The Combinatorial Assignment Problem: Approximate Competitive Equilibrium from Equal Incomes

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Job Market Paper

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Abstract

The combinatorial assignment problem has three principal features: (i) agents require bundles of indivisible objects; (ii) monetary transfers are prohibited; and (iii) the market administrator cares about both efficiency and fairness. An example of this problem is the assignment of course schedules to students. Impossibility theorems have established that the only efficient and strategyproof mechanisms in this environment are dictatorships. Any non-dictatorship solution will involve compromise of efficiency or strategyproofness.

This paper proposes a solution to the combinatorial assignment problem. Since we lack attainable criteria of fairness for this environment, I begin by formalizing two such criteria. The *maximin share guarantee*, based on the idea of divide-and-choose, generalizes and weakens fair share. *Envy bounded by a single good* weakens envy freeness. Both criteria recognize that indivisibilities complicate fair division, but exploit the fact that bundles of indivisible objects are somewhat divisible. Dictatorships fail both criteria.

Second, I propose a specific mechanism, the Approximate Competitive Equilibrium from Equal Incomes Mechanism, which satisfies the fairness criteria while maintaining attractive compromises of efficiency and strategyproofness. An exact CEEI may not exist due to indivisibilities and complementarities. I prove existence of an approximate CEEI in which: (i) incomes (in artificial currency) must be unequal but can be arbitrarily close together; (ii) the market clears with some error, which approaches zero in the limit and is small for realistic problems. I then show that this approximation satisfies the fairness criteria, so long as income inequality is set sufficiently low. Also, the mechanism based on this approximation satisfies an intuitive relaxation of strategyproofness, *strategyproof in a large market*. The theoretical case for the proposed mechanism is complemented with empirical analysis, using data on course allocation at Harvard.

1 Introduction

How can a set of indivisible objects be allocated efficiently and fairly amongst agents who require bundles of the objects? An instance of this problem occurs at many educational institutions. The objects are seats in courses, which may be scarce due to limits on class size. The agents are students, each of whom requires a schedule (bundle) of courses.[1] The allocation problem is complicated by an exogenous restriction against using monetary transfers to balance supply and demand.[2] A closely related problem, within firms, is the assignment of fixed-wage shifts (or tasks) to interchangeable workers.[3]

This combinatorial assignment problem[4] (or course-allocation problem) is one feature removed from several canonical market design problems that have received considerable attention and have compelling solutions. It is like a combinatorial auction problem, except for the restriction against using real money.[5] It is like a school assignment problem, except that agents demand bundles of goods, rather than single goods.[6] It is like a matching problem except that preferences are one-sided: course seats do not have preferences over the students.[7] And it is like a classical fair-division (or cake-cutting) problem, except that the goods are indivisible.[8]

Yet, progress on this problem has been elusive. There are two related reasons. First, we lack realistic criteria of fairness for this environment. Second, even if we can articulate criteria of fairness, there is likely to be a tension between these criteria and efficiency. A series of papers has shown that the only mechanisms for this problem that are ex-post Pareto efficient and strategyproof are

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[1] As an illustration of the problem and the market-designer's objectives, consider the opening sentences of the document that describes Wharton's course-allocation procedure: "Beginning with the 1996-97 academic year, we introduced an auction-based process for registering in Wharton's MBA electives. The auction is designed to achieve an *equitable* and *efficient* allocation of seats in elective courses when demand exceeds supply." (Wharton, 2007. Emphasis in original.)

News coverage of the course-allocation problem can be found in Bartlett (2008), Guernsey (1999), and Neil (2008).

[2] Levitt (2008a, b) describes instances where university administrators sanctioned students who used real money in trades of course seats. Roth (2007) and Sandel (1998) describe numerous other markets that have exogenous constraints against the use of money.

[3] The health care services company McKesson (2008) offers an "equitable open shift management" software product, eShift, that is used by many hospitals to assign nurses to vacant shifts based on their preferences. Interestingly, the software has both a fixed-price version and an auction version, depending on whether the client hospital has discretion to use flexible wages (e.g., due to union restrictions).

[4] The term "assignment" is used in allocation problems with and without monetary transfers. A more specific name of the problem I study would be "combinatorial assignment problem without monetary transfers."

[5] This literature traces to Vickrey (1961). See Milgrom (2004) and Cramton et al (2006) for textbook treatments that discuss both theory and applications.

[6] See Shapley and Scarf (1974), Roth (1982), Abdulkadiroglu and Sonmez (1998), Sonmez (1999), Pathak (2007), and Abdulkadiroglu et al (2006). This problem is often called the "house allocation problem."

[7] The seminal reference is Gale and Shapley (1962). See Roth and Sotomayor (1990) for a textbook treatment and Roth (1984) and Roth and Peranson (1999) on the most well-known instance of such a market.

[8] The classical references are Steinhaus (1948) and Dubins and Spanier (1961). See Moulin (1995), Brams and Taylor (1996), and Robertson and Webb (1998) for textbook treatments.

dictatorships.[9] But dictatorships result in very unequal outcomes: some students get to choose *all* their courses before others get to choose *any.* Any non-dictatorship solution will involve compromise of efficiency or strategyproofness.

This paper proposes a solution to the combinatorial assignment problem. First, I propose two realistic criteria of outcome fairness, the *maximin share guarantee* and *envy bounded by a single good*. An agent's maximin share is defined as the most preferred bundle he could guarantee himself as divider in divide-and-choose against adversaries. Maximin share generalizes and weakens fair share (Steinhaus, 1948) to accomodate environments with indivisibilities and complementarities. An allocation satisfies the maximin share guarantee if all agents obtain a bundle they weakly prefer to their maximin share. An allocation satisfies envy bounded by a single good if, whenever agent i envies agent j's bundle (Foley, 1967), by removing some single object from j's bundle we can eliminate the envy. Both criteria explicitly recognize that indivisibilities complicate fair division, but exploit the fact that *bundles* of indivisible objects are somewhat divisible.

To illustrate, suppose there are two agents and four indivisible objects: two diamonds (big and small) and two rocks (pretty and ugly).[10] The criteria are satisfied by allocations in which one agent gets the bundle {big diamond, ugly rock} while the other gets {small diamond, pretty rock}. (The procedural fairness requirement of symmetry ensures that each agent has an equal chance at the bundle with the big diamond.) The agent who gets the small diamond may envy the other, but he does as well as he could have as divider in divide-and-choose, and his envy is bounded by a single object (the big diamond). Dictatorships fail both criteria, because whichever agent chooses first obtains {big diamond, small diamond}, leaving the other agent with just the rocks. The criteria thus formalize what is so unfair about dictatorships in multi-unit assignment. Note however that dictatorships actually satisfy the proposed criteria for *single*-unit assignment problems, for which they are often used in practice, e.g., school choice.

Second, I propose a specific mechanism, the Approximate Competitive Equilibrium from Equal Incomes Mechanism, which satisfies the fairness criteria while maintaining attractive compromises of efficiency and strategyproofness. The mechanism is based on an old idea, Competitive Equilibrium from Equal Incomes ("CEEI"; Foley 1967; Varian, 1974). CEEI uses *artificial money* – i.e., money with no value outside the problem at hand – and so does not violate the constraint against monetary transfers. In divisible-goods environments, CEEI satisfies fair share and is envy free. Unfortunately,

[9] Variants of this result are obtained in Sonmez (1999); Papai (2001); Ehlers and Klaus, (2003); and Hatfield, (2005). For discussion of these and related results see Budish and Cantillon (2008) and Section 2.1.

Whereas ex-post efficiency is incompatible with fairness, ex-ante efficiency may be altogether impossible to achieve. In an environment that is a special case of ours, Zhou (1990) shows that there is no mechanism that is ex-ante efficient and strategyproof. See further discussion on ex-ante efficiency at the end of the introduction.

[10] This example is presented formally as Example 1. Agents require at most two objects each, and have the preferences we would expect given the names of the objects.

CEEI may not exist in our environment, due to indivisibilities and complementarities.

My main theoretical result (Theorem 1) recovers existence by approximating CEEI in two ways. First, incomes must be unequal, but can be arbitrarily close together. The market designer can assign these close but unequal budgets to the agents however she likes; assigning them randomly ensures the procedural fairness requirement of symmetry. Second, I allow for a small amount of market clearing error. Specifically, I show existence of a price vector such that the Euclidean distance of the corresponding excess demand vector is smaller than $\frac{\sqrt{2kM}}{2}$, where k is the number of courses per student, and M is the number of courses. This bound is tight, and it is small in two respects. First, it does not grow with the number of students or the number of seats per class. So, in the limit, market-clearing error as a fraction of the endowment goes to zero.[11] Second, it is actually a small number for practical-sized problems. For instance, in a single semester at Harvard Business School $k = 5$ and $M = 50$, and so $\frac{\sqrt{2kM}}{2} \approx 11$. This corresponds to a maximum market-clearing error of 11 seats in one class, or of 3 seats in each of 13 classes (since $\sqrt{13 \cdot 3^2} \approx 11$), etc., as compared with about 4500 course seats allocated each semester. Such error can easily be accommodated in practice by adding or removing a few chairs to or from a few classrooms.

The payoff to the existence result is that we can exhibit an approximate CEEI that satisfies the proposed criteria of outcome fairness. The key is to set the degree of budget inequality sufficiently low, as measured by the ratio $(1 + \beta)$ of the largest budget to the smallest budget. Theorem 2 shows that for small enough β the approximation to CEEI approximately satisfies the maximin share guarantee. At worst, each student receives her maximin share in a hypothetical economy in which there is one additional student. Often the approximation is even tighter. For instance, in the two diamonds - two rocks example, we can guarantee that each student receives a diamond, but the approximation error is that the student who gets the small diamond might also get the ugly rock. Theorem 3 shows that for small enough β the approximation to CEEI satisfies envy bounded by a single good.

A mechanism based on the Approximate CEEI is not strategyproof. But it satisfies an intuitive relaxation of strategyproofness, which I call *strategyproof in a large market*. A mechanism satisfies this notion of approximate implementation if in a limit market where agents are zero measure, agents can do no better than to report their preferences truthfully. This is a mild criterion, which essentially asks that a mechanism act in an agent's best interest given her preference report and realized "opportunity set". Yet, none of the course-allocation mechanisms currently found in practice satisfy this criterion. For instance, in the HBS Draft Mechanism studied by Budish and Cantillon (2008),

[11] This notion of approximate competitive equilibrium was emphasized in an old literature on general equilibrium with non-convexities (Starr 1969, Dierker 1971). These papers study exchange economies, in which the notion of equal incomes may not be well defined given indivisibilities.

a student who reports her preferences truthfully risks wasting early-round draft choices on courses that do not reach capacity until a much later round. By contrast, many non-strategyproof market designs that have found widespread use in practice satisfy this incentives criterion. Examples include the Double Auction, the Assignment Exchange, the Uniform Price Auction, Probabilistic Serial, and the Deferred Acceptance Algorithm.[12]

Putting this all together, my solution to the combinatorial assignment problem is the following procedure: (1) agents report their preferences over permissible schedules; (2) agents are randomly assigned budgets in the interval $[1, 1+\beta]$ for suitably small β; and (3) a computer finds and implements an approximate CEEI, taking care to break ties in a way that preserves incentives in a limit market. The mechanism is approximately ex-post efficient, in the sense that it is efficient with respect to the allocated goods and allocation error is small. In exchange for compromising efficiency and using a milder notion of incentive compatibility than strategyproofness, we obtain attractive fairness properies: it is symmetric, guarantees approximate maximin shares, and bounds envy by a single good.

The proposed mechanism has an element of randomness and so it might be argued that efficiency and fairness should be assessed ex-ante rather than ex-post. With fairness, ex-post is actually the more stringent perspective: for instance, a random serial dictatorship is ex-ante envy free even though it results in very unequal outcomes ex-post. Additionally, ex-post is likely to be the perspective that matters to real-life market administrators. A procedure that is fair at some interim stage but not ex-post might have difficulty being adopted for practice. With efficiency, ex-ante is the more stringent perspective. In assignment contexts without transferable utility, a necessary but not sufficient condition for a lottery over allocations to be ex-ante efficient is that all realizations of the lottery are ex-post efficient.

I leave theoretical analysis of the tradeoff between ex-post fairness and ex-ante efficiency to future work.[13] In this paper I provide empirical evidence that the mechanism has attractive ex-ante efficiency properties in a specific course-allocation environment, that at Harvard Business School.[14] The key feature of the HBS data is that it contains not only the preferences students reported under the HBS draft mechanism, but also their underlying truthful preferences, from

[12] See Cripps and Swinkels (2006) on Double Auctions; Milgrom (2008) on Assignment Exchanges; Milgrom (2004) and Cramton et al (2006) on Uniform Price Auctions; Bogomolnaia and Moulin (2001) and Kojima and Manea (2008) on Probabilistic Serial; and Roth and Peranson (1999), Immorlica and Mahdian (2005), and Kojima and Pathak (2008) on Deferred Acceptance Algorithms. Roberts and Postlewaite (1976) study the large-market incentive properties of Walrasian equilibrium itself.

[13] Budish, Che, Kojima and Milgrom (2008) study course-allocation procedures that are ex-ante efficient when students' preferences for courses can be described by assignment messages (Milgrom, 2008). Their proposed mechanism does not satisfy the fairness criteria developed in this paper.

[14] The existence theorem is non-constructive. Othman, Budish and Sandholm (2008) develop a computational procedure that finds approximate CEEI prices in life-size problems. See Section 8.1.

surveys.[15] It is these latter preferences I use to analyze the Approximate CEEI Mechanism. First, I show that market-clearing error is substantially smaller than the Theorem 1 bound; this suggests that ex-post inefficiency is small. Second, I show that students' outcomes are not too sensitive to the randomly drawn budget ordering, unlike in random serial dictatorship. This suggests ex-post efficiency is a reasonable proxy for ex-ante efficiency.[16] Finally, I compare the distribution of realized utilities under the proposed mechanism to that from HBS's own procedure. The Approximate CEEI Mechanism first-order stochastically dominates the actual strategic play of the HBS procedure. Interestingly, it performs a bit worse than a non-equilibrium counterfactual in which students play the HBS procedure truthfully. The difference arises because the Approximate CEEI Mechanism respects incentive constraints that the non-equilibrium counterfactual ignores.

Aren't CEEI-like Mechanisms Already Used In Practice? Some readers may worry that the procedure proposed in this paper is not new. Many prominent universities – including Berkeley, Chicago, Columbia, Michigan, Northwestern, Penn, Princeton and Yale – use equal income artificial-currency procedures to allocate courses.[17]

A clue that something is funny in these mechanisms is that they all use exactly equal incomes, even though we know that a CEEI need not exist. Each of these mechanisms is making a variant of the following conceptual error: *they treat fake money as real money that directly enters the utility function.* (Currency is fake if it has no value outside the allocation problem at hand). This causes the mechanism to allocate incorrect bundles – i.e., not the bundle the student actually demands at the realized prices – and creates incentives to misreport.

With fake money, student i's correct demand, given a budget of b_i and a price vector of $\mathbf{p}$, is

$$x = \arg\max_{x'}(u_i(x') : \mathbf{p} \cdot x' \leq b_i) \tag{1}$$

where u_i is some arbitrary cardinal representation of student i's preferences over schedules. Prices that clear the market according to (1) may not even exist, and even if they did a sophisticated

[15] Budish and Cantillon (2008) study this data and find that students indeed strategically misreport their preferences to game the HBS draft mechanism, and that this misreporting harms welfare in the aggregate: the distribution of utilities under the actual strategic play is (second-order stochastically) dominated by that under a non-equilibrium counterfactual in which students play truthfully.

[16] This is not always the case. Budish and Cantillon (2008) show that random serial dictatorship, which is ex-post efficient, has very poor ex-ante efficiency performance due to a phenomenon they call "callousness".

[17] Sonmez and Unver (2008) describe the Bidding Points Mechanism used at the University of Michigan Business School, and close variants used at the Haas School of Business at UC Berkeley, Columbia Business School, Kellogg Graduate School of Management at Northwestern, Princeton University, and Yale School of Management. Graves et al (1993) describe the Primal-Dual Linear Programming procedure used at the University of Chicago Graduate School of Business. Information on the Bidding Points Double Auction procedure used at the Wharton School of the University of Pennsylvania can be found in Bartlett (2008), Guernsey (1999), Wharton (2007), and http://technology.wharton.upenn.edu/auction/.

fixed-point algorithm would be required to find them. My proposed mechanism finds prices that approximately clear the market according to (1) when students have approximately equal incomes. An algorithm for finding these prices is described in Othman, Budish and Sandholm (2008) and section 8.1.

What each of the schools listed above do is the following. First, they require each student i to report his preferences in a reporting language that ensures that $u_i(x) \leq b_i$, for all possible schedules x. Then they treat his preferences as quasi-linear over courses and fake money! That is, they allocate student i the incorrect bundle

$$x = \arg\max_{x'}(u_i(x') - \mathbf{p} \cdot x') \tag{2}$$

Prices that clear the market according to the incorrect demands (2) are easy to compute. For instance, here is how the widely used Bidding Points Mechanism works. Each student submits bids for classes, the sum of their bids not to exceed some fixed budget amount like 1000 points. A course with q seats is allocated to the q students who bid the most for it.[18] The qth highest bid is described as a "clearing price". Implicitly, the BPM interprets bids as reports of an additive-separable utility function, and then "clears the market" at these prices with respect to demands of the form (2).

