us compare it with the first-best welfare.

First-best Under first-best, students with $x_A \geq x_B + \frac{M}{2}$ should be assigned to school A, and all others should go to B. Note that because $\Pr(x_A \geq x_B + \frac{M}{2}) = \frac{1}{2}$ there is no rationing at either school, which obviously should be the case under first-best. The probability of $x_A \geq x_B + \frac{M}{2}$ being $\frac{1}{2}$, the joint conditional densities are $\frac{1}{M^2}$. The expected utility of getting into A, conditional on $x_A \geq x_B + \frac{M}{2}$ is,

$$E[x_A \mid x_A \geq x_B + \frac{M}{2}] = \int_0^M \int_{x_B+\frac{M}{2}}^{2M} x_A \frac{1}{M^2} dx_A dx_B = \frac{35}{24}M,$$

and the expected utility of getting into B, conditional on $x_A \leq x_B + \frac{M}{2}$, is

$$E[x_B \mid x_A \leq x_B + \frac{M}{2}] = \int_0^M \int_0^{x_B+\frac{M}{2}} x_B \frac{1}{M^2} dx_A dx_B = \frac{7}{12}M.$$

Thus, first-best welfare is given as

$$W^{FB} = \frac{1}{2}\left[\frac{35}{24}M + \frac{7}{12}M\right] = \frac{49}{48}M = 1.02M.$$

Comparing this number to the welfare achieved under the dynamic mechanism, we see that welfare in period one under the dynamic mechanism is only three percent less than first-best. So, adding the complication of uncertainty about future ordinal preferences does not appear to have a big impact on the efficiency properties of the dynamic mechanism.

6 Correlations

We now relax the assumption that utility draws are i.i.d. over time. We show that even with positive correlation, expected welfare under the dynamic mechanisms exceeds welfare under the static mechanism for any correlation coefficient less than one. For perfectly positively correlated utility, welfare is the same under the dynamic and under the static mechanism. Finally, we briefly analyze the model with negative correlation.

6.1 Assumptions

There are two schools A and B, and total mass of students is two. As above, capacity of A is α and capacity of B is $2-\alpha$. There are two periods, and utility for school B is zero for both periods. Utility for A is greater than zero for all individuals and both periods. (We will be more precise about how utility for

A is generated shortly.) Denote by x the first period utility draw and by y the utility for the second period.

6.2 Positive Correlation

So as to model positive correlation, we follow Jackson and Sonnenschein (2004, Appendix 3, p. 45-6) and make the following assumptions. After period one utility x is realized, period two utility y is

$$y = x$$

with probability $\rho \in [0,1]$ and drawn from the distribution $G[0,M]$ with probability $(1-\rho)$. Thus,

$$E[y \mid x] = \rho x + (1-\rho)Eu.$$

Existence for the Case with Positive Correlation We now show that a result that is closely related to Proposition 1 holds for the case with positive correlation.

Proposition 8 *Provided α and ρ are such that $\rho \leq \min\left\{\frac{\alpha}{2-\alpha}, \frac{\frac{2-\alpha}{\alpha}M-Eu}{M-Eu}\right\}$, there is an interior cutoff equilibrium under the dynamic mechanism. A sufficient condition for the existence of an interior cutoff equilibrium for any $\alpha \in (0,2)$ and $\rho \in [0,1]$ is $Eu = m$.*

Proposition 8 generalizes Proposition 1 to the case with positive correlation. The only additional restriction we need to take care of is $x_1(\rho) > 0$, which holds if $\rho < \frac{\alpha}{2-\alpha}$. On the other hand, $x_2(\rho) < M$ is guaranteed by $\rho < \frac{\frac{2-\alpha}{\alpha}M-Eu}{M-Eu}$. This is a generalization of the condition $\alpha < \frac{2M}{Eu+M}$ of Proposition 1.[14] Therefore, the proof is analogous to the one of Proposition 1, which is why we only sketch it.

Sketch of Proof: It is straightforward to derive the candidate equilibrium cutoffs $x_i(\rho)$ with $i = 1,..,4$:

$$\begin{aligned}
x_1(\rho) &= \frac{2-\alpha}{\alpha-\rho(2-\alpha)}(1-\rho)Eu \qquad && F(x_1) \leq \alpha \leq 2-F(x_1) \\
x_2(\rho) &= \frac{\alpha}{2-\alpha-\rho\alpha}(1-\rho)Eu \qquad && 2-F(x_2) \leq \alpha \leq F(x_2) \\
x_3(\rho) &= \frac{2-F(x_3)}{F(x_3)-\rho(2-F(x_3))}(1-\rho)Eu \qquad && \alpha \leq \min\{F(x_3), 2-F(x_3)\} \\
x_4(\rho) &= \frac{F(x_4)}{2-F(x_4)-\rho F(x_4)}(1-\rho)Eu \qquad && \alpha \geq \max\{F(x_4), 2-F(x_4)\},
\end{aligned}$$

[14] To see this, set $\rho = 0$ in the condition of Proposition 8 to get the condition of Proposition 1.

where we have dropped ρ's inside $x_i(.)$ on the right-hand side for notational ease.

Next proceed as in the case with $\rho = 0$. That is, separate the problem into the case with $\alpha \leq 1$ and $\alpha > 1$ and consider the former first.

1. Choose x close to the median m, such that $\alpha \leq \min\{F(x), 2 - F(x)\}$ holds. If in addition $x = \frac{2-F(x)}{F(x)-\rho(2-F(x))}(1-\rho)Eu$, we are done.

2. So assume it does not hold for any x that satisfies $\alpha \leq \min\{F(x), 2 - F(x)\}$. Then, for all these x's either (a) $x > \frac{2-F(x)}{F(x)-\rho(2-F(x))}(1-\rho)Eu$ or (b) $x < \frac{2-F(x)}{F(x)-\rho(2-F(x))}(1-\rho)Eu$ holds.

Consider first (a). Decrease x until $\tilde{x}$, where $F(\tilde{x}) = \alpha$ and note that

$$\tilde{x} > \frac{2-\alpha}{\alpha - \rho(2-\alpha)}(1-\rho)Eu = \frac{2-F(\tilde{x})}{F(\tilde{x}) - \rho(2-F(\tilde{x}))}(1-\rho)Eu.$$

So, decrease x further until $\hat{x} = \frac{2-\alpha}{\alpha-\rho(2-\alpha)}(1-\rho)Eu > 0$ holds. That such a $\frac{2-\alpha}{\alpha-\rho(2-\alpha)}(1-\rho)Eu > 0$ exists is guaranteed by the assumption $\rho < \frac{\alpha}{2-\alpha}$.

Consider now (b). Increase x until $x = \tilde{x}$, where $\alpha = 2 - F(\tilde{x})$. Note that $\tilde{x} < \frac{\alpha}{2-\alpha-\rho\alpha}(1-\rho)Eu$. So, increase x further until $\hat{x} = \frac{\alpha}{2-\alpha-\rho\alpha}(1-\rho)Eu < M$. The inequality is satisfied because of the assumption $\rho < \frac{\frac{2-\alpha}{\alpha}M-Eu}{M-Eu}$, which is satisfied for any $\alpha \leq 1$ since for these α's $\frac{\frac{2-\alpha}{\alpha}M-Eu}{M-Eu} > 1$ holds.

3. Assume $\alpha > 1$. Choose x close to m. Assume that for no such x, $x = \frac{F(x)}{2-F(x)-\rho F(x)}(1-\rho)Eu$ holds for otherwise we are done. So, either (c) $x > \frac{F(x)}{2-F(x)-\rho F(x)}(1-\rho)Eu$ or (d) $x < \frac{F(x)}{2-F(x)-\rho F(x)}(1-\rho)Eu$ holds. Consider first (c). Decrease x until $\tilde{x}$, where $\alpha = 2 - F(\tilde{x}) > F(\tilde{x})$. Decrease x further until $\hat{x} = \frac{2-\alpha}{\alpha-\rho(2-\alpha)}(1-\rho)Eu$ holds. Because for the larger $\tilde{x}$, we had $\alpha = 2 - F(\tilde{x}) > F(\tilde{x})$, the restriction $F(\hat{x}) < \alpha < 2 - F(\hat{x})$ is clearly satisfied. Because of the assumption $\rho < \frac{\alpha}{2-\alpha}$, $\hat{x} > 0$ exists.

Finally, consider (d). Increase x in case the restriction is never satisfied until $x = \tilde{x}$, where $\alpha = F(\tilde{x})$. Increase x further until $\hat{x} = \frac{\alpha}{2-\alpha-\rho\alpha}(1-\rho)Eu$.

Like the conditions in Proposition 1, the condition $\rho \leq \min\left\{\frac{\alpha}{2-\alpha}, \frac{\frac{2-\alpha}{\alpha}M-Eu}{M-Eu}\right\}$ in Proposition 8 is only sufficient. To see this, consider a distribution G satisfying $Eu = m$. Then, $x_3(\rho) = x_4(\rho) = Eu$ for any $\rho \in [0, 1]$. Clearly, case 3 is an equilibrium for any $\alpha \leq 1$, and case 4 is an equilibrium for any $\alpha > 1$. Thus, there will be an interior cutoff equilibrium for any α and ρ, including those for which the condition is not met. ■

Welfare with Positive Correlation Next we compare expected utility in the interim stage under the dynamic mechanism with expected interim utility under a static mechanism. The following proposition is the analogue to, and a generalization of, Proposition 2 for the case with positively correlated utilities.

Proposition 9 *In the interim stage of a cutoff equilibrium with positive correlation, expected utility under the dynamic mechanism is larger than under the static mechanism.*

Proof: The proof mimics the one for Proposition 2. As in Proposition 2, denote by $U(x) = \frac{\alpha}{2}(x + E[y \mid x]) = \frac{\alpha}{2}((1+\rho)x + (1-\rho)Eu)$ the expected utility in the interim stage under the static mechanism. Under the dynamic mechanism, we have:

Case 1: $U_A(x) = \frac{\alpha}{\mu}x + \frac{\alpha-(2-\mu)}{\mu}E[y \mid x] \geq \frac{\alpha}{2}(x + E[y \mid x]) = U(x) \Leftrightarrow x \geq \frac{2-\alpha}{\alpha-\rho(2-\alpha)}(1-\rho)Eu = x_1(\rho)$. On the other hand, $U_B(x) = E[y \mid x] \geq \frac{\alpha}{2}(x+E[y \mid x]) = U(x) \Leftrightarrow x \leq \frac{2-\alpha}{\alpha-\rho(2-\alpha)}(1-\rho)Eu = x_1(\rho)$.

Case 2: $U_A(x) = x \geq \frac{\alpha}{2}(x + E[y \mid x]) = U(x) \Leftrightarrow x \geq \frac{\alpha}{2-\alpha-\rho\alpha}(1-\rho)Eu = x_2(\rho)$. On the other hand, $U_B(x) = \frac{\alpha-\mu}{2-\mu}x + \frac{\alpha}{2-\mu}E[y \mid x] \geq \frac{\alpha}{2}(x + E[y \mid x]) = U(x) \Leftrightarrow x \leq \frac{\alpha}{2-\alpha-\rho\alpha}(1-\rho)Eu = x_2(\rho)$.

Case 3: $U_A(x) = \frac{\alpha}{\mu}x \geq \frac{\alpha}{2}(x + E[y \mid x]) = U(x) \Leftrightarrow x \geq \frac{\mu}{2-\mu-\rho\mu}(1-\rho)Eu = x_3(\rho)$. On the other hand, $U_B(x) = \frac{\alpha}{2-\mu}E[y \mid x] \geq \frac{\alpha}{2}(x + E[y \mid x]) = U(x) \Leftrightarrow x \leq \frac{\mu}{2-\mu-\rho\mu}(1-\rho)Eu = x_3(\rho)$.

Case 4: $U_A(x) = x + \frac{\alpha-(2-\mu)}{\mu}E[y \mid x] \geq \frac{\alpha}{2}(x + E[y \mid x]) = U(x) \Leftrightarrow x \geq \frac{\mu}{2-\mu-\rho\mu}(1-\rho)Eu = x_4(\rho)$. On the other hand, $U_B(x) = \frac{\alpha-\mu}{2-\mu}x + E[y \mid x] \geq \frac{\alpha}{2}(x + E[y \mid x]) = U(x) \Leftrightarrow x \leq \frac{\mu}{2-\mu-\rho\mu}(1-\rho)Eu = x_4(\rho)$. ■

An immediate corollary to Proposition 9 is:

Corollary 2 *Whenever an interior cutoff equilibrium exists with positive correlation, ex ante expected utility under the dynamic mechanism exceeds ex ante expected utility under a static mechanism.*

6.3 Negative Correlation

Though the case with positive correlation may seem more relevant for real world applications, there are also situations where period one and two utilities may be negatively correlated. Consider the server-client example discussed in the Introduction. If the server supports clients who experience stress due to exogenous shocks (e.g. deadlines) and if these shocks are negatively correlated over time (which is likely to be the case for deadlines), then there will be negative correlation.

We capture negative correlation by assuming that $y = M - x$ with probability $\rho \in [0,1]$ and drawn from $G[0,M]$ with probability $(1-\rho)$. Moreover, the distribution $G[0, M]$ from which utility for A is drawn is now assumed to be symmetric and continuous, i.e. $g(x) = g(M - x)$. This assumption makes sure that the unconditional expectation of period two utility is the same as the expectation of period one utility.

Existence Before we prove existence, some preliminary observations are helpful. We write $E_X[.]$ for the expectation taken with respect to the distribution of x.

`Claim`: $E_X[x] = E_X[M - x] = \frac{M}{2}$ when $g(.)$ is symmetric.

`Proof`: First, $E_X[M - x] = M - E_X[x]$, which is immediate. Second, $E_X[x] = E_X[M - x]$. (Together with the first observation this then proves that $E_X[x] = \frac{M}{2}$.) To see that $E_X[x] = E_X[M - x]$ is true, note that

$$E_X[x] = \int_0^M xg(x)dx = \int_0^M (M - x)g(M - x)dx = \int_0^M (M - x)g(x)dx,$$

where the last equality holds because of symmetry. But $\int_0^M (M - x)g(x)dx = E_X[M - x]$. ■

Due to symmetry, it is also true that the median m is equal to $\frac{M}{2} = E[x] \equiv Eu$, where we drop the subscript X if there is no danger of confusion. Moreover, $E[y \mid x] = \rho(M - x) + (1 - \rho)Eu = (1 + \rho)Eu - \rho x$, where the second equality follows because $M = 2Eu$.

We can now state the general existence result, which is straightforward to prove because G is symmetric.

Proposition 10 *With negative correlation (and G symmetric), an interior cutoff equilibrium always exists.*

Proof: Consider the four cases of Proposition 1. Replace Eu by $E[y \mid x] = (1 + \rho)Eu - \rho x$ and solve for the respective cutoff $x_i(\rho)$, $i = 1, .., 4$ to get

$$\begin{aligned}
x_1(\rho) &= \frac{2 - \alpha}{\alpha + \rho(2 - \alpha)}(1 + \rho)Eu \qquad && F(x_1) \leq \alpha \leq 2 - F(x_1) \\
x_2(\rho) &= \frac{\alpha}{2 - \alpha + \rho\alpha}(1 + \rho)Eu && 2 - F(x_2) \leq \alpha \leq F(x_2) \\
x_3(\rho) &= \frac{2 - F(x_3)}{F(x_3) + \rho(2 - F(x_3))}(1 + \rho)Eu && \alpha \leq \min\{F(x_3), 2 - F(x_3)\} \\
x_4(\rho) &= \frac{F(x_4)}{2 - F(x_4) + \rho F(x_4)}(1 + \rho)Eu && \alpha \geq \max\{F(x_4), 2 - F(x_4)\}.
\end{aligned}$$

Because G is symmetric, $Eu = m$ holds. Thus, for $\alpha \leq 1$ case 3 is always an equilibrium, and for $\alpha > 1$, case 4 is always an equilibrium. ■

Welfare Next we address interim expected utility.