Here is a simple illustration of the harm this can cause. Suppose there are four courses (A, B, C, D) and Alice bids $u_{Alice} = (700, 200, 50, 50)$. Interpret this bid as a truthful report of her additive-separable preferences. Suppose budgets are 1000 and prices under the BPM turn out to be $\mathbf{p}^* = (900, 250, 100, 75)$. At these prices, Alice's most-preferred affordable bundle is $\{A, C\}$ (i.e., (1)) but under the BPM (i.e., (2)) she gets *none* of the courses she bid on. The BPM prices simply do not clear the market with respect to students' actual preferences, and Alice will regret reporting truthfully. Implicitly, the mechanism expects her to take consolation in a large bank account of unspent fake money.[19]

[18] More precisely, bids for all courses are sorted in descending order, and are either filled or rejected one at a time depending on whether (i) the course still has capacity for the student; and (ii) the student still has capacity for the course. Because of (ii), a student whose bid for course j is amongst the q_j highest might not get it, meaning some bids lower than the q_j^{th} are successful.

Strategic issues aside, (ii) can lead to inefficient allocations. Sonmez and Unver (2008) and Krishna and Unver (2008) propose a mechanism that eliminates the inefficiencies that arise from this specific aspect of the Bidding Points Mechanism.

[19] Some of the institutions named above allow fake money to be carried over from one period (e.g. semester) to the next. If there were infinitely many such periods, then fake money would be like real money, because it always has a future use (see Abdulkadiroglu and Bagwell (2007) for a related problem and see Athey and Miller (2006) on some limitations even if using real money). In the course-allocation context, the number of periods each student participates in is finite. Each of these institutions treats students in their final period as if they have quasi-linear preferences over courses and fake money.

Organization of the Paper The remainder of this paper is organized as follows. Section 2 describes the environment and reviews the relevant impossibility results (dictatorship theorems) from the social choice literature. Section 3 proposes the new criteria of outcome fairness: the maximin share guarantee and envy bounded by a single good. Section 4 proposes the approximate incentives criterion: strategyproof in a large market. Section 5 defines the approximation to Competitive Equilibrium from Equal Incomes (CEEI) and states the main existence theorem (Theorem 1). The proof is sketched in the body and is contained in full in an appendix. Section 6 explores the outcome fairness properties of an approximate CEEI (Theorems 2 and 3). Section 7 proposes the Approximate CEEI Mechanism and compares its properties to those of other mechanisms known in theory and practice. Section 8 empirically studies the ex-ante efficiency performance of the proposed mechanism, using course-allocation data from Harvard Business School. I conclude with open questions and a note on methodology.

2 Environment

The Combinatorial Assignment Problem A combinatorial assignment problem consists of a set of objects, each with integral capacity, and a set of agents, each with scheduling constraints and preferences. I emphasize the course-allocation application, in which the objects are "courses" and the agents are "students". The elements of a problem $(\mathcal{S}, \mathcal{C}, \mathbf{q}, (\Psi_i)_{i=1}^N, (u_i)_{i=1}^N)$, also called an economy, are defined as follows.

Students There is a set of N students, $\mathcal{S} = \{s_1, ..., s_i, ..., s_N\}$.

Courses There is a set of M courses, $\mathcal{C} = \{c_1, ..., c_j, ..., c_M\}$. There are no other goods in the economy other than seats in courses. In particular, there is no divisible numeraire like money.

Capacities Each course has integral capacity. The capacity vector is $\mathbf{q} = (q_1, ..., q_j, ..., q_M)$.

Schedules A schedule consists of 0 or 1 seats in each course. For each student s_i there is a set $\Psi_i \subseteq \{0,1\}^M$ of permissible schedules. The set Ψ_i encodes any universal scheduling constraints (e.g., cannot take two courses that meet at the same time), and also encodes any scheduling constraints specific to student s_i (e.g., prerequisites). Each Ψ_i includes the empty schedule $(0, 0, ..., 0)$. Notationally, I will use x to denote a generic schedule (i.e., $x \in \{0,1\}^M$), $x_i \in \Psi_i$ to denote a generic schedule in student s_i's permissible set, and $x_{ij} \in \{0,1\}$ as an indicator of whether schedule x_i contains course c_j. Despite the fact that each individual schedule $x \in \{0,1\}^M$ is a vector I do not use boldface, reserving boldface $\mathbf{x}$ for an $N-$agent allocation, defined below.

Let k denote the maximum number of courses any student is allowed to take, i.e., $k \equiv \max_i \max_{x_i \in \Psi_i} \sum_j x_{ij}$. The constants k and M will play a role in the approximation bound of Theorem 1.

Preferences Each student s_i is endowed with a von-Neumann Morgenstern utility function that indicates her utility from each of her permissible schedules: $u_i : \Psi_i \to \mathbb{R}_+$. This is a private values or no peer effects assumption: each student cares only about her own allocation, and perfectly knows her own preferences. Preferences are private information.

I assume agent s_i's utility is zero both for the empty schedule and for any schedule not in Ψ_i. Otherwise, preferences over bundles are strict: for $x_i \neq x'_i \in \Psi_i$, $u_i(x_i) \neq u_i(x'_i)$. Indifferences can be accomodated but at some notational burden without much additional insight.

No further restrictions are placed on the utility function: in particular, students are free to regard courses as complements and substitutes. This is the reason the assignment problem is called "combinatorial" as opposed to "multi-unit".

Feasible Allocations An allocation assigns a schedule to each student. An allocation $\mathbf{x} = (x_i)_{i=1}^N$ is feasible in economy $(\mathcal{S}, \mathcal{C}, \mathbf{q}, (\Psi_i)_{i=1}^N, (u_i)_{i=1}^N)$ if: (i) $x_i \in \Psi_i$ for each student s_i; (ii) $\sum_{i=1}^N x_{ij} \leq q_j$ for each course c_j.

Mechanisms A (direct) combinatorial assignment mechanism (or course-allocation mechanism) is a systematic procedure, possibly with a random element, that selects a feasible allocation for each problem. Formally, it is a function χ which associates a probability distribution over feasible allocations with each problem $(\mathcal{S}, \mathcal{C}, \mathbf{q}, (\Psi_i)_{i=1}^N, (u_i)_{i=1}^N)$.

Students' preferences are private information, and so they might misreport their preferences. Student s_i can report any utility function $\widehat{u}_i : \Psi_i \to \mathbb{R}_+$, the set of which we call U_i. We are often interested in how students' outcomes under a mechanism vary with the profile of reports, holding the other elements of the problem fixed. For fixed $\mathcal{S}, \mathcal{C}, \mathbf{q}$, and $(\Psi_i)_{i=1}^N$, we will use $\chi_i(\widehat{u}_i, \widehat{\mathbf{u}}_{-i})$ to denote s_i's distribution over schedules under χ in economy $(\mathcal{S}, \mathcal{C}, \mathbf{q}, (\Psi_i)_{i=1}^N, (\widehat{u}_i)_{i=1}^N)$. Her expected utility from this lottery is written as $\mathbb{E}u_i[\chi_i(\widehat{u}_i, \widehat{\mathbf{u}}_{-i})]$.

The revelation principle indicates that restricting attention to direct mechanisms, i.e., in which students directly report a vNM utility function, is without loss of generality. The Approximate CEEI Mechanism that will be developed in this paper does not actually utilize all of this preference information. Instead, it uses only students' *ordinal* preferences over bundles. In practice it may actually be useful to have students report cardinal rather than ordinal preferences, because there exist good cardinal reporting languages.[20]

[20] Milgrom (2008) proposes a bidding language in which agents report their cardinal preferences as a linear program.

Most other course-allocation mechanisms found in practice can also be thought of as direct revelation mechanisms that discard some utility information. The HBS Draft Mechanism and the widely used Bidding Points Mechanism use only preferences over singletons. The Chicago Primal-Dual Mechanism uses only preferences over a limited number of schedules per student.

Generality While I emphasize the course-allocation application, the environment is easily seen to be quite general.

For instance, to obtain the problem of "fair division of indivisible objects" as defined in Brams et al (2003), set $\mathbf{q} = (1, 1, ..., 1)$, and $\Psi_i = \{0, 1\}^M$ for all i. The multi-unit assignment problem studied by Budish and Cantillon (2008) sets each Ψ_i to be the set of schedules containing at most k courses, and rules out most forms of complementarity and substitutability.

In some combinatorial allocation settings there is no requirement that each agent consumes at most one of each kind of object. All stated results remain valid for this generalization of the sets $(\Psi_i)_{i=1}^N$, but the market-clearing bound of Theorem 1 may be less compelling. See discussion at the end of section 5.2.

In a shift-assignment setting feasibility might require that *at least* a certain number of agents are assigned to each shift (or task), rather than *at most*. This can be accomodated by setting $q_j < 0$ equal to the negative of the minimum requirement, and letting each $\Psi_i \subseteq \{0, -1\}^M$ rather than $\{0, +1\}^M$.

2.1 Efficiency, Strategyproofness, and the Dictatorship Theorems

A feasible allocation $\mathbf{x}$ is (ex-post) Pareto efficient if there is no other feasible allocation $\mathbf{x}'$ such that $u_i(x_i') \geq u_i(x_i)$ for all i, with at least one strict.

A combinatorial assignment mechanism χ is strategyproof in economy $(\mathcal{S}, \mathcal{C}, \mathbf{q}, (\Psi_i)_{i=1}^N, (u_i)_{i=1}^N)$ if, for all $s_i \in \mathcal{S}$, $\mathbb{E}u_i[\chi_i(u_i, \widehat{\mathbf{u}}_{-i})] \geq \mathbb{E}u_i[\chi_i(\widehat{u}_i, \widehat{\mathbf{u}}_{-i})]$ for any $\widehat{u}_i, \widehat{\mathbf{u}}_{-i}$. In words, no matter the reports of the agents other than s_i, it should be expected-utility maximizing for s_i to report her preferences truthfully. A mechanism is strategyproof if it is strategyproof in any economy.

Serial Dictatorship is a deterministic mechanism in which each student is endowed with a serial number and then, in order of their serial number, each student chooses her most preferred set of courses out of those courses still available. Sequential Dictatorship is similar, except that the

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A language along those lines for the course-allocation problem is sketched in Othman et al (2008).

By contrast there are no known methods that directly elicit ordinal preference relations, except for the case of unit demand. In the National Resident Matching Program (Roth and Peranson, 1999), hospitals with multi-unit demand for doctors express only their ordinal preferences for individual doctors. This leaves their preferences over bundles of these doctors ambiguous; this ambiguity does not harm mechanism performance under an additive-separability assumption (responsiveness) on hospitals' preferences.

choosing order is allowed to be endogenous, in the sense that the first l choices determine who gets the $l+1^{st}$ choice. The Random Serial Dictatorship is a random mechanism in which serial numbers are drawn uniform randomly, and then the corresponding Serial Dictatorship is implemented.

It is easy to see that dictatorships are strategyproof and Pareto efficient. A series of papers suggests that dictatorships may be the *only* mechanisms that satisfy these two criteria for combinatorial assignment problems. Papai (2001) shows that these two properties, combined with non-bossiness, characterize Sequential Dictatorships. Ehlers and Klaus (2003) show that non-bossiness can be replaced with coalitional strategyproofness, and Hatfield (2005) shows that the Papai (2001) result obtains even for additively-separable preferences.[21]

Klaus and Miyagawa (2001), Konishi et al (2001), and Sonmez (1999) obtain similar negative results under slightly different conditions, including existing endowments. Kojima (2007) obtains related negative results for random mechanisms.

3 Fairness

3.1 Indivisibilities Complicate Fair Division

Moulin (1995) writes: "In fair division, the two most important tests of equity are fair share guaranteed and no envy".

In a divisible-goods fair division problem, an agent is said to receive his "fair share" if he receives a bundle he likes at least as well as his per-capita share of the endowment. Formally, if $\mathbf{q}$ is an endowment of *divisible* goods, an allocation $\mathbf{x}$ satisfies the fair-share guarantee if $u_i(x_i) \geq u_i(\frac{\mathbf{q}}{N})$ for all i. Early papers on the cake-cutting problem (Steinhaus 1948, Dubins and Spanier 1961) actually defined fairness itself as this guarantee. The appeal of the fair-share guarantee is that it expresses the ideal of common ownership of the goods which are to be divided. Agents might expect to do better than this ideal, due to heterogeneity in preferences, but certainly they should not do worse.

An allocation $\mathbf{x}$ is said to be envy free if $u_i(x_i) \geq u_i(x_j)$ for all i, j (Foley, 1967). In words, envy-freeness requires that each agent likes his own bundle weakly better than anyone else's. Arnsperger (1994) describes envy freeness is an "ordinalist version of egalitarianism" (see also Thomson and Varian, 1985), i.e., a way to operationalize egalitarianism without inter-personal comparability of utilities.

[21] A deterministic mechanism is non-bossy if, whenever some agent s_i's outcome is the same under (u_i', u_{-i}) as under (u_i'', u_{-i}), all other agents have the same outcomes under these two report profiles as well. A deterministic mechanism is coalitionally strategyproof if no set of agents ever has a manipulation that weakly improves the outcome of all agents in the set, with at least one strict improvement.

In divisible-goods economies, Competitive Equilibrium from Equal Incomes ("CEEI"; Foley, 1967; Varian, 1974) satisfies both criteria. These criteria form the core of the argument that CEEI is an attractive procedure for fair division of divisible goods.

Indivisibilities complicate fair division. With indivisibilities, fair share is not even well defined: $\frac{\mathbf{q}}{N}$ is not a valid consumption bundle. Envy freeness remains well defined, but will be impossible to guarantee: what if there are two agents and a single indivisible object?

This section proposes two new criteria of outcome fairness: the maximin share guarantee (Section 3.3), which generalizes fair share, and envy bounded by a single good (Section 3.4) which weakens envy freeness. The criteria explicitly recognize that indivisibilities complicate fair division, but exploit the fact that *bundles* of indivisible objects are somewhat divisible. We begin though by reviewing, in Section 3.2, previous approaches to defining fairness in environments with indivisibilities.

3.2 Previous Approaches to Outcome Fairness with Indivisibilities

There have been several previous approaches to defining fairness in environments with indivisibilities.

A first approach is simply to ignore outcome fairness altogether, and look solely to procedural fairness. Klaus and Ehlers (2003) take this approach to argue that a dictatorship might be fair for multi-unit assignment: "Dictatorships can be considered to be fair if the ordering of agents is fairly determined."

A second approach is to assume that there is a divisible numeraire good, like money, in addition to the endowment of indivisible goods. Moulin (1995) and Alkan, Demange and Gale (1991) study such problems, and propose definitions of fair share and envy-freeness, respectively, that include transfers. This approach is ruled out by exogenous constraint in the context of course allocation.

A third approach is to follow Hylland and Zeckhauser (1979) and transform indivisible objects into perfectly divisible "probability shares" of the objects. An agent is said to receive his fair share if he receives a bundle of probability shares that he likes at least as well as his per-capita share of the probability-share endowment. An allocation is said to be envy free if, at the interim stage, no agent envies the lottery of any other agent. Brams and Taylor (1996) and Pratt (2007) take this approach. There are two reasons this approach is unattractive for combinatorial assignment problems. First, it requires restrictive assumptions on preferences.[22] Second, it can result in outcomes that seem

[22] The Brams and Taylor (1996) and Pratt (2007) procedures require that preferences are additive-separable and that there are no scheduling constraints. Budish, Che, Kojima and Milgrom (2008) show that probability-shares allocations can be resolved into deterministic assignments under more general circumstances that include certain kinds of scheduling and substitutability constraints.

unacceptably unfair ex post. (See footnote 24 to Example 1 below).

3.3 The Maximin Share Guarantee

I explicitly accept that indivisibilities complicate fair division and propose a weakening of the fair-share common-ownership ideal:

Definition 1. *Fix an economy $(\mathcal{S}, \mathcal{C}, \mathbf{q}, (\Psi_i)_{i=1}^N, (u_i)_{i=1}^N)$. Agent s_i's maximin share, $\underline{u}_i$, is*

$$\begin{aligned} \underline{u}_i &= \max_{(x_l)_{l=1}^N} [\min(u_i(x_1), ..., u_i(x_N))] \text{ subject to} \\ x_l &\in \Psi_i \text{ for all } l = 1...N \\ \sum x_{lj} &\leq q_j \text{ for all } j = 1...M \end{aligned} \tag{3}$$

A course-allocation mechanism satisfies the maximin-share guarantee if, for any allocation $\mathbf{x}$ *selected with positive probability under the mechanism, $u_i(x_i) \geq \underline{u}_i$ for all $i = 1...N$.*

There are two ways to think about the definition of maximin shares. First, each agent's maximin share corresponds to the utility level he could obtain for himself as divider in an $N-$player game of divide-and-choose against adversarial opponents. As divider, the agent will propose a division such that his least favorite of the N bundles is as attractive as possible (in particular, each of the N bundles will be from his own permissible set Ψ_i). Divide-and-choose is perhaps the oldest method of fair division, with accounts of its use appearing in the old testament and in Greek mythology (see Brams and Taylor, 1996; Crawford, 1977).

Second, maximin share is a Rawlsian guarantee from behind what Moulin (1991, 1992) calls a "thin veil of ignorance". The agent knows his own preferences and knows what resources are available to be divided (this is what makes the veil "thin"), but does not know other agents' preferences.

Note that when preferences are convex and goods are divisible maximin share and fair share coincide. The allocation that maximizes $\min[(u_i(x_1), ..., u_i(x_N)]$ sets each x_k equal to $\frac{1}{N}$ of the endowment.

The maximin share guarantee is somewhat pessimistic about the possibilities for fair division when there are indivisibilities. For instance, if two agents are to divide two objects – a diamond and a rock – then even the agent who receives the rock is said to have received his maximin share. (The procedural fairness property of symmetry ensures that each agent has an equal chance at the diamond.)

Despite its pessimism the maximin-share guarantee has bite. The following example shows that dictatorships fail the criterion.

Example 1. (Two Diamonds, Two Rocks.) There are two students (s_1, s_2), four classes (A, B, C, D), one seat in each class, and each student can consume at most two classes. Students have additive-separable utility functions. That is, for $i = 1, 2$ there exists $v_i = (v_{iA}, v_{iB}, v_{iC}, v_{iD})$ such that $u_i(x_i) = \sum x_{ij} v_{ij}$ for all $x_i \in \Psi_i$. Scale v such that $\sum_j v_{1j} = \sum_j v_{2j} = 100$. Label A as the "Big Diamond", B as the "Small Diamond", C as the "Pretty Rock", and D as the "Ugly Rock". Course values are as given by the following table (e.g., $v_{1A} = 70$):

	Big Diamond	Small Diamond	Pretty Rock	Ugly Rock
s_1	70	25	3	2
s_2	52	40	5	3

Student s_1's maximin share is calculated as:

$$\begin{aligned} \underline{u}_1 &= \max_{\mathbf{x}' \text{ feasible}} \min[u_1(x_1'), u_1(x_2')] \\ &= \min[u_1(\{A, D\}), u_1(\{B, C\})] \\ &= u_1(\{B, C\}) \end{aligned}$$

Similarly $\underline{u}_2 = u_2(\{B, C\})$. In a dictatorship, whichever student gets to choose first will obtain $\{A, B\}$ (the diamonds), while the other student is left with $\{C, D\}$ (the rocks), which fails her maximin share guarantee.[23]

Example 1, in conjunction with the characterization theorems of Papai (2001) and Ehlers and Klaus (2003) (see Section 2.1) yields the following simple result.

Proposition 1. *There is no combinatorial assignment mechanism that satisfies the maximin-share guarantee, is ex-post Pareto efficient, and is either coalitionally strategyproof or both strategyproof and non-bossy.*

3.4 Envy Bounded by a Single Good

I propose a weakening of the envy free test that acknowledges that indivisibilities complicate fair division.

[23] The Brams and Taylor (1996) and Pratt (2007) procedures are not well defined for this example, due to the schedule constraints. If we eliminated the schedule constraints, each of these procedures assigns s_1 the bundle $\{A\}$ with probability a and the empty bundle with probability $1 - a$ (Brams and Taylor: $a = .833$; Pratt: $a = .827$). Student s_2 gets $\{B, C, D\}$ with probability a and $\{A, B, C, D\}$ with probability $1 - a$. At an interim phase this lottery satisfies certain fairness criteria, but ex post it is possible that s_1 gets zero objects, so these procedures fail the maximin share guarantee.

Definition 2. *An allocation* $\mathbf{x}$ *satisfies* ***envy bounded by a single good*** *if*

$$\text{For any } s_i, s_j \in \mathcal{S}\text{: There exists some } c_{j'} \in x_j \text{ s.t. } u_i(x_i) \geq u_i(x_j \backslash \{c_{j'}\}) \tag{4}$$

In words, if student s_i envies s_j, we require that by removing some single good from s_j's bundle we can eliminate the envy. A student who obtains the small diamond in Example 1 above may envy his fellow student who gets the big diamond, but his envy is bounded by a single good: by removing the big diamond from his fellow student's bundle the envy is eliminated.

Example 1 shows that dictatorships fail this test, because some agent might get both diamonds. Dictatorships allow for more envy than is necessary given the level of indivisibility in the economy. Even though the big diamond and small diamond are indivisible goods, the *bundle* {big diamond, small diamond} is divisible. This implies the following result, analogous to Proposition 1.

Proposition 2. *There is no combinatorial assignment mechanism that satisfies envy bounded by a single good, is ex-post Pareto efficient, and is either coalitionally strategyproof or both strategyproof and non-bossy.*

3.5 Procedural Fairness

This section has focused on outcome fairness. The other main aspect of fairness is procedural fairness, and all course-allocation mechanisms found in practice satisfy its fundamental requirement, symmetry. Symmetry, also called equal treatment of equals, requires that if two agents are identical in all dimensions relevant to a course-allocation mechanism – that is, they have identical permissible sets, and report the same preferences – they should receive the same distribution over outcomes. This property rules out that a mechanism discriminates on the basis of non-relevant information.

Definition 3. *Let* $\pi : \{1, ..., N\} \to \{1, ..., N\}$ *be a permutation of the* N *agents. A course-allocation mechanism* χ *is symmetric if, for any economy* $(\mathcal{S}, \mathcal{C}, \mathbf{q}, (\Psi_i)_{i=1}^N, (u_i)_{i=1}^N)$ *and any feasible allocation* $\mathbf{x} = (x_1, ..., x_N)$*,* $\mathbf{x}$ *is as likely to be selected by* χ *under* $(\mathcal{S}, \mathcal{C}, \mathbf{q}, (\Psi_i)_{i=1}^N, (u_i)_{i=1}^N)$ *as* $\pi(\mathbf{x}) = (x_{\pi(1)}, ..., x_{\pi(N)})$ *is under* $(\mathcal{S}, \mathcal{C}, \mathbf{q}, (\Psi_{\pi(i)})_{i=1}^N, (u_{\pi(i)})_{i=1}^N)$*.*

This paper takes the view that procedural fairness is necessary but not sufficient for fairness. For instance, the random serial dictatorship is symmetric but not fair.

3.6 Discussion: Dictatorships and Fairness

In the context of course allocation, dictatorships violate the maximin share guarantee and the requirement of envy bounded by a single good. These criteria formalize the sense in which dictatorships are unfair for multi-unit assignment. Furthermore, they suggest an explanation for why we do not observe dictatorships used in practice for multi-unit assignment.

By contrast, dictatorships are frequently observed as allocation mechanisms in single-unit assignment contexts. Examples include school choice and house allocation problems. (See references in footnote 6). But in single-unit assignment we have the following simple observation.

Remark 1. *In single-unit assignment problems, dictatorships satisfy the maximin share guarantee and envy bounded by a single good.*

Hence, these properties help to formalize the intuition that dictatorships may be fair for single-unit assignment.

4 Strategyproof in a Large Market

If a mechanism satisfies the following two properties it is strategyproof:

(i) an agent's report never affects her opportunity set (defined formally in Section 4.2).

(ii) for any fixed opportunity set, truthful reporting selects the agent's favorite outcome from the set.

Mechanisms that are not strategyproof violate either (i) or (ii). In a wide variety of economic contexts, it has been found that an agent's ability to influence her opportunity set diminishes with market size.[24] As a result, mechanisms that fail to be strategyproof due to a failure of (i) may nevertheless have good incentives properties in realistic market environments. By contrast, if a mechanism is manipulable because it fails (ii), it may have a more fundamental incentives problem.

This section proposes a notion of approximate incentive compatibility that requires (ii) but not (i). A mechanism is *strategyproof in a large market* if it is strategyproof in a certain limit market in which all agents are zero measure (Section 4.1). A necessary condition for a mechanism to satisfy this criterion is (ii) (Section 4.2). I argue in Section 4.3 that strategyproof in a large market is useful both as a conceptual device and as a desideratum in practical market design.

4.1 The Continuum Replication

Definition 4. *The continuum replication of* $(\mathcal{S}, \mathcal{C}, \mathbf{q}, (\Psi_i)_{i=1}^N, (u_i)_{i=1}^N)$*, written* $(\mathcal{S}^\infty, \mathcal{C}, \mathbf{q}, (\Psi_i^\infty)_{i=1}^N, (u_i^\infty)_{i=1}^N)$ *is constructed as follows*

[24] For instance, the most familiar kind of opportunity set is the budget set in a Walrasian mechanism, and it is well known that agents become price takers in limit markets (Roberts and Postlewaite, 1976). Other studies of incentives in large markets include Rustichini et al (1994) and Cripps and Swinkels (2006) on Double Auctions; Roth and Peranson (1999), Immorlica and Mahdian (2005) and Kojima and Pathak (2008) on deferred acceptance algorithms; and Kojima and Manea (2008) on the Probabilistic Serial mechanism.

- *The set of students is* $\mathcal{S}^{\infty} = \{\widetilde{s}_i\}_{i\in(0,N]}$
- *The set of courses and their capacities are left as is, except now we understand a course's capacity constraint as a Lebesgue measure of students in the course.*
- *Student* $\widetilde{s}_i \in \mathcal{S}^{\infty}$ *in the continuum-replication economy has the same permissible-schedule set and utility function as student* $s_{\lceil i \rceil}$ *in the original economy, where* $\lceil \cdot \rceil$ *is the ceiling operator. That is, students numbered* $(0,1]$ *in the continuum are identical to* s_1 *in the original, students numbered* $(1,2]$ *in the continuum are identical to* s_2 *in the original, etc.*

Definition 4 combines elements of the classic Debreu and Scarf (1963) and Aumann (1964) conceptions of a large market. As in Debreu and Scarf (1963) there is a finite number of types, but as in Aumann (1964) each agent is zero measure.

For the remainder of this section we restrict attention to mechanisms that are well-defined in the continuum replication of Definition 4.[25] All course-allocation mechanisms found in practice satisfy this requirement, as do many widely-known allocation procedures (see Table 1).

Definition 5. *A course-allocation mechanism* $\boldsymbol{\chi}$ *is strategyproof in a large market if, for any* $(\mathcal{S}, \mathcal{C}, \mathbf{q}, (\Psi_i)_{i=1}^N, (u_i)_{i=1}^N)$, *it is strategyproof in the continuum replication* $(\mathcal{S}^{\infty}, \mathcal{C}, \mathbf{q}, (\Psi_i^{\infty})_{i=1}^N, (u_i^{\infty})_{i=1}^N)$. *That is, for all* $\widetilde{s}_i \in \mathcal{S}^{\infty}$, $\mathbb{E}u_i[\chi_i(u_i, \widehat{\mathbf{u}}_{-i})] \geq \mathbb{E}u_i[\chi_i(\widehat{u}_i, \widehat{\mathbf{u}}_{-i})]$ *for any* $\widehat{u}_i, \widehat{\mathbf{u}}_{-i}$, *where* $\widehat{\mathbf{u}}_{-i}$ *indicates the reports of all students in* $\mathcal{S}^{\infty} \backslash \{\widetilde{s}_i\}$.

This property requires that truthful reporting is a dominant strategy for the kinds of agents we think of as "price takers." Note though that prices do not explicitly appear in the definition, and so we can accomodate both price and non-price allocation mechanisms. For instance, to encode combinatorial auctions we add an $M+1^{st}$ good, money, set q_{M+1} to be very large, and then redefine permissible-schedule sets and utility functions in terms of combinations of both goods and money. Matching problems can be accomodated with slight modification to Definitions 4 and 5.[26]

[25] Formally, an allocation $(x_i)_{\widetilde{s}_i \in \mathcal{S}^{\infty}}$ is feasible in $(\mathcal{S}^{\infty}, \mathcal{C}, \mathbf{q}, (\Psi_i^{\infty})_{i=1}^N, (u_i^{\infty})_{i=1}^N)$ if (i) $x_i \in \Psi_i$ for all $\widetilde{s}_i \in \mathcal{S}^{\infty}$, and (ii) $\int_{\mathcal{S}^{\infty}} x_{ij} d\widetilde{s}_i \leq q_j$ for each course c_j. For a mechanism χ to be well defined in continuum economies, for any continuum replication $(\mathcal{S}^{\infty}, \mathcal{C}, \mathbf{q}, (\Psi_i^{\infty})_{i=1}^N, (u_i^{\infty})_{i=1}^N)$, $\chi[(\mathcal{S}^{\infty}, \mathcal{C}, \mathbf{q}, (\Psi_i^{\infty})_{i=1}^N, (u_i^{\infty})_{i=1}^N)]$ must be a measurable probability distribution over feasible allocations.

[26] A one-to-one matching problem is defined as the tuple $(F, M, (u_f)_{f\in F}, (u_m)_{m\in M})$, where F is a set of females, M is a set of males, u_f are the preferences of female f over the males, and likewise for u_m. The key to defining the continuum replication is to exploit the fact that males have a dominant strategy of reporting their preferences truthfully. Specifically, we treat F analogously to the set of students $\mathcal{S}$, but treat M like the set of courses, $\mathcal{C}$. The continuum replication is $(F^{\infty}, M, (u_f^{\infty})_{f\in F}, (u_m)_{m\in M})$, where the superscript $^{\infty}$ plays the same role as in Definition 4. A matching mechanism is strategyproof in a large market if the zero-measure females never wish to misreport their preferences. (Men are non-strategic, and each type of man has capacity for a measure one set of females).

4.2 A Simple Interpretation of Strategyproof in a Large Market

Definition 6. *Consider a mechanism* $\boldsymbol{\chi}$ *and finite economy* $(\mathcal{S}, \mathcal{C}, \mathbf{q}, (\Psi_i)_{i=1}^N, (\widehat{u}_i)_{i=1}^N)$. *Create the continuum replication* $(\mathcal{S}^\infty, \mathcal{C}, \mathbf{q}, (\Psi_i^\infty)_{i=1}^N, (\widehat{u}_i^\infty)_{i=1}^N)$. *For student* s_i *in the finite economy and any report* u_i' *in her report set* U_i, *let* $(u_i', \widehat{u}_{-i}^\infty)$ *denote the strategy profile in the continuum replication in which*

- *Student* $\widetilde{s}_j \in \mathcal{S}^\infty \backslash \{\widetilde{s}_i\}$ *plays strategy* $\widehat{u}_{\lceil j \rceil}$
- *Student* $\widetilde{s}_i$ *plays strategy* u_i'

That is, students numbered $(0, 1]$ *play* $\widehat{u}_1$, *students numbered* $(1, 2]$ *play* $\widehat{u}_2$, *etc., except for student* $\widetilde{s}_i$ *who plays* u_i'. *We say that student* s_i's **opportunity set** *in the finite economy is*

$$\{\boldsymbol{\chi}_i(u_i', \widehat{u}_{-i}^\infty) : u_i' \in U_i\} \tag{5}$$

That is, s_i's opportunity set in the finite economy is defined as the set of outcomes (possibly random) she can achieve in the continuum replication in which all agents but for herself play according to $\widehat{\mathbf{u}}$. For instance, consider a competitive equilibrium mechanism in which the realized price vector depends in a deterministic way on the distribution of agents' reports. In a finite economy, s_i's report $\widehat{u}_i$ affects this distribution, and so affects price. Say the price under $\widehat{\mathbf{u}}$ is $\mathbf{p}^*$. In the continuum replication, there is a set of measure one of agents whose preferences are identical to s_i's, $\{\widetilde{s}_j\}_{j \in (i-1, i]}$. We have the set of agents $\{\widetilde{s}_j\}_{j \in (i-1, i)}$ continue to report $\widehat{u}_i$, but we let $\widetilde{s}_i$ vary her report. Now, her report no longer affects $\mathbf{p}^*$, because she is zero measure. By varying over all of $\widetilde{s}_i$'s possible reports in U_i, we obtain s_i's opportunity set. The concept of opportunity set is not restricted just to price-based mechanisms; for instance, in a one-to-one Male-Proposing Deferred Acceptance Algorithm the opportunity set for a female agent is the set of proposals she receives.[27]

In either a competitive equilibrium mechanism or a deferred acceptance algorithm, for any fixed opportunity set, reporting truthfully selects the agent's favorite outcome from that set. The following simple result shows that this feature is a necessary condition for a mechanism to be strategyproof in a large market.[28]

Proposition 3. *If a mechanism* χ *is strategyproof in a large market, then for any economy and any set of reports* $(\widehat{u}_i)_{i=1}^N$, *each student* s_i*'s most-preferred element in her realized opportunity set*

[27] Her report affects which proposals she *rejects*, and in a finite economy each rejection has a positive probability of causing a "rejection chain" that causes her to receive new proposals that she would not otherwise have received. (Kojima and Pathak, 2008).

[28] It is not sufficient, because it restricts attention to strategy profiles in which, for each $l = 1, ..., N$, a set of measure one plays $\widehat{u}_l$. Strategyproofness makes no such restriction.

is $\chi_i(u_i, \widehat{\mathbf{u}}^{\infty}_{-i})$*, i.e., the element that corresponds to her truthful report. Formally*

$$\mathbb{E}u_i[\chi_i(u_i, \widehat{\mathbf{u}}^{\infty}_{-i})] \geq \mathbb{E}u_i[\chi_i(\widehat{\widehat{u}}_i, \widehat{\mathbf{u}}^{\infty}_{-i})] \text{ for any } (\widehat{u}_i)_{i=1}^{N}, \text{ any } s_i, \text{ and any } \widehat{\widehat{u}}_i \in U_i$$

Proof. Follows immediately from the definition of strategyproof in a large market. QED.

The Bidding Points Mechanism described in the introduction fails this necessary condition. At the prices $\mathbf{p}^* = (900, 250, 100, 75)$ Alice's realized opportunity set was the set of bundles that cost weakly less than her budget of 1000, i.e., $\{\{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\}, \{A\}, \{B\}, \{C\}, \{D\}, \emptyset\}$. Her favorite bundle, $\{A, C\}$, corresponds to the report $\widehat{\widehat{u}}_i(c_1) = 750$, $\widehat{\widehat{u}}_i(c_2) = 250$. By contrast, the truthful report $u_{Alice} = (700, 200, 50, 50)$ causes her to obtain zero courses. So, reporting truthfully does not select the most preferred element in her opportunity set, and the mechanism is not strategyproof in a large market.

4.3 The Usefulness of Strategyproof in a Large Market

The incentives criterion of strategyproof in a large market is useful both as a conceptual device and as a desideratum in practical market design.

Conceptually, it is a simple-to-apply criterion that separates market designs that are certainly manipulable in large finite markets from market designs that may not be. All course-allocation mechanisms currently found in practice fail the criterion. By contrast, many widely-used non-strategyproof mechanisms satisfy the criterion. See Table 1.

Table 1. Which Market Designs are Strategyproof in a Large Market?