Proposition 11 *With negative correlation (and G symmetric), interim expected utility under the dynamic mechanism exceeds interim expected utility under a static mechanism.*

The proof is omitted because it is a one-to-one replication of the proofs of Propositions 2 and 9. The only additional thing that one needs to keep in mind is that $M = 2Eu$ because the distribution G is symmetric. The proposition also implies that ex ante expected welfare under the dynamic mechanism exceeds expected welfare under a static mechanism.

Welfare with Equal Capacities Assume $\alpha = 1$. Then, the cutoff in period one is $x_1 = m$, and we have first-best in period one:

$$W_1^{DM} = E[x \mid x \geq m] = W_1^{FB}.$$

In period two, welfare is

$$W_2^{DM} \quad = \quad 2\int_0^m \left(\rho(M-x) + (1-\rho)E[x]\right) g(x)dx,$$

where $2g(x)$ is the density conditional on $x \leq m$. Thus,

$$W_2^{DM} = (1-\rho)E[x] + 2\rho\int_0^m (M-x)g(x)dx = (1-\rho)E[x] + \rho E[x \mid x \geq m].$$

Therefore, overall, welfare under the dynamic mechanism is

$$W^{DM}(\rho) = (1+\rho)E[x \mid x \geq m] + (1-\rho)E[x],$$

which for $\rho = 1$ is equal to first-best welfare $W^{FB} = 2E[x \mid x \geq m]$.

7 Switching Costs

In many settings, it is natural to assume that there is some kind of inertia. For example, given that a student has already been assigned to a school, he or she may be less inclined to go to the other school even if the other school is, a priori, perceived as better. Similarly, when assigning tasks within an organization or assigning houses to individuals, there may be economies of scale because handing over the task or apartment from one individual to another is costly. So as to take this possibility into account, we now introduce a switching cost $s \geq 0$ that is borne by every student upon switching from one school to another.

However, before turning to the details of the model, a few comments are in order. Clearly, any cost of switching, say, schools must be put into proportion with the utility generated by a school. Though this cost may be relatively high when schools were to be switched, say, every year, it is arguably much smaller when students switch from one school to another every five or six years, as is the case under the Boston school assignment scheme discussed in the Introduction. In particular, when periods are sufficiently long, switching costs may be close to zero or even become negative.[15]

7.1 The Model with Switching Costs

The assumptions are now slightly modified to incorporate switchings costs. As before, there are two periods, two schools A and B and students have homogenous and time-invariant ordinal preferences for the two schools. Utility for school B is normalized to zero, while utility for A is drawn from the distribution G with support [0,M]. Upon changing from one school to another, a student bears a switching cost $s \geq 0$.

As in the case with correlation, when assessing welfare one can no longer restrict attention to the period one allocation. Students who are re-assigned in $t = 2$ bear switching costs, which are a dead weight loss and hence work against the desirability of our mechanism. On the other hand, in $t = 2$ some students who are in B and who have low preference intensity for A prefer to stay in B, which all else equal enhances the efficiency of the mechanism.

Introducing inertia in the form of switching costs does not only have the effect of reducing the desirability of the dynamic mechanism. It also renders the model much less tractable. We therefore proceed as follows. Instead of treating the model in complete generality, we derive first the cutoff x^* for an equilibrium related to a case 2 equilibrium (as in Section 3) under the assumption that such an equilibrium exists. Second, when G is uniform we derive sufficient conditions for such an equilibrium to exist. Third, we discuss the welfare properties of the dynamic mechanism and compare them with those for a static mechanism and with first-best. Fourth, we show that for equal capacities and the distributions G with $Eu = m$ there is no case 3 and case 4 equilibrium.[16] Finally, we briefly discuss a modified dynamic mechanism that induces less switching and yields larger welfare than the standard dynamic mechanism used so far.

[15] This would be the case if parents feel that it is a good thing if their children learn to adopt to a new environment every once in a sufficiently long while.

[16] Equilibria of type 1 with switching costs, on the other hand, turn out to be very hard to derive, which is why we chose not address them here.

7.2 Case 2 Equilibrium Conditions

Recall that a case 2 equilibrium is such that $\mu < \alpha < 2-\mu \Leftrightarrow 2-\mu > 2-\alpha$. In this equilibrium, all students who apply to A in $t=1$ will get into A in $t=1$ and into B in $t=2$ with certainty. With a positive cost of switching schools, though, only the fraction $1-G(s)$ of the $(2-\mu)$ students in B want to get into A because the students who have been assigned to B in $t=1$ and whose utility draw x is less than s in $t=2$ prefer to stay in B. So as to preserve the above mentioned properties of a case 2 equilibrium, the condition

$$(2-\mu)(1-G(s)) > \alpha \tag{1}$$

has therefore to hold additionally, which will hold only if s is not too large. The adjusted conditions for a case 2 equilibrium are thus

$$\mu < \alpha < 2-\mu \quad \text{and} \quad (2-\mu)(1-G(s)) > \alpha.$$

Some additional notation is useful. Temporarily, let β denote the probability of getting into A when applying to B and denote consequently by $(1-\beta)$ the probability of getting into B when applying to B in $t=1$. Similarly, let δ and $(1-\delta)$ be the probability of getting into A and B in $t=2$, respectively, when applying to B in $t=1$.[17] Also, denote by $E[x \mid x > s]$ the expected utility for school A conditional on the utility for A being greater than the switching cost s, while the unconditional expected utility for school A is still denoted as Eu.

With this notation at hand, the expected utility of applying to B given utility draw x in $t=1$ is

$$U_B(x) = \beta\left[x + \delta Eu - (1-\delta)s\right] + (1-\beta)\delta(1-G(s))\left(Eu[x \mid x > s] - s\right), \tag{2}$$

while the expected utility when applying to A is simply

$$U_A(x) = x - s. \tag{3}$$

Because in an (adjusted) case 2 equilibrium, those who apply to A get into A for sure in $t=1$ and into B for sure in $t=2$, $U_A(x) = x - s$ follows immediately. Equality (2) is also easily understood. With probability β, the individual is assigned to A in $t=1$, in which case he gets utility x in $t=1$. With probability δ, he will remain in A in $t=2$, whence he derives an expected utility of Eu. With probability $(1-\delta)$, though, he has to switch schools in

[17] Of course, these probabilities are endogenous and depend on the behavior of the individuals under the mechanism. They will be expressed in terms of the primitives of the model shortly.

$t = 2$, which costs him s. This explains the first expression on the right-hand side of (2). As to the second term, note first that the individual is assigned to school B for both periods with probability $(1-\beta)(1-\delta)$, in which case he gets zero utility but bears also no switching costs, which is why there are only two terms on the right-hand side. With probability $(1-\beta)\delta$ the individual who was in B in $t = 1$ switches to A in $t = 2$. His expected gross utility of being in school A is $Eu[x \mid x > s]$ since he only applies to school B in $t = 2$ when $x > s$. In this case, though, he bears also the switching cost s, so that his net utility is $Eu[x \mid x > s] - s$. Because this happens only with probability $(1 - G(s))$, the expected net utility has to be multiplied by $(1 - G(s))$.

The equilibrium cutoff x^* is such that $U_A(x^*) = U_B(x^*)$. Solving for x^* yields

$$x^* = \frac{\beta}{1-\beta}\left[\delta Eu - (1-\delta)s\right] + \delta(1-G(s))\left(Eu[x \mid x > s] - s\right) + \frac{1}{1-\beta}s. \quad (4)$$

Under the adjusted assumptions for case 2,

$$\beta = 1 - \frac{2-\alpha}{2-\mu} \quad \text{and} \quad \delta = \frac{\alpha}{(2-\mu)(1-G(s))}.$$

Moreover, $\mu = 2 - 2G(x^*)$ must hold for x^* to be an equilibrium. Replacing these probabilities in (4) reveals that there will be no neat general solution.