Manipulable in Large Markets	**Strategyproof in Large Markets**
Bidding Points Mechanism	Deferred Acceptance
HBS Draft Mechanism	Double Auctions
Boston Mechanism	Assignment Exchange
All-Pay Auctions	Probabilistic Serial
Discriminatory Auctions	Uniform Price Auctions

A seeming exception to this pattern is the single-unit first-price sealed bid auction, a widely used auction format (e.g., in procurement) that is manipulable. First-price auctions are a special case of multi-unit Discriminatory Auctions, which are manipulable in large markets (even though an agent is zero measure, his own bid amount determines what he pays when he wins). But first-price auctions are also a special case of multi-unit Uniform Price Auctions, which are strategyproof in large markets.[29]

As a practical matter, the criterion may be a sufficient condition for market administrators to feel comfortable advising market participants that it is in their interest to report their preferences truthfully; this saves market participants any costs of strategizing.[30] Note too that if a mechanism satisfies this criterion, then at whatever opportunity set is realized ex post, agents are happiest if they reported their preferences truthfully. That is, unless they understand the specific way that their misreport would have affected their opportunity set,[31] truthful agents are unlikely to experience ex-post regret.

5 The Approximate Competitive Equilibrium from Approximately Equal Incomes

Competitive Equilibrium from Equal Incomes ("CEEI")[32] is an attractive solution to the problem of efficient and fair division of *divisible* goods. It is Pareto efficient by the first welfare theorem. It satisfies the fair share guarantee and is envy free. A CEEI mechanism can be defined to satisfy the procedural fairness requirement of symmetry, and the incentives criterion of strategyproof in a large market. Arnsperger (1994) writes "essentially, to many economists, [CEEI is] the description of perfect justice."[33]

Unfortunately CEEI need not exist. Either indivisibilities or complementarities alone would make existence problematic, and our economy features both. In order to recover existence we will need to approximate both the "CE" and the "EI" of CEEI:

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[29] In a recent presentation on auction design for the United States Treasury's Troubled Asset Relief Program, Ausubel and Cramton (2008) wrote: "General assessement is that uniform price performs at least as well as pay-as-bid [i.e., discriminatory price] for financial instruments ... *Bidders hate pay-as-bid auctions, as they look foolish (or unemployed) after selling at unnecessarily low prices.*" (Emphasis added).

[30] By contrast, here is an excerpt from the advice that Wharton provides its students on how to play its course-allocation mechanism, which is a multi-round variant on the Bidding Points Mechanism (in the first round students buy courses; in subsequent rounds they can both buy and sell):

"Look at past results and the price history of individual courses over all rounds in which it was offered; Last round results are a good measure of final market equilibrium. Each semester is generally similar to the corresponding semester last year in terms of the demand for courses. Beware of the mid-auction "bubble". Don't pay too high a price for a course that will eventually open up. Look at the 8th round of the Fall 2006 auction to see where prices eventually settled. This should help mitigate some of the anxiety that middle rounds inevitably create with their high prices and low liquidity." (Wharton, 2007)

[31] Which they likely *would* in first-price auctions, but likely would not for competitive equilibrium and deferred acceptance mechanisms.

[32] See Foley (1967), Varian (1974), and several other seminal references summarized in Thomson and Varian (1985).

[33] The philosopher Ronald Dworkin (1981, 2000) argues extensively that CEEI is fair, using CEEI as the motivation for his theory that fairness is "Equality of Resources".

5.1 Definition of Approximate CEEI

Definition 7. *Fix an economy* $(\mathcal{S}, \mathcal{C}, \mathbf{q}, (\Psi_i)_{i=1}^N, (u_i)_{i=1}^N)$. *The allocation* $\mathbf{x}^* = (x_1^*, ..., x_N^*)$, *budgets* $\mathbf{b}^* = (b_1^*, ..., b_N^*)$ *(with* $\min_i b_i^* = 1$, *wlog), and prices* $\mathbf{p}^* = (p_1^*, ..., p_M^*)$ *constitute an* (α, β)*-approximate competitive equilibrium from equal incomes (Approximate CEEI) of this economy if:*

(i) $x_i^* = \arg\max_{x_i' \in \Psi_i}[u_i(x_i') : \mathbf{p}^* \cdot x_i' \leq b_i^*]$ *for all* $i = 1, ..., N$

(ii) $||\mathbf{z}^*||_2 \leq \alpha$ *where* $\mathbf{z}^* = (z_1^*, ..., z_M^*)$ *and*

$z_j^* = \sum_i x_{ij}^* - q_j$ *if* $p_j^* > 0$

$z_j^* = \max(\sum_i x_{ij}^* - q_j, 0)$ *if* $p_j^* = 0$

(iii) $\max_i(b_i^*) \leq 1 + \beta$

Condition (i) indicates that, at the competitive equilibrium prices and budgets, each agent chooses her most-preferred schedule that costs weakly less than her budget. Observe that agents consume sure bundles rather than probability shares of objects as in Hylland and Zeckhauser (1979).

Condition (ii) is where we approximate "CE". The market is allowed to clear with some error, α, calculated as the Euclidean distance (squareroot of sum of squares) of market clearing error. If a course has a strictly positive price both excess demand and excess supply count as error. If its price is zero then only excess demand counts as error. (Preferences are non-monotone, and so it is quite possible for a course to have excess supply at price zero).

Condition (iii) is where we approximate "EI". The largest budget can be no more than β proportion larger than the smallest budget.

If $\alpha = \beta = 0$ then we have an exact CEEI. Our version of exact CEEI is stated a bit differently from the classical version (Foley, 1967; Varian, 1974), which is formulated as the competitive equilibrium of an exchange economy in which all agents have the same endowment.

5.2 The Existence Theorem

For a price vector $\mathbf{p} \geq \mathbf{0}$, let $B_\delta(\mathbf{p})$ denote its δ-ball in non-negative space. Formally, $B_\delta(\mathbf{p}) = \{\mathbf{p}' \geq \mathbf{0} : ||\mathbf{p}' - \mathbf{p}||_2 \leq \delta\}$.

Definition 8. *Fix an economy* $(\mathcal{S}, \mathcal{C}, \mathbf{q}, (\Psi_i)_{i=1}^N, (u_i)_{i=1}^N)$. *Set* $b_i = 1$ *for all* i *(wlog) and let* $d_i^*(\mathbf{p}) \in \Psi_i$ *denote student* i*'s demanded schedule when prices are* $\mathbf{p}$ *and his budget is* 1. *The Demand Sensitivity of this economy,* σ, *is defined by*

$$\sigma = \sup_{i,\mathbf{p}} \lim_{\delta \to 0^+} \sup_{\mathbf{p}' \in B_\delta(\mathbf{p})} \left(\left\|d_i^*(\mathbf{p}) - d_i^*(\mathbf{p}')\right\|_2\right)^2$$

The Demand Sensitivity of an economy tells us the maximum possible discontinuity in a single agent's demand with respect to price. This discontinuity is measured as the square of Euclidean distance. In the course-allocation context the maximum possible discontinuity occurs when some student's demand changes from one bundle of k courses to an entirely disjoint bundle of k courses (unless $2k > M$ which is unusual in practice).

Remark 2. *In the course-allocation problem, in which each student demands at most one of each course and at most k courses overall, $\sigma \leq \min(2k, M)$.*

The main existence result is:

Theorem 1. *Fix any economy $(\mathcal{S}, \mathcal{C}, \mathbf{q}, (\Psi_i)_{i=1}^N, (u_i)_{i=1}^N)$. For any $\beta > 0$, there exists a $(\frac{\sqrt{\sigma M}}{2}, \beta)-$Approximate CEEI.*

In particular, for any budget vector $\mathbf{b}'$ that satisfies $\max_i(b_i') \leq 1 + \beta$ and $\min_i(b_i') = 1$, and any $\varepsilon > 0$, there exists a $(\frac{\sqrt{\sigma M}}{2}, \beta)-$Approximate CEEI with budgets of $\mathbf{b}^*$ that satisfy $|b_i^* - b_i'| < \varepsilon$ for all i.

Theorem 1 indicates that any strictly positive amount of budget inequality is enough to ensure that there is a price vector whose market clearing error is smaller than $\frac{\sqrt{\sigma M}}{2}$.

The market administrator is free to specify any vector of target budgets $\mathbf{b}'$ and any $\varepsilon > 0$, and is assured that each realized budget b_i^* is within ε of its target. So long as the target budgets are strictly unequal, the perturbation ε can be made small enough so that $\mathbf{b}^*$ preserves the same strict budget order as $\mathbf{b}'$. Two natural choices for how to specify $\mathbf{b}'$ are: (i) randomly assign unequal budgets in $[1, 1+\beta]$; and (ii) assign unequal budgets in $[1, 1+\beta]$ based on some pre-existing priority order like seniority or grade-point average.

In Section 6 we will show that by setting β sufficiently small the market administrator can guarantee attractive outcome fairness properties.

Discussion of the approximation bound The $\frac{\sqrt{\sigma M}}{2}$ bound is small in two respects. First, it does not grow with either N (the number of agents) or $\mathbf{q}$ (the capacity vector). This means that in the continuum replication of any finite economy we can guarantee exact market clearing (market clearing error will be zero measure).

Second, $\frac{\sqrt{\sigma M}}{2}$ is actually a small number for practical problems. For instance, in a semester at Harvard Business School $k = 5$ and $M = 50$, and so $\sigma \leq 10$ and $\frac{\sqrt{\sigma M}}{2} \lesssim 11$. This corresponds to a maximum market-clearing error of 11 seats in one class, or of 3 seats in each of 12 classes, etc., as compared with about 4500 total course seats allocated per semester. Such error can easily be accomodated in practice, by adding or removing a few chairs to or from a few classrooms.

Relation to the Combinatorial Auctions Literature A notable feature of Theorem 1 is that it provides an approximate existence result for item prices in an economy with complex preferences including complementarities. This seems counterintuitive given the combinatorial auctions literature, where results for the existence of market-clearing item prices are negative (Gul and Stachetti, 2000; Milgrom, 2004, 2007; Bikhchandani and Mamer, 1997; Bikhchandani and Ostroy, 2002; Parkes, 2007) and even approximate existence is thought to require non-linear non-anonymous prices (Nisan and Segal, 2006).

The key to understanding the difference is σ, the Demand Sensitivity parameter.

In a general combinatorial auction setting it is possible that some agent gets positive utility only if he consumes the bundle of all goods. If this bundle costs more than his value for it, he consumes nothing. The demand of this agent is highly discontinuous at prices where he is indifferent between the all and nothing bundles. If for simplicity we assume $q_j = q$ for all j, then the demand sensitivity of an economy with such an agent is $\sigma = Mq^2$. So the Theorem 1 bound $\frac{\sqrt{\sigma M}}{2} = \frac{Mq}{2}$. This is a meaningless bound: the market-clearing error from allocating no goods at all ($\sqrt{M}q$) can be lower!

By contrast, in the combinatorial assignment problem studied here, any complementarities must be "small", because each agent consumes at most one of each object. This is what allows item prices to clear the market to within an attractive approximation bound. There may be other combinatorial allocation environments in which σ is small, and so a result along the lines of Theorem 1 can guarantee an attractive approximation with item prices.

5.3 Sketch of Proof

The proof of Theorem 1 is contained in Appendix A. Here we provide a detailed sketch of the proof in order to provide the main intuition for how it works. For the sketch it is convenient to assume that agents' permissible sets are identical ($\Psi_i \equiv \Psi$ for all i). Also, the sketch ignores some boundary issues that are handled in the formal proof.

5.3.1 The Tâtonnement Price-Adjustment Function

Consider a tâtonnement price adjustment function of the form

$$f(\mathbf{p}) = \mathbf{p} + \mathbf{z}(\mathbf{p}) \tag{6}$$

where $\mathbf{z}(\mathbf{p})$ indicates excess demand as in (ii) of Definition 7. If $f(\cdot)$ has a fixed point, this point is a competitive equilibrium price vector.

Unfortunately there is no guarantee that $f(\cdot)$ has a fixed point. Any time any agent's demand changes it does so discontinuously, because goods are discrete. So $\mathbf{z}(\cdot)$, and hence $f(\cdot)$, are not continuous with respect to price and we cannot apply fixed-point theorems to $f(\cdot)$. Instead, we first mitigate the potential discontinuities in $f(\cdot)$ by perturbing budgets, and then we find an *approximate* fixed point of $f(\cdot)$.

5.3.2 The Role of Unequal Budgets

The role of unequal budgets is to mitigate discontinuities in $f(\cdot)$.

Consider a student s_1 whose budget is 1000 and whose favorite bundle at some price vector $\mathbf{p}'$, x', costs 999. If we change the price vector a small amount to $\mathbf{p}''$ so that x' costs 1001 instead, s_1 no longer can afford it. Her demand will change. The size of a single agent's demand discontinuity is bounded by $\sqrt{\sigma}$.

Suppose that students $s_2 ... s_N$ also have a budget of exactly 1000. Then as price changes from $\mathbf{p}'$ to $\mathbf{p}''$ their choice sets vary identically with s_1's. Their demands might change as well. It is possible that all N agents' demands change exactly as s_1's demand changes. That is, it is possible that $f(\cdot)$ has a discontinuity with respect to price of size $N\sqrt{\sigma}$. Note that the size of this potential discontinuity grows with the number of agents, N.

Suppose instead that s_2 has a budget of 1002, s_3 of 1004, etc. Now, as the price of x' rises from 999 to 1001 s_1 can no longer afford it, but $s_2, s_3, ...$ still can. So the change in price that causes s_1's demand to change need not cause other students' demands to change. This is the basic intuition for why even an arbitrarily small amount of budget inequality is so helpful.

The story is a bit more complicated than this sketch involving bundle x' suggests, because our economy uses M item prices, not $|\Psi| >> M$ bundle prices. So any change in the price of x' changes the price of some other bundles as well. Let $H(s_i, x) = \{\mathbf{p} : \mathbf{p} \cdot x = b_i\}$ denote the hyperplane in M-dimensional price space along which agent s_i can exactly afford bundle x. Every time price crosses such a "budget-constraint hyperplane", some agent's choice set changes, and hence their demand might change. In the sketch above, $\mathbf{p}'$ and $\mathbf{p}''$ were on opposite sides of $H(s_1, x')$.

Because the number of agents and the number of permissible bundles are finite, so too is the number of budget-constraint hyperplanes. I define a perturbation scheme – a tiny "tax / credit" specific to each agent-bundle pair – that does two things.[34] First, no more than M budget-constraint hyperplanes intersect at any one point - now the maximum discontinuity in $f(\cdot)$ with respect to price is $M\sqrt{\sigma}$, which no longer grows with N. Second, no two budget-constraint hyperplanes are identical – this means that for any two price vectors $\mathbf{p}'$ and $\mathbf{p}''$, there is a path through price space that connects $\mathbf{p}'$ and $\mathbf{p}''$ and that crosses just one budget-constraint hyperplane at a time, i.e., a path along which all discontinuities are of size at most $\sqrt{\sigma}$.

[34] This perturbation scheme is the reason for the ε in the theorem statement.

5.3.3 Finding an Approximate Fixed Point

How are we going to use the fact that the discontinuities in $f(\cdot)$ are now "small" to obtain an approximate fixed point? Consider the following convexification of $f(\cdot)$:

$$F(\mathbf{p}) = co\{y : \exists \text{ a sequence } \mathbf{p}^w \to \mathbf{p}, \mathbf{p}^w \neq \mathbf{p} \text{ such that } f(\mathbf{p}^w) \to y\} \tag{7}$$

where *co* denotes the convex hull. Cromme and Diener (1991; Lemma 2.4) show that, for any map $f(\cdot)$ on a compact and convex set, correspondences of the form (7) are upper-hemicontinuous. This allows application of Kakutani's fixed-point theorem (the other conditions are trivially satisfied): there exists $\mathbf{p}^*$ such that $\mathbf{p}^* \in F(\mathbf{p}^*)$.

What does (7) tell us about such a $\mathbf{p}^* \in F(\mathbf{p}^*)$? In any arbitrarily small neighborhood of $\mathbf{p}^*$, there must exist a set of points such that some convex combination of the f's of these points is equal to $\mathbf{p}^*$: that is, a convex combination of their excess demands exactly clears the market.

Because agents' demands change only when price crosses one of their budget-constraint hyperplanes, we can put a lot of structure on demands in a small neighborhood of $\mathbf{p}^*$. If $\mathbf{p}^*$ is not on any budget-constraint hyperplane, then in a small enough neighborhood of $\mathbf{p}^*$ demand is unchanging, and so $\mathbf{p}^* \in F(\mathbf{p}^*)$ actually implies $\mathbf{p}^* = f(\mathbf{p}^*)$, and we are done. Suppose instead that $\mathbf{p}^*$ is on L budget-constraint hyperplanes. We know from the perturbation above that $L \leq M$.

The two key ideas are the following. First, for any price $\mathbf{p'}$ in a small enough neighborhood of $\mathbf{p}^*$, demand at $\mathbf{p'}$ is entirely determined by which side of the L hyperplanes $\mathbf{p'}$ is on (the affordable side or the unaffordable side). That is, out of a whole neighborhood, we can limit attention to a finite set of at most 2^L points.

Second, for each of the L agents corresponding to the L hyperplanes, their demand depends only on which side of their *own* budget-constraint hyperplane price is on. For each of the L agents $s_{(i)}$, $i = 1...L$, we can define a "change-in-demand vector" $v_{(i)}$ that describes how their demand changes as price crosses from the affordable to the unaffordable side of their budget-constraint hyperplane. Thus a set of L change-in-demand vectors entirely describe how demand changes near $\mathbf{p}^*$.[35] The set of feasible demands in an arbitrarily small neighborhood of $\mathbf{p}^*$ is:

$$\{a \in \{0,1\}^L : \mathbf{z}(\mathbf{p}^*) + \sum_{i=1}^{L} a_{(i)} v_{(i)}\} \tag{8}$$

[35] There are two exceptions to this statement that are handled in the proof. The first exception is if $\mathbf{p}^*$ is on the boundary of price space. In this case we may need to perturb budgets a tiny bit more in order to cross certain combinations of hyperplanes. The second exception is if multiple hyperplanes belong to a single agent. Then their change in demand close to $\mathbf{p}^*$ is a bit more complicated than can be described by a single change-in-demand vector.