7.3 Equilibrium with Uniform and Equal Capacities

So as to simplify further, we assume now that the two schools have equal capacities (i.e. we set $\alpha = 1$), normalize $M = 1$ and let G be uniform on $[0, 1]$.

Given the normalization $M = 1$, the requirement $x^*(s) < M$ implies

$$s < \frac{4-\sqrt{10}}{3} = 0.279, \quad (5)$$

and the cutoff is given as the relevant solution to the quadratic equation (4), which is

$$x^*(s) \equiv \frac{4s + 1 - 2s^2 + \sqrt{32s^2 - 8s - 20s^3 + 1 + 4s^4}}{4(1-s)}. \quad (6)$$

Also, $G(x^*) = x^*$, so that $\mu^*(s) = 2(1 - G(x^*(s))) = 2(1 - x^*(s))$. Hence,

$$\mu^*(s) = \frac{3 + 2s^2 - 8s - \sqrt{32s^2 - 8s - 20s^3 + 1 + 4s^4}}{4(1-s)}. \quad (7)$$

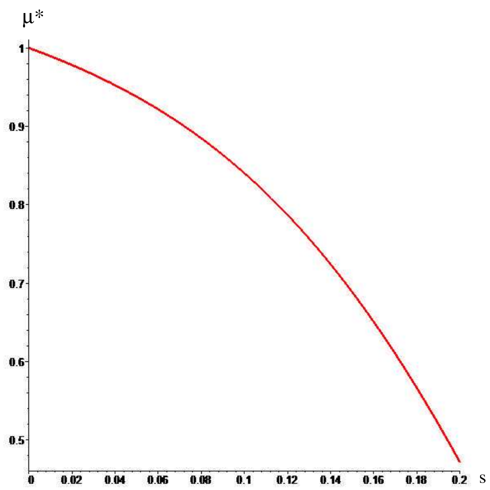

Figure 5: The Number of Applicants to A.

7.4 Welfare

We first look at welfare under a static mechanism. Then we compute welfare under the modified dynamic mechanism and compare them. Lastly, we derive first-best welfare and compare welfare under the static mechanism and under the dynamic mechanism with first-best.

Under the optimal static mechanism, individuals are randomly assigned to a school for two periods. Expected utility of an individual in school A is thus simply $Eu = \frac{M}{2} = \frac{1}{2}$ per period. Over the two periods, students of mass 2 enjoy this utility, so that

$$W^{SM} = 1.$$

Under the dynamic mechanism, overall welfare consists of three components, period one welfare denoted as W_1, period two welfare denoted as W_2 and aggregate switching costs, which we denote by SWC. In general terms, period one and two welfare are given as

$$\begin{aligned} W_1 &= \mu^* E[x \mid x > x^*] + (1 - \mu^*) E[x \mid x \leq x^*] \\ W_2 &= E[x \mid x \geq s], \end{aligned}$$

where the assumption that condition (1) holds, i.e.

$$(2 - \mu^*)(1 - G(s)) \geq 1$$

has been made implicitly. This assumption says that the number of individuals from school B who do not opt out in $t = 2$ exceeds the capacity of school A. Consequently, the period two allocation will assign only individuals from

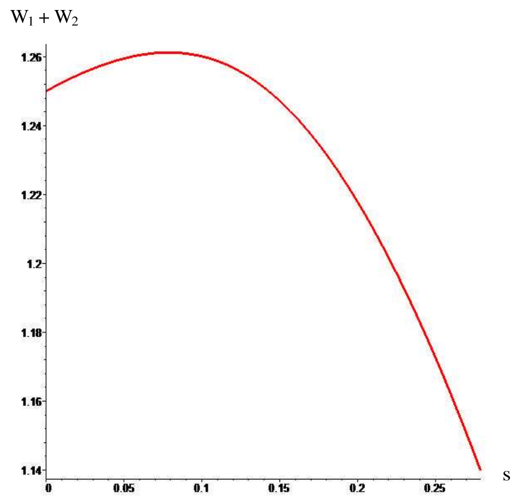

Figure 6: Aggregate Welfare $W_1 + W_2$.

school B to school A. Therefore, under this assumption, in period two students of mass one move from B to A, and all students in A move to B, so that aggregate switching costs are

$$SWC = 2s.$$

Now, $G(.)$ being uniform and $M = 1$, we have $G(s) = s$,

$$E[x \mid x \geq s] = \frac{1+s}{2}, \quad E[x \mid x > x^*] = \frac{1+x^*}{2} \quad \text{and} \quad E[x \mid x \leq x^*] = \frac{x^*}{2},$$

while x^* and μ^* are given by equations (4) and (7) above; Figure 5 depicts μ^*. It is straightforward to check that under these conditions, the assumption $(2 - \mu^*)(1 - G(s)) \geq 1$ does indeed hold.

Computing $W_1 + W_2 - 2s$ yields

$$\frac{14s^2 - 28s + 11 - \sqrt{4s^4 + 32s^2 - 20s^3 + 1 - 8s}}{8(1-s)}.$$

Probably more conclusive is a look at Figures 6 and 7. Figure 6 depicts W_1+W_2. The figure corroborates the conjecture made above that increasing s has also a positive effect in that it makes the period two allocation more efficient if one neglects the actual cost of switching schools. This is illustrated by the increasing part of the curve $W_1 + W_2$ in Figure 6. However, the negative effect of a less efficient selection due to a higher cutoff $x^*(s)$ dominates when s becomes large.

Figure 7 depicts welfare under the modified dynamic mechanism, taking the cost of switching schools into account. The straight line that parallels the horizontal axes is the aggregate welfare under the static mechanism. It is particularly noteworthy that welfare under the dynamic mechanism exceeds welfare under the static mechanism for all s up to approximately 0.127. In other words, for switching costs no larger than approximately one fourth of the expected utility generated by the better school does the dynamic mechanism still outperform the static mechanism.

To complete the analysis, let us compute first-best welfare. In period one, all individuals with $x \geq \frac{1}{2}$ go to school A. Consequently, period one welfare under first-best is

$$W_1^{FB} = \frac{3}{4}.$$

In period two, the first-best allocation is slightly more complicated. Because switching schools is costly, only those students who are in B in $t = 1$ should go to A in $t = 2$ whose utility for A is greater than the utility of those in A whom they replace by $2s$ or more. Otherwise, the cost of switching schools (which is borne by both the student who moves to and the one who moves away from A) would not be offset by the increase in utility. That is, there is a $\underline{x}$ such that all students who have been in A in $t = 1$ and whose second period utility is $x \leq \underline{x}$ move to B and all those students who have been in B in $t = 1$ and whose second period utility exceeds $\underline{x} + 2s$ move to A. Since A's capacity is one and since total mass of students both in A and B is one, the additional constraint for the first-best scheme is

$$1 - G(\underline{x}) + 1 - G(\underline{x} + 2s) = 1 \Leftrightarrow G(\underline{x}) + G(\underline{x} + 2s) = 1.$$

Because $G(.)$ is uniform,

$$\underline{x} = \frac{1}{2} - s$$

is readily established. The total number of students who switch schools is given by $G(\underline{x}) + 1 - G(\underline{x} + 2s)$, which is equal to $1 - 2s$. Thus, aggregate switching costs are

$$SWC^{FB} = s(1 - 2s).$$

Clearly, this requires $s \leq \frac{1}{2}$. For $s > \frac{1}{2}$, the first-best mechanism is static, i.e. it induces no school changes in $t = 2$. Period two welfare is

$$W_2^{FB} = (1 - G(\underline{x}))E[x \mid x \geq \underline{x}] + (1 - G(\underline{x} + 2s))E[x \mid x \geq \underline{x} + 2s].$$

Noting that $(1 - G(\underline{x})) = \frac{1}{2} + s$ and $(1 - G(\underline{x} + 2s)) = \frac{1}{2} - s$ and replacing $E[x \mid x \geq \underline{x}]$ and $E[x \mid x \geq \underline{x} + 2s]$ by $\frac{3}{4} - \frac{s}{2}$ and $\frac{3}{4} + \frac{s}{2}$, one gets after simplifying

$$W_2^{FB} = \frac{3}{4} - s^2.$$

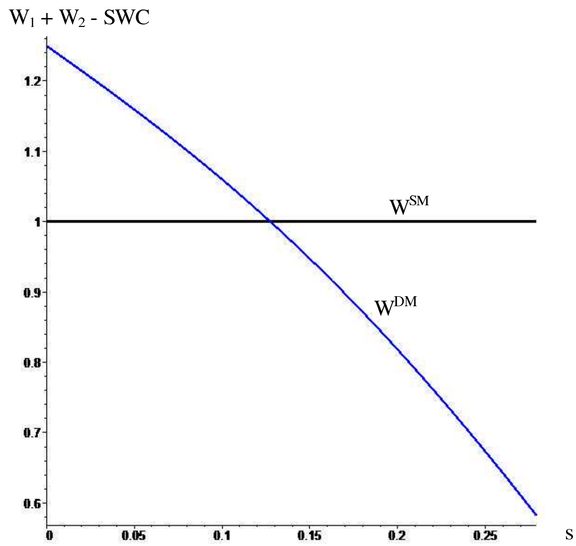

Figure 7: Aggregate Welfare $W_1 + W_2 - SWC$.

Overall, first-best welfare is thus

$$W^{FB} \equiv W_1^{FB} + W_2^{FB} - SWC^{FB} = \frac{3}{2} - s + s^2.$$

Let us compare first-best with welfare under the dynamic and under the static mechanisms. We do this by plotting the welfare under the dynamic and the static mechanism as fraction of the first-best welfare, i.e. by plotting $\frac{W^{DM}}{W^{FB}}$ and $\frac{W^{SM}}{W^{FB}}$ as functions of s in Figure 8. The parallel to the horizontal axes depicts first-best.

For small switching costs, the dynamic mechanism achieves more than eighty percent of the first-best welfare, while the static mechanism achieves less than seventy percent of first-best. Notice also that for $s = 0$ the difference between first-best and the dynamic mechanism is due solely to the second period since in period one, the dynamic mechanism achieves first-best. As seen above, the point of intersection of $\frac{W^{DM}}{W^{FB}}$ and $\frac{W^{SM}}{W^{FB}}$ is at $s = 0.127$, where the static and dynamic mechanism achieve slightly more than seventy percent of first-best.

7.5 Other Equilibria

We now address the question whether there are other equilibria. We show that there is no case 3 and no case 4 equilibrium with equal capacities (i.e. for $\alpha = 1$) and $G(x)$ such that $Eu = m$.

No Case 3 and 4 Equilibrium To see that there is no case 3 and 4 equilibrium when G is such that $Eu = m$, note first that for $\alpha = 1$ the restric-

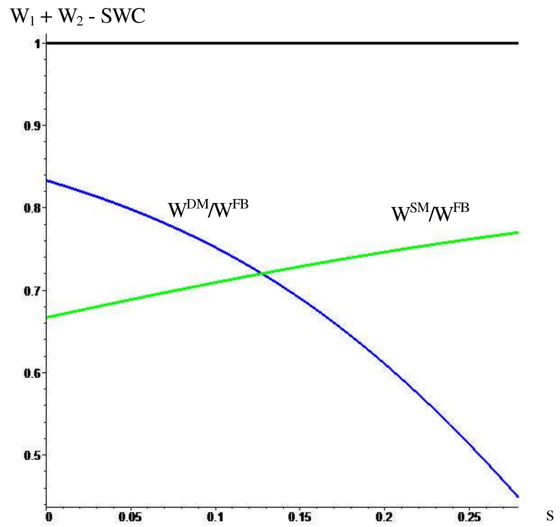

Figure 8: Welfare Comparisons.

tions $\alpha \leq \min\{F(x^*), 2 - F(x^*)\}$ and $\alpha \geq \min\{F(x^*), 2 - F(x^*)\}$ require $F(x^*) = 1 \Leftrightarrow x^* = m = Eu$. Second, the cutoff in a case 3 and 4 equilibrium is given by

$$U_A(x^*) = x^* + G(s)Eu - (1 - G(s))s = (1 - G(s))E[x \mid x > s] - s) = U_B(x^*). \quad (8)$$

This is easily seen when noting that because $\mu = 1$, all applicants to A get into A and all others get into B in period one. In period two, the fraction $G(s)$ of those who were in B in period one have a utility for A that is smaller than the switching cost s. Consequently, they will opt out, so that their overall utility is zero. With probability $(1 - G(s))$ they have utility for A that outweighs the cost of switching, which explains the expected utility of applying to B. Since the fraction $G(s)$ of applicants to B opts out, the probability of staying in A for applicants to A is $G(s)$, in which case their expected utility is Eu. With the probability $(1 - G(s))$, though, they will be assigned to B in period two, in which case they bear the switching cost s. This explains the expression for the expected utility of applying to A.

Solving (8) for x^* yields

$$x^* = (1 - G(s))E[x \mid x > s] - G(s)Eu. \quad (9)$$

So as to complete the proof that there is no case 3 and no case 4 equilibrium, it suffices to show that $x^* < Eu$. To see this, note that $(1 - G(s))E[x \mid x > s] - G(s)Eu < Eu$ implies $(1 - G(s))E[x \mid x > s] < Eu + G(s)Eu$.

But $(1-G(s))E[x \mid x > s] = \int_s^M xdG(x)$, which is certainly less than $Eu + G(s)Eu = \int_0^M xdG(x) + G(s)\int_0^M xdG(x)$ for $s > 0$. Thus, $x^* < Eu$ follows. For the uniform distribution on [0,1], for example, $Eu = \frac{1}{2}$, $G(s) = s$ and $E[x \mid x > s] = \frac{1+s}{2}$. Thus, $x^* = \frac{1-s-s^2}{2}$, which is smaller than $Eu = \frac{1}{2}$ for any $s > 0$.

7.6 Modified Dynamic Mechanism

When looking at the allocation under the standard dynamic mechanism, there is likely to be too much school switching in equilibrium, i.e. students switch schools more often than seems necessary from an incentive point of view: Under the case 2 equilibrium analyzed above, some students who apply to B in $t = 1$ are sent to A in $t = 1$ because of excess demand for B and sent back to B in $t = 2$ because of excess demand for A by students with the same priority. This is likely to be inefficient because the incentives to apply to B would not necessarily be weakened if those rationed at B in $t = 1$ were assigned to A permanently. Such a mechanism would induce less switching in equilibrium and therefore may increase welfare. We are now going to show that such a mechanism exists for a uniform distribution and equal capacities.

New Dynamic Mechanism The new mechanism works as follows. If you apply to B in $t = 1$, you'll have priority over A-applicants in $t = 2$. (This is as before.) However, if you get rationed at the school you apply to in $t = 1$, you'll be in the school you are assigned to after being rationed for both periods.

Equilibrium As in a case 2 equilibrium, assume

$$(A1) \quad \mu < \alpha < 2 - \mu \Leftrightarrow 2 - \mu > 2 - \alpha$$

and define the residual capacity of A as the capacity of A that remains to be allocated after those who are in there permanently have been subtracted. Under assumption (A1), the residual capacity of A is μ since $\alpha - \mu$ seats are occupied permanently by lucky B-applicants who were assigned to A in $t = 1$.

Note that $(2-\alpha)(1-G(s))$ is the number of B-applicants who want to get into A in $t = 2$ because for them, $x > s$ holds. The second assumption is

$$(A2) \quad \mu \leq (2-\alpha)(1-G(s)).$$

This assumption implies that the number of B-applicants who want to get into A in $t = 2$ exceeds the residual capacity of A. Consequently, no one who applied to A in $t = 1$ will be in A in $t = 2$.