Now $\mathbf{p}^* \in F(\mathbf{p}^*)$ tells us something much more useful than above: perfect market clearing is in the convex hull of (8)! Our market-clearing error is the maximum-minimum distance between a vertex of (8) – one of the feasible demands near $\mathbf{p}^*$ – and a point in the convex hull of (8). The worst case distance occurs when there are M change-in-demand vectors, each of the maximum possible length $\sqrt{\sigma}$, the vectors are mutually orthogonal, and the perfect market clearing ideal is equidistant from all 2^M vertices. This worst case distance is half the diagonal length of an M-dimensional hypercube of side length $\sqrt{\sigma}$: $\frac{\sqrt{\sigma M}}{2}$.

5.4 Tightness of Theorem 1

The bound of Theorem 1 is tight. I present an illustrative example and then the formal tightness statement.

Example 2. There are 4 courses $\{A, B, C, D\}$ each with capacity 2. There are 4 students whose budgets are $\mathbf{b}^* = (1203, 1202, 1201, 1200)$ and whose preferences over the scarce courses are:

$$\begin{aligned} u_1(\{A,B,C\}) &> u_1(\{D\}) > ... \\ u_2(\{A,B,D\}) &> u_2(\{C\}) > ... \\ u_3(\{A,C,D\}) &> u_3(\{B\}) > ... \\ u_4(\{B,C,D\}) &> u_4(\{A\}) > ... \end{aligned}$$

Consider the price vector $\mathbf{p}^* = (402, 401, 400, 399)$. Student s_1 can exactly afford $\{A, B, C\}$ at $\mathbf{p}^*$. So along $\mathbf{p}^w \to \mathbf{p}^*$ her demand converges either to $\{A, B, C\}$ or $\{D\}$. Similarly, student s_2's demand converges to either $\{A, B, D\}$ or $\{C\}$, etc. Some feasible total demands in a neighborhood of $\mathbf{p}^*$ are $(1, 1, 1, 1)$, $(2, 2, 2, 0)$, $(3, 3, 1, 1)$, $(4, 2, 2, 2)$, and $(3, 3, 3, 3)$. A convex combination of students' demands in a neighborhood of $\mathbf{p}^*$ exactly clears the market.

Observe that every feasible demand in a neighborhood of $\mathbf{p}^*$ is Euclidean distance 2 from $\mathbf{q} = (2, 2, 2, 2)$. Why?

The change-in-demand matrix at $\mathbf{p}^*$ (i.e., the matrix formed by stacking the $v_{(i)}$'s) is

$$\begin{pmatrix} -1 & -1 & -1 & +1 \\ -1 & -1 & +1 & -1 \\ -1 & +1 & -1 & -1 \\ +1 & -1 & -1 & -1 \end{pmatrix} \tag{9}$$

This is an example of a Hadamard matrix: all of its entries are ± 1 and its rows are mutually orthogonal.[36]

Whenever the change-in-demand matrix at $\mathbf{p}^*$ is a Hadamard matrix, aggregate demand in a neighborhood of $\mathbf{p}^*$ forms a hypercube with sides of length $\sqrt{M}$ (here, $\sigma = M$, so $\frac{\sqrt{\sigma M}}{2} = 2$). If $\mathbf{q}$ is the hypercube's center, as here, we obtain the worst-case bound for market clearing.

The Hadamard matrix (9) has an additional feature, called regularity, which requires that each row has the same number of $+1$'s. It can be shown that regular Hadamard matrices exist for all powers of 4.[37] So we can construct examples that are analogous to Example 2 – in which all courses have approximately the same price and each agent's two favorite bundles are disjoint – for $M = 16, 64, 256, ...$ We summarize this sense that the Theorem 1 bound is tight as:

Proposition 4. *For any M', there exists an economy with $M'' \geq M'$ courses such that, for $\alpha < \frac{\sqrt{\sigma M}}{2}$ and some $\beta > 0$, there does not exist an (α, β)-Approximate CEEI.*

Note that the preferences in Example 2 seem unrealistic. This gives some hope that in practice we will be able to find approximations that are better than the bound.

6 Theoretical Properties of the Approximate CEEI

The purpose of this section is to show that the approximation to Competitive Equilibrium from Equal Incomes (CEEI) guaranteed by Theorem 1 can be the basis of a mechanism that approximates the desirable efficiency and outcome-fairness properties of CEEI.

[36] See Wallis et al (1972) or http://en.wikipedia.org/wiki/Hadamard_matrix.

[37] I thank Neil Sloane for the proof. Let A be the matrix defined in (9). The tensor product of two Hadamard matrices is itself a Hadamard matrix, and the tensor product preserves the "same number of +1s per row" property. So $A \otimes A$ is a 16-Dimensional Hadamard matrix with the same number of +1s per row, $A \otimes (A \otimes A)$ is a 64-Dimensional example, etc. It is conjectured that there exist regular Hadamard matrices of order $(2n)^2$ for any integer n. Useful references are http://www.research.att.com/~njas/hadamard/ and http://www.research.att.com/~njas/sequences/A016742.

6.1 Efficiency

An Approximate CEEI is not Pareto efficient, because a small number of positive-priced goods might go unallocated. It is easy to see though that it is efficient with respect to whatever goods *are* allocated; that is, there will not be any ex-post Pareto improving trades amongst the students.[38]

6.2 Theorem 2: Approximate CEEI Guarantees Approximate Maximin Shares

What is unusual about this definition of CEEI is that it is possible that agents' optimal bundles do not exhaust their budgets. And yet, the currency is artificial; it has no use outside the allocation problem at hand and does not enter the utility function. So we should worry: to what extent do equal budgets actually guarantee that agents will receive acceptable outcomes?

We begin by showing that exact CEEI's guarantee exact maximin shares. The proof will be instructive for the main result of this section, which is that an approximate CEEI guarantees an approximation to maximin shares that is based on adding one more agent to the divide-and-choose procedure.

Proposition 5 *(CEEIs guarantee Maximin Shares). If* $\mathbf{x}^*, \mathbf{b}^*$, and $\mathbf{p}^*$ *constitutes an exact CEEI, then each student receives at least her maximin share.*

Proof. Suppose there exists some agent s_i such that, for

$$\begin{aligned}
\widetilde{\mathbf{x}} &\in \arg\max_{(x_l)_{l=1}^N} \min[u_i(x_1'), ..., u_i(x_N')] \text{ subject to} \\
x_l' &\in \Psi_i \text{ for all } k = 1...N \\
\sum x_{lj}' &\leq q_j \text{ for all } j = 1...M
\end{aligned}$$

we have $u_i(x_i^*) < u_i(\widetilde{x}_l)$ for each $\widetilde{x}_l \in \widetilde{\mathbf{x}}$. By conditions (i) and (iii) of a CEEI we have $\mathbf{p}^* \cdot \widetilde{x}_l > b_i^*$ for each $\widetilde{x}_l \in \widetilde{\mathbf{x}}$, and $\mathbf{p}^* \cdot x_l^* \leq b_i^*$ for each $x_l^* \in \mathbf{x}^*$. But by condition (ii) of a CEEI, any course that has positive price under $\mathbf{p}^*$ is at full capacity under $\mathbf{x}^*$, so $\widetilde{\mathbf{x}}$ cannot cost more in total than $\mathbf{x}^*$. This yields a contradiction:

$$Nb_i^* \geq \sum_l \mathbf{p}^* \cdot x_l^* \geq \sum_l \mathbf{p}^* \cdot \widetilde{x}_l > Nb_i^*$$

[38] Proof. Suppose that there exists a feasible allocation $\widehat{\mathbf{x}}$ that Pareto improves upon $\mathbf{x}^*$ and that uses at most the same number of each strictly-positive priced good as $\mathbf{x}^*$. By condition (i) of the definition of Approximate CEEI, for any agent i for which $\widehat{x}_i \neq x_i^*$ it must be the case that $\mathbf{p}^* \cdot \widehat{x}_i > \mathbf{p}^* \cdot x_i^*$. This implies that the allocation $\widehat{\mathbf{x}}$ has a total cost at $\mathbf{p}^*$ strictly greater than that of $\mathbf{x}^*$, which is a contradiction since prices are non-negative. QED.

If agents have indifferences then Condition (i) of Definition 7 needs to be modified for this proof to work: each agent buys the least cost bundle out of those that maximize their utility subject to the budget constraint.

QED.

The proof that a CEEI guarantees maximin shares relies on two facts about CEEI's: (i) EI means each student has $\frac{1}{N}$ of the endowment; (ii) at a CE price vector $\mathbf{p}^*$ the goods endowment costs weakly less than the income endowment.

The approximate CEEI jeopardizes both of these properties. We need to use two approximation parameters to ensure that the error from (i) and (ii) is minimized. The β budget approximation is the first. The second is defined as follows.

Definition 9. *Fix an economy* $(\mathcal{S}, \mathcal{C}, \mathbf{q}, (\Psi_i)_{i=1}^N, (u_i)_{i=1}^N)$. *For* $\delta \geq 0$ *and budgets* $\mathbf{b}$, *the set* $P(\delta, \mathbf{b}) = \{\mathbf{p} \in [0, \max_i(b_i)]^M : \sum_{j=1}^M p_j q_j \leq \sum_{i=1}^N b_i(1+\delta)\}$.

In words, at any price vector in $P(\delta, \mathbf{b})$, the goods endowment costs at most δ proportion more than the income endowment.

Lemma 1. *Fix an economy* $(\mathcal{S}, \mathcal{C}, \mathbf{q}, (\Psi_i)_{i=1}^N, (u_i)_{i=1}^N)$. *For any* $\delta > 0$ *and any set of target budgets* $\mathbf{b}'$ *there exists an* $(\alpha, \beta)-$A*pproximate CEEI* $\mathbf{x}^*, \mathbf{b}^*, \mathbf{p}^*$ *that satisfies all of the conditions of Theorem 1 and additionally* $\mathbf{p}^* \in P(\delta, \mathbf{b}^*)$.

The proof of Lemma 1 is in Appendix B. The key fact is that the approximate CEEI price vector $\mathbf{p}^*$ guaranteed by Theorem 1 is near a fixed point of $F(\cdot)$ (see 5.3.3). At each price "near" to the fixed point agents can afford their demands, and a convex combination of these demands is feasible. Hence, since $\mathbf{p}^*$ is nearly a fixed point, agents can approximately afford the endowment.

By choosing β, δ small enough we will be able to ensure that each agent's budget is at least $\frac{1}{N+1}$ of the cost of the endowment at $\mathbf{p}^*$. This guarantees that the approximate CEEI approximately satisfies the maximin share guarantee.

Definition 10. *Fix an economy* $(\mathcal{S}, \mathcal{C}, \mathbf{q}, (\Psi_i)_{i=1}^N, (u_i)_{i=1}^N)$. *For* $\widehat{N} \geq N$, *agent* s_i*'s* $\widehat{N}-$*maximin share is*

$$\begin{aligned} \underline{u}_i^{\widehat{N}} &= \max_{(x_l)_{l=1}^{\widehat{N}}} [\min(u_i(x_1), ..., u_i(x_N), ...u_i(x_{\widehat{N}}))] \text{ subject to} \\ x_l &\in \Psi_i \text{ for all } l = 1...N...\widehat{N} \\ \sum x_{lj} &\leq q_j \text{ for all } j = 1...M \end{aligned} \tag{10}$$

Agent i's $\widehat{N}-$*maximin split is the allocation* $x_1, ..., x_N, ...x_{\widehat{N}}$ *in (10). A course-allocation mechanism satisfies the* $\widehat{N}-$*maximin share guarantee if, for any allocation* $\mathbf{x}$ *selected with positive probability,* $u_i(x_i) \geq \underline{u}_i^{\widehat{N}}$ *for all* $i = 1...N$.

Theorem 2. *Fix an economy* $(\mathcal{S}, \mathcal{C}, \mathbf{q}, (\Psi_i)_{i=1}^N, (u_i)_{i=1}^N)$. *If* $\mathbf{x}^*, \mathbf{b}^*, \mathbf{p}^*$ *is an* $(\alpha, \beta)-$*approximate* *competitive equilibrium from equal incomes where, for some $\delta \geq 0$, $\mathbf{p}^* \in P(\delta, \mathbf{b}^*)$ and $\beta < \frac{1-\delta N}{N(1+\delta)}$, then each agent obtains at least their $N+1$-maximin share.*

Proof. See Appendix B.

Theorem 2 indicates that by using an approximately equal distribution of artificial currency (i.e., β small) we can provide an attractive guarantee on ex-post utilities, despite the fact that goods are indivisible, and so much of this artificial currency might go unspent. Specifically, each agent does at least as well as he could guarantee himself as divider in an $N+1$-way divide-and-choose procedure.

We might worry, especially in small markets, about the difference between N and $N+1$ maximin shares. For instance, if there are two agents and two diamonds, the N-maximin share guarantees each agent a diamond, whereas the $N+1$ share does not. It turns out that we can often provide a slightly stronger guarantee than Theorem 2.

Proposition 6. *If $\alpha = 0$ and $\beta < \frac{1}{N-1}$* then *each agent is guaranteed the weaker of*

(a) an outcome weakly better than her N^{th} favorite bundle in her $N+1$-maximin split

(b) an outcome strictly better than her $N+1^{st}$ favorite bundle in her $N+1$-maximin split (i.e., her $N+1$-maximin share).

Proposition 6 guarantees that each agent receives a diamond in the two-diamond two-rock example (Example 1). The approximation error is that the agent who gets the small diamond may get the ugly rock.[39] By contrast, in a dictatorship whichever agent goes first gets both diamonds, while the other is left with only rocks.

6.3 Theorem 3: Approximate CEEI Guarantees Envy Bounded by a Single Object

Exact CEEIs are envy free because all agents have the same choice set. Formally,

Remark 3. *(CEEIs are Envy Free). If* $\mathbf{x}^*, \mathbf{b}^*$, and $\mathbf{p}^*$ *constitutes a CEEI, then the allocation* $\mathbf{x}^*$ *is envy free.*

Proof. If $x_j^* \in \Psi_i$ then condition (iii) of a CEEI implies that $\mathbf{p}^* \cdot x_j^* \leq b_i^*$. If $x_j^* \neq x_i^*$ then condition (i) of a CEEI implies $u_i(x_i^*) > u_i(x_j^*)$. So s_i does not envy s_j. QED

[39] For these examples it is easy to see that there exists a $(0, \beta)$-Approximate CEEI for any $\beta > 0$. Each agent's $N+1$-maximin split is $\{\{A\}, \{B\}, \{C, D\}\}$, so part (a) of the guarantee is a bundle weakly better than $\{B\}$, and part (b) of the guarantee is a bundle strictly better than $\{C, D\}$ (which happens to coincide with part (a); this is not the case generally). Market clearing then implies each agent gets at least $\{B, D\}$.

When agents have unequal incomes they have different choice sets, and so envy-freeness cannot be assured. The idea of the following result is: if inequality in budgets is sufficiently small, then we can bound the difference in agents' choice sets, and hence the degree of envy.

Theorem 3. *Fix an economy* $(\mathcal{S}, \mathcal{C}, \mathbf{q}, (\Psi_i)_{i=1}^N, (u_i)_{i=1}^N)$*. If* $\mathbf{x}^*, \mathbf{b}^*, \mathbf{p}^*$ *is an* $(\alpha, \beta)-$*approximate competitive equilibrium from equal incomes and* $\beta < \frac{1}{k-1}$ *(where* k *is the maximum number of courses per student) then* $\mathbf{x}^*$ *satisfies envy bounded by a single object.*

Proof. Suppose for a contradiction that s_i envies s_j, and this envy is not bounded by a single object. Let $k' \leq k$ denote the number of courses in s_j's bundle x_j^* and number these courses $(c_1, ..., c_{k'})$. Then we have:

$$\begin{array}{rcl} u_i(x_j^* \backslash \{c_1\}) & > & u_i(x_i^*) \\ & & ... \\ u_i(x_j^* \backslash \{c_{k'}\}) & > & u_i(x_i^*) \end{array}$$

Condition (i) of the definition of the Approximate CEEI indicates that s_i cannot afford any of these k' bundles formed by removing an object from x_j^*:

$$\begin{array}{rcl} \mathbf{p}^* \cdot (x_j^* \backslash \{c_1\}) & > & b_i^* \\ & & ... \\ \mathbf{p}^* \cdot (x_j^* \backslash \{c_{k'}\}) & > & b_i^* \end{array}$$

Since $p_1^* + p_2^* + ... + p_{k'}^* = \mathbf{p}^* \cdot x_j^* \leq b_j^*$ we can sum these inequalities to obtain

$$(k'-1)b_j^* \geq (k'-1)(\mathbf{p}^* \cdot x_j^*) > k' b_i^*$$

which implies that $\frac{b_j^*}{b_i^*} \geq \frac{k'}{k'-1}$. Since $k' \leq k$ we have $\frac{b_j^*}{b_i^*} \geq \frac{k}{k-1}$. So if $\beta < \frac{1}{k-1}$ we have a contradiction. QED.

Note that the degree of budget inequality necessary to guarantee envy bounded by a single object ($\beta < \frac{1}{k-1}$) is different from that required to guarantee agents their approximate maximin shares (essentially, $\beta < \frac{1}{N}$). Theorem 1 allows us to choose β arbitrarily small, so we can ensure that we satisfy the requirements of Theorems 2 and 3.