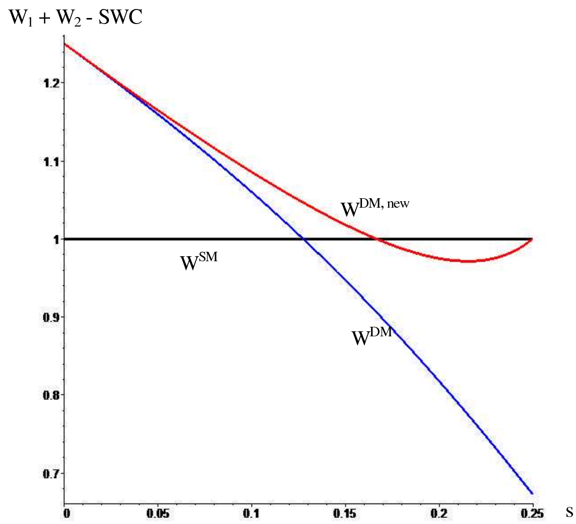

Figure 9: Welfare under the New Dynamic Mechanism.

Under these assumptions, the cutoff equilibrium conditions are as follows. Utility of applying to A is

$$U_A(x) = x - s,$$

while utility of applying to B is

$$\begin{aligned} U_B(x) &= \frac{2-\alpha}{2-\mu}(1-G(s))\frac{\mu}{(2-\alpha)(1-G(s))}\left(E[x \mid x > s] - s\right) \\ &+ \left(1 - \frac{2-\alpha}{2-\mu}\right)[x + Eu], \end{aligned}$$

which simplifies to

$$U_B(x) = \frac{\mu}{2-\mu}\left(E[x \mid x > s] - s\right) + \frac{\alpha-\mu}{2-\mu}[x + Eu].$$

Solving $U_A(x) = U_B(x)$ yields

$$x^* = \frac{\mu(E[x \mid x > s] - 2s) + (\alpha - \mu)Eu + 2s}{2-\alpha}.$$

Notice that for $s = 0$ we have $x^* = \frac{\alpha}{2-\alpha}Eu$, which is the cutoff of a type 2 equilibrium.

For the uniform $G(x) = x$ and $\alpha = 1$, we get

$$x^*(s) = \frac{1-2s}{2(1-3s)}$$

and

$$\mu^*(s) = \frac{1-4s}{1-3s}.$$

Note that $\mu(s) > 0$ requires $s < \frac{1}{4}$. For larger s, there is no interior cutoff, and the mechanism reduces to a static mechanism where all students are randomly and permanently assigned in $t = 1$.[18] The assumption (A1) and (A2) are satisfied because $\mu(s) < 1 - s$ holds for all $s > 0$, which is (A2). Since for $\alpha = 1$ (A1) is contained in (A2), (A1) is also satisfied. Thus, this is an interior cutoff equilibrium for all $s < \frac{1}{4}$.

Welfare First note that aggregate switching costs are

$$SWC = 2\mu(s)s = 2s\frac{1-4s}{1-3s}.$$

As conjectured, the aggregate switching costs under the new dynamic mechanism are smaller than under the standard dynamic mechanism, which are $2s$. Second, welfare W_t in period $t = 1, 2$, neglecting switching costs, is

$$\begin{aligned} W_1 &= \mu E[x \mid x > x^*] + (1-\mu)E[x \mid x < x^*] \\ W_2 &= (1-\mu)Eu + \mu E[x \mid x > s]. \end{aligned}$$

These expressions are readily explained. The number of A-applicants assigned to A in $t = 1$ is $\mu < 1$. The expected utility of each of these applicants is $E[x \mid x > x^*]$, which explains the first term of W_1. The second term is expected utility of students who applied to B. Their expected utility for A is $E[x \mid x < x^*]$ and $1 - \mu$ of them are admitted to A, which gives us the second term. As for W_2, note first that there is no sorting among the $1 - \mu$ students with permanent seats. Thus, their expected utility is just the unconditional expectation Eu, which gives us the first term. The residual capacity of μ is filled with students from B whose utility exceeds s, which explains the second term.

Substituting yields

$$\begin{aligned} W_1 &= \frac{3-10s}{4(1-3s)} \\ W_2 &= \frac{1-2s-4s^2}{2(1-3s)}, \end{aligned}$$

and aggregate welfare $W^{DM,new}$, taking switching costs into account, is

$$W^{DM,new} \equiv W_1 + W_2 - SWC = \frac{5+24s^2-22s}{4(1-3s)}.$$

[18] To see this, note that with $\mu = 0$, every one applies to B. Half of them get rationed and are assigned permanently to A.

Assuming that a static mechanism would assign individuals for two periods, overall welfare under a static mechanism is simply

$$W^{SM} = 1.$$

Solving $W^{DM,new} = W^{SM}$ yields $(1-6s)(1-4s) = 0$, which has two solutions $\frac{1}{6}$ and $\frac{1}{4}$. Thus, in the presence of switching costs the new dynamic mechanism performs better than a static mechanism for all $s < \frac{1}{6}$. For $s \in \left(\frac{1}{6}, \frac{1}{4}\right)$, the static mechanism is better than the new dynamic mechanism, while for $s \geq \frac{1}{4}$ the two mechanism induce the same equilibrium; see Figure 9.

8 Summary

For the various models analyzed in the paper, a number of results were obtained that, coupled with the often subtle differences in the underlying assumptions, may appear confusing to the reader. So as to make clear what results hold under which conditions, we now provide a brief overview over the main results. We broadly separate results into the categories Existence and Welfare.

8.1 Existence

Assume that all individuals agree that school A is better than school B in every period. Proposition 1 guarantees the existence of an interior cutoff equilibrium under fairly general conditions. That is, such an equilibrium exists for all $\alpha < \frac{2M}{M+Eu}$ when utilities are uncorrelated over time, utility for school B is normalized to zero and G has full support and satisfies $G(0) = 0$. Because G has full support, $Eu < M$ follows, implying $\frac{2M}{M+Eu} > 1$. Thus, Proposition 1 asserts in particular existence for all $\alpha \leq 1$. The same proposition states that $Eu = m$ is an alternative sufficient condition for equilibrium existence when $\alpha > \frac{2M}{M+Eu}$. An example that illustrates both the sufficiency and the relevance of these conditions is in Appendix B. Proposition 12 in Appendix A shows that for any $\alpha \in (0, 2)$ an interior cutoff equilibrium exists for any G satisfying $G(0) = 0$ and full support when utility for B is not normalized but drawn independently from $G[0, M]$ while utility for A is drawn from $G[M, 2M]$.

Under only somewhat more restrictive assumptions on α and ρ, the existence result of Proposition 1 extends directly to the case when utilities are positively correlated. This is the content of Proposition 8. For negative correlation, Proposition 10 guarantees existence for any ρ and α, provided G is symmetric. It may be worth mention at this point that symmetry of G guarantees existence for all models (the only exception is the model with switching costs) independent of the degree or sign of correlation. Symmetry implies $Eu = m$. Consequently, case 3 is an equilibrium for $\alpha \leq 1$ and case 4 is an equilibrium otherwise.

We have discussed uniqueness vs. multiplicity issues only for the model of Section 3, where utilities are uncorrelated. The main result (Proposition 4) is that under the dynamic mechanism the unique equilibrium for $\alpha \leq 1$ is a cutoff equilibrium. For $\alpha > 1$, there may be multiple equilibria with an interior cutoff, or all students apply to B in period one.

The existence of a cutoff equilibrium when individuals differ with respect to their ordinal preferences has been addressed in Section 4, where the assumptions were that some individuals have a time invariant preference for school A and some for B and that utility of the disliked school is zero, while utility for the preferred school is drawn i.i.d. from $G[0, M]$. For this model, Lemma 2 shows that there is an appropriately adjusted dynamic mechanism such that all individuals reveal their true ordinal preferences in phase 1 of this mechanism. Assume that there is excess demand for one school after these ordinal preferences have been revealed (otherwise, the problem is trivially solved). Then, Proposition 6 says that in phase 2 of the adjusted mechanism, the game reduces to the one where individuals of mass two have an ordinal preference for the school with excess demand, which was studied in Section 3.

As for the model with switching costs, no general existence results were obtained. Nonetheless, we showed for an example with uniform distribution and equal capacities that a cutoff equilibrium exists when switching costs are not too large.