7 The Approximate CEEI Mechanism

This section develops the Approximate CEEI Mechanism. The existence theorem ensures that such a mechanism will be well defined. We will ensure that budget inequality is sufficiently small that we can apply Theorems 2 and 3 to guarantee approximate outcome fairness. By allocating budgets randomly and then choosing randomly amongst multiple equilibria, we can ensure that the mechanism is symmetric and strategyproof in a large market. We conclude the section by summarizing the mechanism's properties and contrasting with those of other known mechanisms.

7.1 Definition: the Approximate CEEI Mechanism

Approximate CEEI Mechanism. *Fix an economy* $(\mathcal{S}, \mathcal{C}, \mathbf{q}, (\Psi_i)_{i=1}^N, (u_i)_{i=1}^N)$*. The Approximate CEEI Mechanism is the following procedure:*

1. *Agents report utility functions* $(\widehat{u}_i)_{i=1}^N$.[40]
2. *The mechanism computes the set of CEEI prices of the economy* $(\mathcal{S}, \mathcal{C}, \mathbf{q}, (\Psi_i)_{i=1}^N, (\widehat{u}_i)_{i=1}^N)$*, normalizing budgets to* $\mathbf{b}^* = (1, ..., 1)$*.*
3. *If the set of CEEI prices is non-empty, the mechanism chooses uniform randomly from this set and announces the prices* $\mathbf{p}^*$*, the corresponding allocation* $\mathbf{x}^*$*, and budgets* $\mathbf{b}^*$*.*
4. *If the set of CEEI allocations is empty, the mechanism chooses a target budget vector* $\mathbf{b}'$ *by choosing non-identical budgets uniform randomly from* $[1, 1+\beta]$*, for some* $\beta < \min(\frac{1}{N}, \frac{1}{k-1})$*. Then, for* ε *smaller than the smallest difference between budgets,* $\delta < 1 - N\beta$*, and* $\alpha \leq \frac{\sqrt{\sigma M}}{2}$*, the mechanism computes the set of* $(\alpha, \beta)-$*Approximate CEEI price vectors in* $P(\delta)$ *that have corresponding budget vectors within* ε *of the target* $\mathbf{b}'$*.*
5. *The set of Approximate CEEI price vectors is guaranteed to be non-empty by Theorem 1. The mechanism chooses uniform randomly amongst those in this set with the smallest* α*, and announces the prices* $\mathbf{p}^*$*, and the corresponding budgets* $\mathbf{b}^*$ *and allocation* $\mathbf{x}^*$*.*

Steps 2 and 3 seek an exact CEEI, which is particularly attractive but may not exist. Step 4 uses Theorems 2 and 3 to compute a set of Approximate CEEI's that satisfy the criteria of outcome fairness. Theorem 1 guarantees that the set asked for in Step 4 is non-empty. This set is computable in principle because the number of budget-constraint hyperplanes is finite, hence so is

[40] The number of possible schedules can be quite large, so as a practical matter the procedure must provide a way for students to express their preferences concisely. A natural starting point for the reporting language is Milgrom's (2008) class of assignment messages. See the discussion in Othman, Budish and Sandholm (2008).

the number of regions of price space we have to check. A specific algorithm that computes prices will be discussed in Section 8.1.

7.2 The Approximate CEEI Mechanism is Strategyproof in a Large Market

In this section we will show that the Approximate CEEI Mechanism is strategyproof in a large market. The first step is to define Approximate CEEI for continuum economies.

Definition 11. *Fix an economy* $(\mathcal{S}, \mathcal{C}, \mathbf{q}, (\Psi_i)_{i=1}^N, (u_i)_{i=1}^N)$, *and consider its continuum replication* $(\mathcal{S}^\infty, \mathcal{C}, \mathbf{q}, (\Psi_i^\infty)_{i=1}^N, (\widehat{u}_i^\infty)_{i=1}^N)$. *The allocation* $\mathbf{x}^* = (x_i^*)_{i\in(0,N]}$, *budgets* $\mathbf{b}^* = (b_i^*)_{i\in(0,N]}$ *(normalized so that* $\inf_i(b_i^*) = 1$*) and prices* $\mathbf{p}^* = (p_1^*, ..., p_M^*)$ *constitute an* $(\alpha, \beta)-$*Continuum Approximate CEEI if:*

(i) $x_i^* = \underset{x_i' \in \Psi_i}{\arg\max}[u_i(x_i') : \mathbf{p}^* \cdot x_i' \leq b_i^*]$ *for all* $i \in (0, N]$

(ii) $||\mathbf{z}^*|| \leq \alpha$ *where* $\mathbf{z}^* = (z_1^*, ..., z_M^*)$ *and*

$z_j^* = \int_{\mathcal{S}^\infty} x_{ij}^* ds_i - q_j$ *if* $p_j^* > 0$

$z_j^* = \max(\int_{\mathcal{S}^\infty} x_{ij}^* di - q_j, 0)$ *if* $p_j^* = 0$

(iii) $\sup_i b_i^* \leq 1 + \beta$

Notice that we have defined α as the measure of market-clearing error. Theorem 1 suggests that we can find market-clearing error of zero in continuum economies. In fact, there is a much simpler proof available in continuum economies, because, given an atomless budget distribution and a finite set of possible consumption bundles, it is easy to show that excess demand is continuous with respect to price. Standard fixed-point arguments then imply the existence of an exact competitive equilibrium price vector; in particular, so long as the target budget distribution $\mathbf{b}'$ is atomless, there is no need to perturb budgets by ε and so we just set $\mathbf{b}^* = \mathbf{b}'$. It is also simple to see that the Approximate CEEI Mechanism has attractive incentive properties in the continuum limit.

Proposition 7. *The Approximate CEEI Mechanism is Strategyproof in a Large Market.*

Proof. Consider student s_i. Fix an arbitrary set of reports $\widehat{\mathbf{u}} = (\widehat{u}_i, \widehat{\mathbf{u}}_{-i}^\infty)$ where $\widehat{\mathbf{u}}_{-i}^\infty$ denotes the reports of students in $\mathcal{S}^\infty \backslash \{s_i\}$. The distribution of the reports $\widehat{\mathbf{u}}$ (more specifically, the distribution of the ordinalizations of these reports) determines the set of CEEI price vectors in Step 2. If this set is empty, then in Step 4 s_i is randomly allocated a budget b_i^*, and then the distribution of $\widehat{\mathbf{u}}$ and the randomly assigned budgets determines the set of Approximate CEEI price vectors. Student s_i's own report cannot affect the distribution of $\widehat{\mathbf{u}}$, nor the amount of market-clearing error for any price vector, because he is zero measure. So the price vector $\mathbf{p}^*$ is chosen independently of his report, and his budget b_i^* similarly is independent of his report. If s_i reports $\widehat{u}_i(\cdot)$ he is allocated

$x_i^* = \underset{x_i' \in \Psi_i}{\arg\max}[\widehat{u}_i(x_i') : \mathbf{p}^* \cdot x_i' \leq b_i^*]$. Clearly he can do no better than to report his preferences truthfully. QED

It is possible to generalize Step 5 of the Approximate CEEI Mechanism in certain ways without jeopardizing large-market strategyproofness. For instance, a market administrator might specify a penalty function in terms of α and β (or more complicated statistics of market-clearing error and budget inequality) and seek the best such Approximate CEEI. All that is important for the property of strategyproof in the limit is that the tiebreaking is based on aggregate features of the Approximate CEEI rather than individuals' allocations, so that in a continuum economy the probability that any agent affects the market administrator's choice of prices or budgets is zero.

7.3 Summary: the Case for the Approximate CEEI Mechanism

Properties of the Approximate CEEI Mechanism. *In any economy, the Approximate CEEI Mechanism has the following properties.*

Efficiency

- *Ex-post efficient with respect to the allocated goods.*

Fairness

- *Symmetric*
- *N+1 Maximin Share Guaranteed*
- *Envy Bounded by a Single Good*

Incentives

- *Strategyproof in the Limit*

The best way to argue that the Approximate CEEI Mechanism is an attractive compromise of the competing objectives is by comparing it to the other course-allocation mechanisms used in practice and proposed in theory. We begin by noting its relationship to Random Serial Dictatorship.

Remark 4. *In single-unit assignment problems, the Approximate CEEI Mechanism coincides with Random Serial Dictatorship. In course-allocation (i.e., multi-unit assignment) problems, RSD corresponds to a competitive equilibrium mechanism in which* $\mathbf{b}^*$ *is a uniform random permuation of* $(1, (1+k), (1+k)^2, ..., (1+k)^{N-1})$*, with k the maximum number of courses per student.*

Proof. (Single-unit assignment) A CEEI exists if and only if all agents have a different favorite object. In this case, RSD and the Approximate CEEI Mechanism clearly coincide. Suppose a CEEI does not exist. Then the strict budget order selected in Step 4 of the Approximate CEEI Mechanism plays the same role as the serial order selected in RSD. Specifically, the RSD allocation can be supported as a $(0, \beta)$-Approximate CEEI by the price vector in which an object's price is equal to the budget of the last student to obtain a copy of it, or zero if it does not reach capacity. Any other allocation has strictly-positive market-clearing error at any price vector: if the agent with the l^{th} highest budget obtains an allocation at some price vector $\mathbf{p}'$ that is strictly better than he receives under the serial dictatorship then one of the first l objects selected in the serial dictatorship must be over-allocated at $\mathbf{p}'$, and vice versa. So the RSD allocation is selected in Step 5 of the Approximate CEEI Mechanism.

(Multi-unit assignment) Run a serial dictatorship with the same serial order as the randomly selected budget order. If a course reaches capacity at the l^{th} student's turn, set its price equal to $\frac{1}{k}$ times the l^{th} highest budget. If a course never reaches capacity, set its price equal to zero. At this price vector, each agent consumes her most preferred bundle that consists of courses available at her turn in the serial dictatorship, and market-clearing error is zero. QED.

Remark 4 may help us further understand why RSD is observed often in practice for single-unit assignment (e.g., school choice) but not for multi-unit assignment. The following table compares the Approximate CEEI Mechanism's properties to those of all other known course-allocation mechanisms, both from theory and practice.

Insert Table 2: Comparison of Alternative Mechanisms

Table 2 shows that the Approximate CEEI Mechanism is on the "efficiency-fairness-incentives frontier". Every other known mechanism is unfair ex-post, manipulable in large markets, or both unfair and manipulable.

8 Ex-Ante Efficiency of the Approximate CEEI Mechanism

The proposed mechanism has an element of randomness, and so it can be argued that efficiency should be assessed ex-ante, rather than ex-post. In random assignment contexts such as this one, ex-ante efficiency is a strictly stronger criterion: a necessary but not sufficient condition for a lottery over allocations to be ex-ante efficient is that all realizations of the lottery are ex-post efficient.[41]

In this section I empirically assess the ex-ante efficiency properties of the proposed mechanism in a specific course-allocation environment. Theorem 1 is non-constructive, and so I begin by describing the algorithm used to compute approximate CEEI prices (Section 8.1). Section 8.2 describes the data, which comes from the course-allocation procedure at Harvard Business School.

[41] As discussed in the introduction, fairness criteria should be assess ex-post, both because ex-post is the perspective that matters to market administrators, and because ex-post is actually the more stringent perspective. For instance, a random serial dictatorship is ex-ante envy free even though it results in very unequal outcomes ex post.

Section 8.3 examines market-clearing error: it is substantially smaller than even the Theorem 1 bound, suggesting that ex-post inefficiencies are small. Section 8.4 examines how sensitive students' outcomes are to the randomness of the mechanism. This exercise suggests that ex-post efficiency is likely to be a reasonable proxy for ex-ante efficiency. Finally, in Section 8.5 I directly assess the ex-ante efficiency of the proposed mechanism, by comparing it to that used at HBS. The proposed mechanism is superior to the actual strategic play of the HBS draft mechanism, but, notably, is a bit worse than a non-equilibrium counterfactual in which students play the HBS draft mechanism truthfully.

8.1 Approximate CEEI Algorithm

Theorem 1 is non-constructive and so computing Approximate CEEI prices is non-trivial. There are two computational challenges. The first is that calculating demands is NP Hard: the problem of solving for an agent's demand at a particular price vector is formally equivalent to a set-packing (i.e., knapsack) problem. The complexity of solving for an agent's demand grows with the number of bundles he must consider, which itself grows exponentially with the maximum number of courses per bundle. The second is that even if excess demand were easy to compute, finding an approximate zero of excess demand is a challenging search problem.

Othman, Budish and Sandholm (2008) develop a computational procedure that overcomes these two challenges in life-size problems. Agents' demands are calculated using an integer program solver, CPLEX. Our search procedure takes a traditional tâtonnement search process – which Scarf (1960) showed can cycle even in economies with divisible goods and convex preferences – and enhances it using an artificial-intelligence method called Tabu Search. There are two basic ideas to the enhancement. First, we consider not only a tâtonnement adjustment of the form $\mathbf{p}^{t+1} = \mathbf{p}^t + \mathbf{z}(\mathbf{p}^t)$ but also adjustments that raise or lower just a single price at a time. This set of potential adjustments is called the neighborhood of $\mathbf{p}^t$. Second, of this neighborhood, the algorithm travels to the price vector that has the lowest market-clearing error, except that we avoid prices that have an excess demand vector that has been encountered recently (the "Tabu List"). That is, the algorithm often travels in a seemingly less attractive direction, in an attempt to avoid cycles.[42]

The algorithm can currently handle problems that are the size of a single life-size semester (5 courses per student; 50 courses overall of which 20 are scarce; 456 students).

[42] Russell and Norvig (2002; Chapter 4) provide an overview of Tabu Search. See also http://en.wikipedia.org/wiki/Tabu_search. Our algorithm stops when it has (i) found a price vector with market clearing error within the Theorem 1 bound; (ii) gone 100 iterations without further improvement. The algorithm does not explicitly calculate the full set of Approximate CEEI price vectors.

8.2 Data and Key Assumptions

I use the Budish and Cantillon (2008) data on course allocation at Harvard Business School (HBS) for the 2005-2006 academic year. The data consist of students' true and stated ordinal preferences over 50 Fall semester courses and 47 Spring semester courses, as well as these courses' capacities. Data on true course preferences are generally difficult to obtain because all of the course-allocation mechanisms used in practice are easy to manipulate. Budish and Cantillon (2008) use a survey conducted by HBS a few days' prior to the run of its mechanism as the proxy for true preferences, and then corroborate this assumption by running an additional survey several months after the run of the mechanism. We have the stated preferences for all 916 HBS students, and true preferences for the 456 who filled out the survey. In the analysis we consider an economy with just the 456, adjusting course capacities proportionally. Robustness checks reported in Budish and Cantillon (2008) indicate that there are no systematic differences in strategic preferences between the 456 who filled out the survey and the 460 who did not.

In order to convert the HBS data on ordinal preferences over individual courses into data on students' preferences over lotteries for bundles of courses, I need to make additional assumptions. I assume: (A1) preferences are additively-separable; and (A2) students care about the average rank of the courses they receive (e.g. they prefer their 2nd and 3rd favorite courses to their 1st and 5th because $2.5 < 3$). These assumptions seem reasonable for handling the data incompleteness problem for two reasons: (i) the HBS elective-year curriculum is designed to avoid complementarities and overlap between courses; and (ii) in the HBS draft mechanism[43] students are unable to express the intensity of their preference for individual courses beyond ordinal rank. Preliminary exploration of preferences more complex than average-rank suggest that all of the reported results are robust. In particular, the performance of the HBS draft mechanism deteriorates relative to the Approximate CEEI Mechanism when there are complementarities or intense preferences, making the welfare difference found in Section 8.5 more pronounced.

The other main substantive assumption is (A3) that students report their preferences truthfully under the Approximate CEEI Mechanism. While we know that the mechanism is strategyproof in continuum markets, we have no way of assessing whether 916 students is large enough to provide incentives for truthful reporting. The main obstacles are: (i) the number of potential misreports is large;[44] and (ii) we have no theoretically-motivated way to restrict attention to some subset of

[43] The HBS Draft Mechanism works as follows. Students report their ordinal preferences over individual courses. A computer assigns each student a random priority number. Then, over a series of rounds, it chooses courses for the students one at a time based on their reported preferences. In rounds 1, 3, 5, ... the computer proceeds through students in ascending order of the random priority numbers, whereas in rounds 2, 4, 6, ... it proceeds in descending order. At each turn, the choosing student is given his most-preferred course that (i) he has not yet received; (ii) is not yet at capacity.

[44] There are 50 courses per semester, and each student ranks about 15 courses per semester. So there are about

these potential manipulations, unlike in Roth and Peranson (1999).[45]

Two other small assumptions are necessary for the analysis. First, I treat each semester's allocation problem separately. This is due to computational limitations of the Othman et al (2008) algorithm, which currently can handle semester-size problems (five courses per student) but not full-year size problems. Second, I ignore scheduling constraints, but for the constraint that each student takes at most five courses per semester.

8.3 How Large is Market-Clearing Error?

Theorem 1 indicates that, so long as budgets are unequal, there exist prices that clear the HBS course-allocation market to within market-clearing error of $\frac{\sqrt{2kM}}{2}$ (Euclidean distance), where k is the number of courses per student, and M is the number of courses. Here, $k = 5$ and $M = 50, 47$ for the Fall and Spring semesters, respectively. So the market clearing bound is $\frac{\sqrt{2kM}}{2} \approx 11$.

Figure 1 reports the actual market clearing error over 100 runs of the Approximate CEEI Mechanism on the Fall and Spring semesters of course allocation at HBS. Each run corresponds to a different random budget ordering. There are 456! possible orderings, and each run takes about one hour of computational time, so we are able to explore just a tiny fraction of them.

[Insert Figure 1: Distribution of Market Clearing Errors]

The actual error is substantially smaller than the bound implied by Theorem 1. The maximum observed error in Euclidean Distance is $\sqrt{14}$ in the Fall and $\sqrt{15}$ in the Spring (versus the Theorem 1 bound of $\sqrt{125}$ and $\sqrt{117.5}$). In terms of seats, the maximum observed error is 14 seats in the Fall and 11 seats in the Spring, with averages of 6.04 and 5.50, respectively.

8.4 How Sensitive is Ex-Post Utility to Income?

There are two elements of randomness in the proposed mechanism. First, budgets are distributed uniform randomly on $[1, 1+\beta]$. Second, for a given realization of budgets there might be multiple approximate CEEI price vectors with equivalent market-clearing error.

$\frac{50!}{(50-15)!} \approx 2 \times 10^{12}$ possible reports, even within the restricted class of additive-separable average-rank preferences.

[45] I have run computational experiments on a highly stylized economy in which the number of reports is so small that we can search exhaustively for profitable manipulations. Specifically, there are four courses; students have additive-separable preferences, with student i's value for course j given by $u_{ij} = v_j + \varepsilon_{ij}$, with average course values of $\mathbf{v} = (1, 2, 3, 4)$ and $\varepsilon_{ij} \sim N(0, 1)$ i.i.d. across students and courses. There are N agents, and each course has capacity $\lceil \frac{1.1N}{2} \rceil$, i.e., there is 10% excess capacity.

Student i has a Bayesian manipulation if he has a misreport that is profitable on average over a large number of opponent value draws and random budget draws. I find some Bayesian manipulations for small N, but not for $N \geq 200$. The results are very preliminary, and it is difficult to extrapolate much from this highly stylized environment, but at least these results suggest that the mechanism provides exact incentives in some finite markets.

If students' outcomes are invariant across trials, then ex-post efficiency is an exact proxy for ex-ante efficiency. We explore how sensitive students' outcomes are to the randomness of the mechanism by looking, student by student, at the difference in utility between their single best and single worst outcome over the 100 trials. See Figure 2.

[Insert Figure 2: Distribution of Difference Between Best and Worst Outcomes]

Around half of the students' outcomes are invariant over the trials. For another 15%, the difference between their best and worst outcome is a single rank, i.e., the difference between getting one's l^{th} and $l+1^{st}$ favorite course, for some l. For about 3.5% of students in the Fall and 5% in the Spring, the maximum difference is 4-5 ranks, and the maximum observed difference is 7.

These differences seem small, especially by contrast with dictatorships. Students who get to choose early in a dictatorship get their top-five favorite courses, whereas students who choose late might get a very poor selection. In simulations of a Random Serial Dictatorship, over 80% of students have an observed maximum difference in excess of 20 ranks.[46]

8.5 How Does Ex-Ante Efficiency Compare Against an Alternative?

In order to assess the welfare performance (i.e., distribution of realized utilities) of the Approximate CEEI Mechanism we need a benchmark. The HBS Draft Mechanism is a natural candidate for comparison. First, variants of the mechanism are used at other universities and in many other allocation contexts (see Brams and Straffin, 1979). Second, our data consist of students' actual strategic reports under this mechanism, so we do not need to solve for equilibrium play. Third, Budish and Cantillon (2008) show that even though the HBS Draft Mechanism is manipulable in theory and manipulated in practice, it performs better than a strategyproof dictatorship on several measures of ex-ante efficiency. So in a sense the HBS Draft Mechanism is the current "high-water mark" in course-allocation mechanism design.

We have assumed that students' ordinal preferences over bundles are based on the average rank of the courses contained in each bundle. We have not yet made any additional assumption about how their cardinal utilities depend on average rank. For instance, if x_i' has a lower average rank for s_i than x_i'', then we have assumed $u_i(x_i') > u_i(x_i'')$ but have not made any assumptions on the magnitude of this difference. This in mind, let us now look at the distribution of average ranks for

[46] Budish and Cantillon (2008) show that the variability in outcomes in RSD translates to substantial ex-ante inefficiency under the assumption of average-rank preferences. The theoretical intuition is simple: lucky students who get early serial numbers make their *last* choices independently of whether these courses would be some unlucky student's *first* choice. The lucky students "callously" disregard the preferences of the unlucky students.

Ex-ante, students do not know whether they will be early or late in the choosing order, and regard the distribution over callous outcomes as unattractive.

each of the two mechanisms. See Figure 3.

[Insert Figure 3: Average Rank Comparison: Approximate CEEI versus HBS]

In each semester, the distribution of average ranks under the Approximate CEEI Mechanism first-order stochastically dominates that under the HBS Draft Mechanism. First-order stochastic dominance is an especially strong comparison relation: we do not need to make any further assumptions on how utility responds to average rank to reach a welfare comparison.

There are two equivalent ways to interpret the f.o.s.d. finding. First, utilitarian school administrators should prefer the Approximate CEEI Mechanism to the HBS Draft Mechanism. Second, a student who knows the distribution of outcomes but does not yet know his own preferences – i.e., a student behind a veil of ignorance in the sense of Harsanyi (1953) or Rawls (1971) – should prefer the Approximate CEEI Mechanism to the HBS Draft Mechanism.

Notably, the HBS Draft Mechanism would be a bit *better* than the Approximate CEEI Mechanism if students submitted their true preferences under the HBS procedure. The mean average ranks are 4.09 in the Fall and 4.40 in the Spring under HBS-Truthful, versus 4.24 and 4.44 for Approximate CEEI.[47] The cause of this difference is that the Approximate CEEI Mechanism respects students' incentives to misreport their preferences. This has a cost in terms of average welfare.[48]

[47] In the Fall there is a second-order stochastic dominance relation in addition to the difference in means. There is no dominance relation in the Spring.

[48] For instance, suppose that student s_i's four favorite courses are very unpopular her fifth favorite course, c_j, is popular. She has a natural strategic manipulation under the HBS draft mechanism - by asking for c_j in the first round, she is likely to get it, without undermining her chances of getting the other courses she likes. This is likely to be bad for welfare as measured by average rank – because c_j is so popular, there is probably some student who likes it much higher than fifth who no longer gets it, because of s_i. The Approximate CEEI Mechanism respects s_i's incentive constraint: because the top four courses he likes will have price zero, he is very likely to get c_j even if he ranks it fifth. By contrast, under truthful play of the HBS draft mechanism he probably does not get c_j, which is good for welfare as measured by average rank, because it allows someone else who ranks it more highly to get it instead.

9 Conclusion

Combinatorial assignment is a problem of theoretical and practical importance. Most of what is known about the problem are impossibility theorems which indicate that there is no perfect solution.

This paper's solution gets around the impossibility theorems by seeking tight approximations of the ideal properties a mechanism should satisfy. Ideally, a course-allocation mechanism would be both ex-post and ex-ante efficient. My proposed solution is approximately ex-post efficient in theory (Theorem 1), and has attractive ex-ante efficiency performance in a specific empirical environment. Ideally, a course-allocation mechanism would satisfy the outcome fairness criteria of maximin share guarantee and envy freeness. My proposed solution approximates these two ideals (Theorems 2 and 3). Ideally, a course-allocation mechanism would be strategyproof. My proposed solution satisfies a large-market notion of strategyproofness.

There are many open questions for future research. Can we improve the market-clearing approximation bound of Theorem 1 for restrictive classes of preferences (e.g. additive-separable)? How closely can we approximate fair outcomes if we require exact as opposed to approximate market clearing? What are the tradeoffs between ex-ante efficiency and ex-post fairness? Is there a Bayesian argument that the mechanism is difficult to manipulate in finite markets? What if there are multiple sets of objects that cannot be allocated simultaneously (e.g., shift allocation over multiple months)?

I close on a methodological note. Practical market-design problems often prompt the development of new theory that enhances and extends old ideas. To give a prominent example, the elegant matching model of Gale and Shapley (1962) was not able to accommodate several complexities found in the practical design problem of matching medical residents to residency positions. This problem prompted the development of substantial new theory (summarized in Roth (2002)) and a new market design described in Roth and Peranson (1999). Similarly, the beautiful theory of Competitive Equilibrium from Equal Incomes developed by Foley (1967), Varian (1974) and others is too simple for practice because it assumes perfect divisibility. This paper proposes a richer theory that accommodates indivisibilities, and develops a market design based on this richer theory. I hope that, just as a concrete application renewed interest in Gale and Shapley's remarkable deferred-acceptance algorithm, this paper and its motivating application will renew interest in CEEI as a framework for market design.

A Proof of Theorem 1

Preliminaries

Fix an economy $(\mathcal{S}, \mathcal{C}, \mathbf{q}, (\Psi_i)_{i=1}^N, (u_i)_{i=1}^N)$, and fix $\beta > 0, \varepsilon > 0$. Let $\mathbf{b}' = (b_1', ..., b_N')$ be any vector of budgets that satisfies $\max_i(b_i') \leq 1 + \beta$ and $\min_i(b_i') = 1$. In particular, $\mathbf{b}'$ can be the target budgets specified by the market administrator.

Notationally, throughout the proof I will use markers like $'$ and * (e.g., $\mathbf{b}'$ and $\mathbf{b}^*$) to denote specific instances of objects, and will not use such markers (e.g., $\mathbf{b}$) when defining functions. Boldface is used to indicate objects that pertain to all students (e.g., an allocation $\mathbf{x} = (x_1, ..., x_N)$ or price vector $\mathbf{p} = (p_1, ..., p_M)$). Vectors are not bolded when they pertain to just a single student; e.g., an individual schedule $x_i = (x_{i1}, ..., x_{iM})$ is not bolded. The generic student/agent s_i is sometimes referred to simply as i when this does not cause confusion.

Let $\bar{b} = 1 + \beta + \varepsilon$. Define an $M-$dimensional price space by $P = [0, \bar{b}]^M$. For much of the proof we will work with an enlargement of this space $\widetilde{P} = [-1, \bar{b} + 1]^M$ in order to handle a boundary issue that arises because excess supply is allowable for goods at price zero but not for goods with strictly positive prices.

Define a truncation function $t : \widetilde{P} \to P$ that takes any price vector in $\widetilde{P}$ and truncates all prices to be within $[0, \bar{b}]$. Formally, $t(\mathbf{p}) = (t_1(\mathbf{p}), ..., t_M(\mathbf{p}))$ with $t_j(\mathbf{p}) = \min(\bar{b}, \max(0, p_j))$.

In Step 2 we will assign to each agent i -bundle x pair a small reverse-tax $\tau_{ix} \in (-\varepsilon, \varepsilon)$ that affects i's cost of purchasing x: at prices $\mathbf{p}$ her total cost is $\mathbf{p} \cdot x - \tau_{ix}$ (that is, a positive τ_{ix} decreases the price of x to i).

Demand and excess demand are defined on all prices in $\widetilde{P}$ (including negative prices). Agent i's demand $d_i(\cdot)$ depends on prices $\mathbf{p}$, her budget b_i, and the set of taxes $\tau_i \equiv (\tau_{ix})_{x \in \Psi_i}$:

$$d_i(\mathbf{p}, b_i, \tau_i) = \underset{x' \in \Psi_i}{\arg\max}(u_i(x') : \mathbf{p} \cdot x' \leq b_i + \tau_{ix'}) \tag{11}$$

Let $\boldsymbol{\tau} \equiv (\tau_i)_{i=1,...,N}$. Excess demand $\mathbf{z}(\cdot)$ is defined by

$$\mathbf{z}(\mathbf{p}; \mathbf{b}, \boldsymbol{\tau}) = \sum_{i=1}^{N} d_i(\mathbf{p}, b_i, \tau_i) - \mathbf{q}. \tag{12}$$

We will suppress the $\mathbf{b}$ and $\boldsymbol{\tau}$ arguments from $d_i(\cdot)$ and $\mathbf{z}(\cdot)$ when their values are clear from the context. (Usually we are interested in how $d_i(\cdot)$ and $\mathbf{z}(\cdot)$ move with price).

Since each agent consumes either 0 or 1 of each object, it is without loss of generality to assume $q_j \in \{1, ..., N\}$ and so $-N \leq z_j \leq N - 1$ for all $j = 1...M$.

For agent $i = 1, ..., N$, schedule $x \in \Psi_i$, define the budget-constraint-hyperplane $H(i, x)$ by

$H(i,x) \equiv \{\mathbf{p} \in \widetilde{P} : \mathbf{p} \cdot x = b_i + \tau_{ix}\}$. Each budget-constraint hyperplane is of dimension $M-1$.

Both the taxes and the enlarged price space play a role that is entirely internal to the proof. At the end we will have a price vector in P and set all of the taxes to zero.

Step 1. *Define a standard tâtonnement price-adjustment function f on $\widetilde{P}$.* If f has a fixed point $\widetilde{\mathbf{p}}^* = f(\widetilde{\mathbf{p}}^*)$, *then its truncation $\mathbf{p}^* = t(\widetilde{\mathbf{p}}^*)$ is an exact competitive equilibrium.*

We define a standard tâtonnement price-adjustment function on the enlarged price space $\widetilde{P}$. Let $\gamma \in (0, \frac{1}{N})$ be a small positive constant. Given budgets $\mathbf{b}$ and taxes $\boldsymbol{\tau}$ define $f : \widetilde{P} \to \widetilde{P}$ by:

$$f(\widetilde{\mathbf{p}}) = t(\widetilde{\mathbf{p}}) + \gamma \mathbf{z}(t(\widetilde{\mathbf{p}}); \mathbf{b}, \boldsymbol{\tau}) \tag{13}$$

The reason we impose $\gamma < \frac{1}{N}$ is to ensure the image of f lies in $\widetilde{P}$.

Suppose, for budgets of $\mathbf{b} = \mathbf{b}'$ and taxes of $\boldsymbol{\tau} = \mathbf{0}$, that f has a fixed point $\widetilde{\mathbf{p}}^* = f(\widetilde{\mathbf{p}}^*)$. Then its truncation $\mathbf{p}^* = t(\widetilde{\mathbf{p}}^*)$ is an exact competitive equilibrium price vector for budgets of $\mathbf{b}'$.

First, note that at any fixed point no individual price $p_j^* \geq \bar{b}$. Given the definition of $\bar{b}$ no agents can afford a seat in course c_j at price $\bar{b}$. So $\widetilde{p}_j^* \geq \bar{b}$ implies $z_j(\mathbf{p}^*; \mathbf{b}', \mathbf{0}) \leq 0 - q_j < 0$ which contradicts $\widetilde{p}_j^* \geq \bar{b}$ being part of a fixed point since $f_j(\widetilde{\mathbf{p}}^*) = \bar{b} + \gamma z_j(\mathbf{p}^*; \mathbf{b}, \boldsymbol{\tau}) < \bar{b}$.

Second, note that $p_j^* \in (0, \bar{b}]$ implies $z_j(\mathbf{p}^*; \mathbf{b}', \mathbf{0}) = 0$. Finally, $p_j^* = 0$ implies that $z_j(\mathbf{p}^*; \mathbf{b}, \boldsymbol{\tau}) \leq 0$. So, if $f(\cdot)$ has a fixed point we have perfect market clearing.

Discussion.

Unfortunately f is not continuous, so there is no guarantee that such a fixed point will exist. In particular, f is potentially discontinuous at every price vector that is on a budget-constraint hyperplane. Every time price crosses some budget-constraint hyperplane $H(i,x)$, agent i's choice set changes: he can afford bundle x if $\mathbf{p} \cdot x \leq b_i'$ but not if $\mathbf{p} \cdot x > b_i'$. If price crosses just a single such hyperplane the discontinuity is "small" - it is bounded by how much a single agent can change their demand (i.e. $\sqrt{\sigma}$). If agents have equal budgets their budget-constraint hyperplanes coincide, so it is impossible for price to cross just one at a time. Discontinuities in f might be "large".

The purpose of step 2 is to perturb the budget-constraint hyperplanes using taxes $\boldsymbol{\tau}$ so that the potential for discontinuities in f is mitigated in two ways: (i) no two hyperplanes coincide; and (ii) no more than M hyperplanes intersect at a single point.

Step 3 then defines a convexification of f that smoothes f as price crosses the budget-constraint hyperplanes. The convexification, F, is guaranteed to have a fixed point by Kakutani's theorem.

The remainder of the proof uses the hyperplane structure provided by step 2 and the definition of F in step 3 to show that there must be a price vector arbitrarily close to the fixed point of F

that clears the market to within the Theorem 1 approximation bound of $\frac{\sqrt{\sigma M}}{2}$. Care is taken along the way to ensure the other conditions of Theorem 1 are satisfied.

Step 2. *Choose taxes* $(\tau^*_{ix})_{i,x\in\Psi_i}$ *such that*

(i) $-\varepsilon < \tau^*_{ix} < \varepsilon$ *(taxes are small)*

(ii) $\tau^*_{ix} > \tau^*_{ix'}$ *if* $u_i(x) > u_i(x')$ *(taxes favor more-preferred bundles)*

(iii) $\max_{i,x}(b'_i + \tau^*_{ix}) \leq \max_i(b'_i)$, $\min_{i,x}(b'_i + \tau^*_{ix}) \geq \min_i(b'_i)$; *(inequality bound is preserved)*

(iv) $b'_i + \tau^*_{ix} \neq b'_{i'} + \tau^*_{i'x'}$ *for any* $i \neq i', x \in \Psi_i, x' \in \Psi_{i'}$); *(no two perturbed budgets are equal)*

(v) there is no price $\mathbf{p} \in \widetilde{P}$ *at which more than* M *perturbed budget-constraint hyperplanes intersect.*

It will be important for obtaining the approximation bound that no two hyperplanes coincide (iv), and no more than M of the hyperplanes intersect at any particular price vector (v). We could ensure the former by perturbing just the budgets, but to ensure (v) we need to perturb each budget-constraint hyperplane separately.