8.2 Welfare

Assume that all individuals agree that school A is better than school B in every period. Proposition 2, 9 and 11 address welfare under a dynamic and a static mechanism for independent, positively and negatively correlated utility draws, respectively. These propositions assert that in the interim stage of any cutoff equilibrium, every individual has a greater expected utility under a dynamic mechanism than in the equilibrium under a static mechanism. A corollary of these propositions is that ex ante expected utility under the dynamic mechanism is larger for any individual than under a static mechanism, provided, of course, that there is a cutoff equilibrium.

When utility draws are independent over time, Proposition 3 says that whenever the allocation is first-best in period one under the dynamic mechanism, it is second-best overall.[19] This proposition relies heavily on independence of utility draws over time because then achieving first-best in period one has no opportunity cost in period two. This is not the case when utilities are correlated, and consequently, we cannot make an equivalent statement for the case with correlated utilities. For the model with i.i.d. utilities, Proposition 5 provides a lower bound for welfare that can be achieved in a $T > 2$ period model by using dynamic mechanisms whenever a cutoff equilibrium exists in the two period model.

Like with existence we haven't obtained any general results with respect to welfare when there are switching costs. For the example with the uniform distribution and equal capacities, we showed that welfare under the dynamic mechanism exceeds welfare under a static mechanism as long as switching costs are moderate.

9 Conclusions

This paper studies the potential of dynamic mechanism to allocate indivisible goods (e.g. houses or seats in a school) to individuals (e.g. students) when these allocations are made repeatedly and when individuals face uncertainty about the intensity of their future preferences.

For a two period two school model, where all individuals agree that one school is better than the other, an equilibrium under the dynamic mechanism exists under fairly general conditions. Moreover, we show that at the interim stage every individual expects greater utility under the dynamic mechanism than under a static mechanism, no matter what the degree of positive correlation of first and second period utility. For the special case when both schools have the same capacity and when the distribution that generates individuals' instantaneous utilities is symmetric, equilibrium welfare is first-best in period one and second-best over both periods.

In practice, a severe problem in school assignments with static mechanisms is that bad schools are underdemanded since there are no incentives to apply to these schools. This contrasts with assignments under the dynamic mechanism, which provides exactly this type of incentives and thereby induces individuals to apply to schools that are perceived as bad. Therefore, the dynamic mechanism has also the potential of mitigating the problem that some schools have too little demand.

[19] First-best in period one is achieved, for example, when schools have equal capacities ($\alpha = 1$) and when G satisfies $Eu = m$.

At least two questions remain open and require further research. First, though we derive conditions under which a dynamic mechanism establishes the second-best allocation over both periods, little is known about the optimal incentive compatible mechanism in our model when these conditions are not met. This is a question we are currently working on. Second, the model with more than two types of schools remains to be analyzed. An immediate extension of the model with two schools and two periods is the following. Assume that the number of schools N, each with a capacity of one, is equal to the number of periods T and that the total mass of students is equal to N. Students unanimously agree about the ordering of schools, which is $A \succ B \succ ... \succ N$, and the cardinal utility x_k for school k with $k = A, .., N$ is drawn i.i.d. over time and schools, so that, e.g., $x_A \sim G[(N-1)M, NM]$ and $x_N \sim G[0, M]$. This model is balanced in an obvious sense.[20] Consider the simple dynamic mechanism: "Each student can apply to every school exactly once." That is, after a student applied to A in $t = 1$ his choice set in $t = 2$ is $\{N, N-1, ..., B\}$, which is a direct extension of the dynamic mechanism of this paper to a balanced $N = T$ problem. There is an equilibrium under this mechanism which induces no rationing at any stage in any school and in which welfare is first-best in $t = 1$, better than under a random allocation in any $t < T$ and equal to welfare under a random allocation in $t = T$. In this sense, the results of the present paper carry over to any balanced problem. Therefore, balanced models are a natural starting point for the analysis of models with an arbitrary number of schools and periods, possibly varying capacities and heterogenous preferences. This analysis remains to be done.

[20] Appendix A contains this model with $N = 2$, except that it allows for varying capacities.

Appendix

A The Model without the Normalization

Throughout, we have maintained the assumption that utility for the worse school is zero for all individuals and periods. We now relax this normalization by making the following assumptions.

Total mass of students is two, and there are two periods and two schools A and B. Capacity of A is $\alpha \in (0,2)$ and capacity of B is $(2-\alpha)$. Instantaneous utilities are i.i.d. draws from the distribution $G(.)$ with support $[0,M]$ for B and $[M,2M]$ for A.[21] As above, we denote by μ the mass of students who apply to A in $t=1$, and we denote now by $\Delta EU \equiv EU_A - EU_B > 0$ the difference between the two expected utilities. Note that $\Delta EU = M$.

A.1 Equilibrium

The four cases:

As in the case with the normalization, as a function of α and μ four cases are to be distinguished.

Case 1: $2-\mu < \alpha < \mu$

Note that this implies $2-\alpha < \mu$ and $2-\mu < 2-\alpha$. Then:

$$\begin{aligned} U_A(x) &= \frac{\alpha}{\mu}x_A + \left(1-\frac{\alpha}{\mu}\right)x_B + \frac{\alpha-(2-\mu)}{\mu}EU_A + \left(1-\frac{\alpha-(2-\mu)}{\mu}\right)EU_B \\ &= x_B + EU_A = U_B(x). \end{aligned}$$

Re-arranging and simplifying yields

$$x_A^1 = x_B + \frac{2-\alpha}{\alpha}\left[EU_A - EU_B\right] = x_B + \frac{2-\alpha}{\alpha}\Delta EU.$$

Notice that the sole difference to case 1 with the normalization is that x_B appears on the right-hand side and that EU_A is replaced by ΔEU.

Case 2: $\mu < \alpha < 2-\mu$

Note that this implies $2-\mu > 2-\alpha$. Then:

$$\begin{aligned} U_A(x) &= x_A + EU_B \\ &= \frac{2-\alpha}{2-\mu}x_B + \left(1-\frac{2-\alpha}{2-\mu}\right)x_A + \frac{\alpha}{2-\mu}EU_A + \left(1-\frac{\alpha}{2-\mu}\right)EU_B = U_B(x). \end{aligned}$$

Re-arranging and simplifying yields

$$x_A^2 = x_B + \frac{\alpha}{2-\alpha}\Delta EU.$$

[21] Denote by G_A the distribution of x_A and by G_B the distribution of x_B. Then, our assumption is that $G_A(M+x) = G_B(x)$ for all $x \in [0,M]$.

Case 3: $\alpha < \min\{\mu, 2-\mu\}$

Note that this implies $2-\mu < 2-\alpha$ and $\mu < 2-\alpha$. Then:

$$\begin{aligned} U_A(x) &= \frac{\alpha}{\mu}x_A + \left(1-\frac{\alpha}{\mu}\right)x_B + EU_B \\ &= x_B + \frac{\alpha}{2-\mu}EU_A + \left(1-\frac{\alpha}{2-\mu}\right)EU_B = U_B(x). \end{aligned}$$

Re-arranging and simplifying yields

$$x_A^3 = x_B + \frac{\mu}{2-\mu}\Delta EU.$$

Case 4: $\alpha > \max\{\mu, 2-\mu\}$

Note that this implies $2-\alpha < \mu$ and $2-\alpha < 2-\mu$. Then:

$$\begin{aligned} U_A(x) &= x_A + \frac{2-\alpha}{\mu}EU_B + \left(1-\frac{2-\alpha}{\mu}\right)EU_A \\ &= \frac{2-\alpha}{2-\mu}x_B + \left(1-\frac{2-\alpha}{2-\mu}\right)x_A + EU_A = U_B(x). \end{aligned}$$

Re-arranging and simplifying yields

$$x_A^4 = x_B + \frac{2-\mu}{\mu}\Delta EU.$$

Denote by $x(\phi)$ the set of all pairs (x_A, x_B) such that $x_A \geq \phi + x_B$ for $\phi \in [0, 2M]$. Formally,

$$x(\phi) = \{(x_A, x_B) \mid x_A \geq \phi + x_B, \forall x_B \in [0, M], \forall x_A \in [M, 2M]\}.$$

There being a total mass of students equal to two,

$$F(x(\phi)) = 2\int_0^M \int_{\max\{\phi+x_B, M\}}^{2M} dG(x_A)dG(x_B)$$

is the number of students with utility draws below (and to the the right of) the line with slope 1 and the intercept ϕ on the horizontal axis. Because $G(.)$ is continuous in x_B and x_A, $F(x(\phi))$ is also continuous for $\phi \in [0, 2M]$. In particular, $F(x(0)) = 0$ and $F(x(2M)) = 2$. Moreover, since utility draws are i.i.d., $F(x(M)) = 1$. Figure 10 illustrates the set $x(\phi)$ for $\phi = 0$ and $\phi = M$.

Replace μ by $2 - F(x_A^i)$, where $F(x_A^i)$ is the mass of students with utilities below the line given by $x_A^i, i = 1, .., 4$ and let

$$\begin{aligned} \phi_1 &\equiv \frac{2-\alpha}{\alpha}\Delta EU \\ \phi_2 &\equiv \frac{\alpha}{2-\alpha}\Delta EU \\ \phi_3 &\equiv \frac{2-F(x(\phi_3))}{F(x(\phi_3))}\Delta EU \\ \phi_4 &\equiv \frac{F(x(\phi_4))}{2-F(x(\phi_4))}\Delta EU. \end{aligned}$$

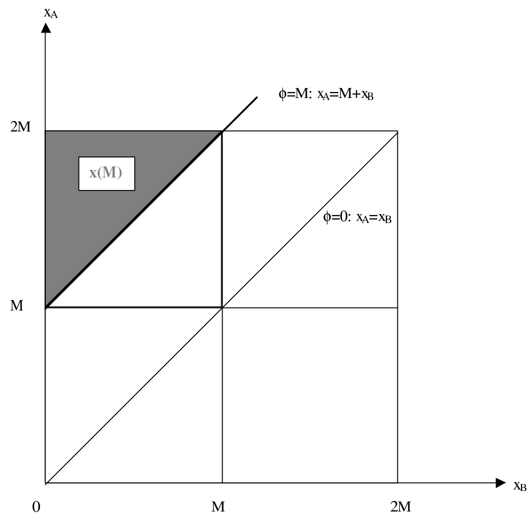

Figure 10: The Model without the Normalization $x_B = 0$.

Summarizing, we then have

$$\begin{aligned} x_A^1 &= x_B + \frac{2-\alpha}{\alpha}\Delta EU && F(x(\phi_1)) \leq \alpha \leq 2 - F(x(\phi_1) \\ x_A^2 &= x_B + \frac{\alpha}{2-\alpha}\Delta EU && 2 - F(x(\phi_2)) \leq \alpha \leq F(x(\phi_2)) \\ x_A^3 &= x_B + \frac{2-F(x(\phi_3))}{F(x(\phi_3))}\Delta EU && \alpha \leq \min\{F(x(\phi_3)), 2 - F(x(\phi_3))\} \\ x_A^4 &= x_B + \frac{F(x(\phi_4))}{2-F(x(\phi_4))}\Delta EU && \alpha \geq \max\{F(x(\phi_4)), 2 - F(x(\phi_4))\}. \end{aligned}$$

Proposition 12 *For any $\alpha \in (0,2)$ and any continuous $G(.)$, there is an interior cutoff equilibrium.*

Proof: Set $\phi = M$. Then, $F(x(\phi)) = F(x(M)) = 1$ and $\frac{2-F(x(M))}{F(x(M))}\Delta EU = M$ and $\frac{F(x(M))}{2-F(x(M))}\Delta EU = M$. Thus, for $\alpha \leq 1$, case 3 is an equilibrium, and for $\alpha > 1$, case 4 is an equilibrium. ■

Remark Notice the difference to Proposition 1, which is valid for the model with the normalization. Without the normalization, no additional restrictions on α and G have to be made. This suggests that if the normalization does anything it works *against* our mechanism.

A.2 Welfare with Equal Capacities

Let us now derive the first-best allocation, and show that this allocation is the same in period one as the allocation under the dynamic mechanism when schools have equal capacities. Thus, the result of the paper, according to which first-best in period one is achieved exactly under these conditions, does not appear to depend on the normalization we made.

First-best Consider an individual with utility draw $x = (x_A, x_B)$. If this individual is assigned to school A, it adds x_A to total welfare. If assigned to B, it adds x_B, so that it adds net welfare of $x_A - x_B$ when assigned to school A. Maximizing aggregate welfare therefore requires assigning all those individuals to A who add the largest net welfare when going to school A, subject to the constraint that their mass does not exceed one, which is school A's capacity. For an illustration, consider Figure 10. Under first-best, all individuals with utility draws above the line with $\phi = M$ are assigned to school A in both periods.

This is exactly what is achieved in the equilibrium under the dynamic mechanism in period one. All the individuals whose net welfare contributions are no less than M go to school A, the other ones to school B. Thus, as in the case with the normalization, first-best is achieved in $t = 1$, and second-best overall.

B Example of Equilibrium Nonexistence

This part of the Appendix provides an example that illustrates that the conditions of Proposition 1 are sufficient and yet have grip.

Consider the piecewise uniform distribution

$$G(x) = \begin{cases} \frac{\theta}{2\theta-1}x & \text{if} \quad 0 \leq x \leq m \\ 1-\theta+\theta x & \text{if} \quad m < x \leq 1 \end{cases},$$

with mean $Eu = \frac{3\theta-1}{4\theta}$, median $m = \frac{2\theta-1}{2\theta}$ and the shape parameter $\theta > \frac{1}{2}$.[22] For $\theta = 1$, we thus have $Eu = m = \frac{1}{2}$, which corresponds to the usual uniform distribution. As θ approaches $\frac{1}{2}$, the median approaches zero and the mean $\frac{1}{4}$. The difference between the expected value and the median is $Eu - m = \frac{1-\theta}{4\theta}$.

First, we demonstrate that an equilibrium may exist even when both sufficient restrictions are violated. Let $\theta = \frac{4}{5}$ and $\alpha = \frac{8}{5}$. Then, $x_2 = \frac{7}{4} > 1 \equiv M$. Thus, there is no case 2 equilibrium in this case. On the other hand, since $\theta \neq 1$, $Eu \neq m$, so the sufficient condition for the case 4 equilibrium is violated as well. Nonetheless, there is a case 1 equilibrium with $x_1 = \frac{7}{64}$. To see this, note that $x_1 = \frac{7}{64} < \frac{3}{16} = m$. Thus, $F(x_1) \equiv \frac{2\theta}{2\theta-1}x_1 = \frac{7}{24} < \frac{8}{5} = \alpha < \frac{17}{24} = 2 - F(x_1)$. Second, to see that the restrictions, though sufficient, have grip, assume now $\theta = \frac{3}{5}$ and keep $\alpha = \frac{8}{5}$. Then, $x_2 = \frac{4}{3} > 1$ and $x_1 = \frac{1}{12} < \frac{1}{6} = m$. Thus,

[22] It is easy to check that $G(0) = 0$ and $G(m) = \frac{1}{2}$ holds.

$F(x_1) = \frac{1}{2} < \frac{8}{5} = \alpha$. However, $2 - F(x_1) = \frac{3}{2} < \frac{8}{5} = \alpha$, so there is no case 1 equilibrium either. Though $Eu \neq m$, there might still be a case 4 equilibrium. However, no such equilibrium exists. For $\theta < 0.86$, there is no solution to $x_4 = \frac{F(x_4)}{2-F(x_4)} Eu$ such that $x_4 > m$. For $x_4 < m$, the solution is easily seen to be $x_4 = \frac{5\theta - 3}{3\theta}$. But for $\theta = \frac{3}{5}$, $x_4 = 0$. Therefore, $2 - F(x_4) = 2 > \frac{8}{5} = \alpha$, violating the restrictions for a case 4 equilibrium. Since case 3 cannot be an equilibrium when $\alpha > 1$, it follows that there is no cutoff equilibrium.

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