At the end of the proof, if agent i is actually assigned bundle x' in the approximate CEEI we will adjust i's budget to $b^*_i = b'_i + \tau^*_{ix'}$. Then we will set all of the taxes to zero. Property (ii) will ensure that x' is i's most-preferred choice at a budget of b^*_i. Property (i) will ensure that $|b^*_i - b_i| < \varepsilon$, and property (iii) will ensure that $\mathbf{b}^*$ preserves the inequality bound β.

Existence of a set of taxes $(\tau^*_{ix})_{s_i\in\mathcal{S},x\in\Psi_i}$ satisfying (i)-(v) is trivial. Choose initial $(\tau'_{ix})_{s_i\in\mathcal{S},x\in\Psi_i}$ that satisfy (i)-(iii) and $b_i + \tau'_{ix} \neq b_{i'} + \tau'_{i'x'}$ for $(i,x) \neq (i',x')$, which is a bit stronger than (iv). There are a finite number of hyperplanes $(H(i,x))_{s_i\in\mathcal{S},x\in\Psi_i}$ and because of $b_i+\tau'_{ix} \neq b_{i'}+\tau'_{i'x'}$ no two hyperplanes are homogeneous (i.e., have the same constant on the RHS). Generically, no more than M of a finite set of inhomogeneous hyperplanes intersect at a single point of an M-dimensional space. If the $(\tau'_{ix})_{s_i\in\mathcal{S},x\in\Psi_i}$ happen to yield an $L > M$-way intersection, perturb $L - M$ of the taxes associated with the hyperplanes in the intersection in a manner that preserves (i)-(iv). Let $(\tau^*_{ix})_{s_i\in\mathcal{S},x\in\Psi_i}$ denote the set of taxes that satisfies (i)-(v).

Step 3. *Define an upper hemicontinuous set-valued correspondence* F *which is a "convexification" of* f*, and which is guaranteed to have a fixed point by Kakutani's theorem. Let* $\widetilde{\mathbf{p}}^* \in F(\widetilde{\mathbf{p}}^*)$ *denote the fixed point and let* $\mathbf{p}^* = t(\widetilde{\mathbf{p}}^*)$ *denote its truncation.*

Fix budgets to $\mathbf{b}'$ and taxes to the τ^* found in step 2. Create the correspondence $F : \widetilde{P} \to \widetilde{P}$. as follows

$$F(\mathbf{p}) = co\{y : \exists \text{ a sequence } \mathbf{p}^w \to \mathbf{p}, \mathbf{p}_i \neq \mathbf{p} \text{ such that } f(\mathbf{p}^w) \to y\} \tag{14}$$

where co denotes the convex hull. Cromme and Diener (1991, Lemma 2.4) show that for any map f, the correspondence F constructed according to (14) is upper hemicontinuous, and hence has a fixed point (the other conditions for Kakutani's fixed point theorem – F is non-empty; $\widetilde{P}$ is compact and convex; and $F(\mathbf{p})$ is convex – are trivially satisfied).

So there exists $\widetilde{\mathbf{p}}^* \in F(\widetilde{\mathbf{p}}^*)$. Let $\mathbf{p}^* = t(\widetilde{\mathbf{p}}^*)$ denote its truncation.

Step 4. *If the price vector $\mathbf{p}^*$ is not on any budget-constraint hyperplane then it is an exact competitive equilibrium price vector and we are done.*

Recall that $H(i,x) = \{\mathbf{p} \in \widetilde{P} : \mathbf{p} \cdot x = b'_i + \tau^*_{ix}\}$ denotes the budget constraint hyperplane associated with agent s_i purchasing bundle x. Let $\widetilde{H}(i,x) = \{\widetilde{\mathbf{p}} \in \widetilde{P} : t(\widetilde{\mathbf{p}}) \cdot x = b'_i + \tau^*_{ix}\}$. Observe that $\mathbf{p}^* \notin H(i,x) \implies \widetilde{\mathbf{p}}^* \notin \widetilde{H}(i,x)$. Agents' choice sets change only when price crosses a budget-constraint hyperplane. So if $\mathbf{p}^*$ is not on any budget-constraint hyperplane, f is continuous at $\widetilde{\mathbf{p}}^*$. So $F(\widetilde{\mathbf{p}}^*)$ is single valued, hence $\widetilde{\mathbf{p}}^* = F(\widetilde{\mathbf{p}}^*) = f(\widetilde{\mathbf{p}}^*)$, and by Step 1 we are done.

Step 5. *Suppose $\mathbf{p}^*$ is on $L \geq 1$ budget-constraint hyperplanes. By Step 2 we know $L \leq M$. Let $\Phi = \{0,1\}^L$. Define a set of 2^L price vectors $\{\mathbf{p}^\phi\}_{\phi \in \Phi}$ satisfying the following conditions:*

(i) Each $\mathbf{p}^\phi$ is close enough to $\mathbf{p}^*$ that there is a path from $\mathbf{p}^\phi$ to $\mathbf{p}^*$ that does not cross any budget-constraint hyperplane (until the moment it reaches $\mathbf{p}$)

(ii) Each $\mathbf{p}^\phi$ is on the "affordable" side of the l^{th} hyperplane if $\phi_l = 0$ and is on the "unaffordable" side if $\phi_l = 1$.

That is, each $\phi \in \Phi$ "labels" a region of price space close to $\mathbf{p}^*$.

For $l = 1, ..., L$ we write the l^{th} hyperplane as $H(i_{(l)}, x_{(l)}) := \{\mathbf{p} \in \widetilde{P} : \mathbf{p} \cdot x_{(l)} = b_{i_{(l)}} + \tau_{i_{(l)} x_{(l)}}\}$. Each hyperplane $H(i_{(l)}, x_{(l)})$ defines two half spaces: $H^0_l := \{\mathbf{p} \in \widetilde{P} : \mathbf{p} \cdot x_{(l)} \leq b_{i_{(l)}} + \tau_{i_{(l)} x_{(l)}}\}$ is the closed half space in which agent $s_{i_{(l)}}$ can (weakly) afford bundle $x_{(l)}$, and $H^1_l := \{\mathbf{p} \in \widetilde{P} : \mathbf{p} \cdot x_{(l)} > b_{i_{(l)}} + \tau_{i_{(l)} x_{(l)}}\}$ is the open half space in which agent $s_{i_{(l)}}$ cannot afford bundle $x_{(l)}$.

We label combinations of half spaces as follows. Let $\Phi = \{0,1\}^L$, with each label $\phi = (\phi_1, ..., \phi_L) \in \Phi$ an L-dimensional vector of $0's$ and $1's$. The convex polytope $\pi(\phi) := \bigcap_{l=1}^{L} H^{\phi_l}_l$ denotes the set of points in $\widetilde{P}$ that belong to the intersection of half spaces indexed by ϕ.

Let $\mathcal{H}$ denote the (finite) set of all hyperplanes formed by any i, x : $\mathcal{H} = \{H(i,x)_{s_i \in \mathcal{S}, x \in \Psi_i}\}$. Let $\delta < \inf_{\mathbf{p}'' \in P,\, H \in \mathcal{H}} ||(\mathbf{p}^* - \mathbf{p}'') : \mathbf{p}'' \in H, \mathbf{p}^* \notin H||_2$. That is, any hyperplane to which $\mathbf{p}^*$ does not belong is strictly further than δ away from $\mathbf{p}^*$ in Euclidean distance. Let $B_\delta(\mathbf{p}^*)$ denote a δ-ball of $\mathbf{p}^*$.

We can now define the set $\{\mathbf{p}^\phi\}_{\phi \in \Phi}$: each $\mathbf{p}^\phi$ is an arbitrary element of $\pi(\phi) \cap B_\delta(\mathbf{p}^*)$.[49]

[49] It is possible, if $\mathbf{p}^*$ is on the boundary of P, that $\pi(\phi) \cap P = \emptyset$ for some combinations ϕ. (For instance, it is impossible to be below $x + y = 1$ and above $y = 1$ while $x, y \geq 0$). In that case $\mathbf{p}^\phi$ might include some prices which

Step 6: *By the way we have constructed F, for any $y \in F(\mathbf{p}^*)$ there exist non-negative weights $\{\lambda^\phi\}_{\phi\in\Phi}$ with $\sum_{\phi\in\Phi}\lambda^\phi = 1$ such that $\sum_{\phi\in\Phi}\lambda^\phi f(\mathbf{p}^\phi) = y$.*

The idea of this step is: for any price $\mathbf{p}'$ close enought to $\mathbf{p}^*$, excess demand at $\mathbf{p}'$ is determined entirely by the combination of budget-constraint hyperplane half spaces to which it belongs.

Consider an arbitrary ϕ and consider any two prices $\mathbf{p}', \mathbf{p}'' \in \pi(\phi) \cap B_\delta(\mathbf{p}^*)$. Since both prices are in $\pi(\phi)$ they are on the same side of each of the L hyperplanes that intersect at $\mathbf{p}^*$. Since both prices are in $B_\delta(\mathbf{p}^*)$, by the way we chose δ, for any other hyperplane in $\mathcal{H}$, $\mathbf{p}'$ and $\mathbf{p}''$ are on the same side. Together, this means that every agent has the same choice set at $\mathbf{p}'$ as at $\mathbf{p}''$. Since we chose $\mathbf{p}'$, $\mathbf{p}''$ arbitrarily, demand at any price vector in $\pi(\phi) \cap B_\delta(\mathbf{p}^*)$ is equal to demand at $\mathbf{p}^\phi$.

Consider any sequence of prices $\mathbf{p}^{w,\phi} \to \mathbf{p}^*$, with each $\mathbf{p}^{w,\phi} \in \pi(\phi) \cap B_\delta(\mathbf{p}^*)$. The preceding argument implies:

$$f(\mathbf{p}^{w,\phi}) \to \mathbf{p}^* + \gamma\mathbf{z}(\mathbf{p}^\phi) \tag{15}$$

Note too that any sequence $\mathbf{p}^w \to \mathbf{p}^*$ for which $f(\mathbf{p}^w)$ converges must converge to $\mathbf{p}^* + \mathbf{z}(\mathbf{p}^{\phi'})$ for some $\phi' \in \Phi$. This follows because $\bigcup\limits_{\phi\in\Phi} \pi(\phi) \cap B_\delta(\mathbf{p}^*) = \widetilde{P} \cap B_\delta(\mathbf{p}^*)$.

Combining these facts, if $y \in F(\mathbf{p}^*)$ then

$$\begin{aligned} \exists\{\lambda_\phi\}_{\phi\in\Phi} \text{ with} \sum_{\phi\in\Phi}\lambda^\phi &= 1 \text{ and } \lambda_\phi \geq 0, \text{ all } \phi \in \Phi \text{ s.t.:} \\ \sum_{\phi\in\Phi}\lambda^\phi[\mathbf{p}^* + \gamma\mathbf{z}(\mathbf{p}^\phi)] &= y \end{aligned} \tag{16}$$

Step 7. *Map from the set of prices $\{\mathbf{p}^\phi\}_{\phi\in\Phi}$ to the set of excess demands $\{\mathbf{z}(\mathbf{p}^\phi)\}_{\phi\in\Phi}$. Step 6 implies that a perfect market clearing excess demand vector lies in the convex hull of $\{\mathbf{z}(\mathbf{p}^\phi)\}_{\phi\in\Phi}$.*

Any price near to $\widetilde{\mathbf{p}}^*$ has a truncation that is near to $\mathbf{p}^*$.[50] Examining the definitions of f and F, i.e. (13) and (14), this means that $F(\widetilde{\mathbf{p}}^*) \subseteq F(\mathbf{p}^*)$. So $\widetilde{\mathbf{p}}^* \in F(\widetilde{\mathbf{p}}^*) \implies \widetilde{\mathbf{p}}^* \in F(\mathbf{p}^*)$. Adapting the argument of Step 6 yields

$$\begin{aligned} \exists\{\lambda_\phi\}_{\phi\in\Phi} \text{ with} \sum_{\phi\in\Phi}\lambda^\phi &= 1 \text{ and } \lambda_\phi \geq 0, \text{ all } \phi \in \Phi \text{ s.t.:} \\ \sum_{\phi\in\Phi}\lambda^\phi[t(\widetilde{\mathbf{p}}^*) + \gamma\mathbf{z}(\mathbf{p}^\phi)] &= \widetilde{\mathbf{p}}^* \end{aligned} \tag{17}$$

are strictly negative. We have defined demand and excess demand to be well defined for such prices, but note that at the final step of the proof we will ensure that all prices are weakly positive.

[50](Note that the converse need not be true; if $\widetilde{p}^*_c < 0$ then every price p' near to $\widetilde{p}^*_c$ has $p'_c = 0$, whereas some prices p'' near to p^* will have $p''_c > 0$).

which in turn implies (using the same λ's)

$$\sum_{\phi\in\Phi}\lambda^{\phi}\mathbf{z}(\mathbf{p}^{\phi})=\frac{\widetilde{\mathbf{p}}^{*}-\mathbf{p}^{*}}{\gamma} \tag{18}$$

By the same argument as in Step 1, demand for a course c_j must be zero at price $\bar{b}$, so $\widetilde{p}_j^* \in [-1,\bar{b})$ for all $j=1...M$. So, for all j, either $\widetilde{p}_j^*=p_j^*$ or $\widetilde{p}_j^*<0=p_j^*$. So we have that $\sum_{\phi\in\Phi}\lambda^{\phi}\mathbf{z}(\mathbf{p}^{\phi})\leq 0$ with $\sum_{\phi\in\Phi}\lambda^{\phi}z_j(\mathbf{p}^{\phi})<0\Longrightarrow p_j^*=0$, as required for market clearing. That is, a convex combination of excess demands for prices near $\mathbf{p}^*$ exactly clears the market at prices $\mathbf{p}^*$.

Consider the set of excess demands $\{\mathbf{z}(\mathbf{p}^{\phi})\}_{\phi\in\Phi}$ and let $\zeta=\sum_{\phi\in\Phi}\lambda^{\phi}\mathbf{z}(\mathbf{p}^{\phi})$. The vector ζ is a "perfect market clearing" ideal at prices $\mathbf{p}^*$ since $\zeta\leq 0$ with $\zeta_j<0\Longrightarrow p_j^*=0$. Clearly $\zeta\in co\{\mathbf{z}(\mathbf{p}^{\phi})\}_{\phi\in\Phi}$, where co denotes the convex hull.

Step 8. *The set of excess demands $\{\mathbf{z}(\mathbf{p}^{\phi})\}_{\phi\in\Phi}$ has a special geometric structure. In particular, if the L hyperplanes correspond to L distinct agents then $\{\mathbf{z}(\mathbf{p}^{\phi})\}_{\phi\in\Phi}$ are the vertices of a zonotope.*

The L intersecting budget-constraint hyperplanes name $L'\leq L$ distinct agents. Renumber these agents $i=1,...,L'$, and renumber the bundles that correspond to them as $(x_i^1,...,x_i^{w_i})_{i=1,...,L'}$ where w_i is the number of intersecting budget-constraint hyperplanes that involve agent s_i, and $u_i(x_i^1)>...>u_i(x_i^{w_i})$. Note that $\sum_{i=1}^{L'}w_i=L$.

We will show that agent s_i purchases at most w_i+1 distinct bundles at prices near to $\mathbf{p}^*$. In the halfspace $H^0(i,x_i^1)$ he can afford x_i^1, his favorite bundle whose affordability is in question near to $\mathbf{p}^*$, and so it does not matter which side of $H(i,x_i^2),...,H(i,x_i^{w_i})$ price is on. Let d_i^0 denote his demand at prices in $H^0(i,x_i^1)\cap B_\delta(\mathbf{p}^*)$.

If price is in $H^1(i,x_i^1)\cap H^0(i,x_i^2)$ then i cannot afford x_i^1 but can afford x_i^2, his second-favorite bundle whose affordability is in question. So it does not matter which side of $H(i,x_i^3),...,H(i,x_i^{w_i})$ price is on. Let d_i^1 denote his demand at prices in $H^1(i,x_i^1)\cap H^0(i,x_i^2)\cap B_\delta(\mathbf{p}^*)$.

Continuing in this manner, define $d_i^2,...,d_i^{w_i}$. The process ends when we have crossed to the unaffordable side of all w_i of s_i's budget-constraint hyperplanes, and so cannot afford any of $x_i^1,...,x_i^{w_i}$.

The demand of any agents other than the L' named on budget-constraint hyperplanes is unchanging near $\mathbf{p}^*$. Call the total demand of such agents $d_{-L'}(\mathbf{p}^*)=\sum_{i=L'+1}^{N}d_i(\mathbf{p}^*;b_i',\tau_i^*)$, and let $z_{-L'}(\mathbf{p}^*)=d_{-L'}(\mathbf{p}^*)-\mathbf{q}$.

We can now characterize the set $\{\mathbf{z}(\mathbf{p}^{\phi})\}_{\phi\in\Phi}$ in terms of the demands of the L' individual agents near $\mathbf{p}^*$:

$$\begin{aligned} \{\mathbf{z}(\mathbf{p}^{\phi})\}_{\phi\in\Phi} &= \{z_{-L'}(\mathbf{p}^{*}) + \sum_{i=1}^{L'}\sum_{f=0}^{w_i} b_i^f d_i^f\} \\ &\quad \text{subject to} \\ b_i^f &\in \{0,1\} \text{ for all } i, f \\ \sum_{f=0}^{w_i} b_i^f &= 1 \text{ for all } i = 1, ..., L' \end{aligned} \tag{19}$$

At any price vector near to $\mathbf{p}^*$, each agent $i = 1, ..., L'$ demands exactly one of their $w_i + 1$ demand bundles. Over the set $\Phi = \{0,1\}^L$ every combination of the L' agents' demands is possible. Informally, it is possible to "walk through price space" near to $\mathbf{p}^*$ in such a way that we cross just a single budget-constraint hyperplane (and hence change just a single agent's demand) at a time. This would not be possible if agents had identical budgets, because then their hyperplanes would coincide. (Also, we would not be able to guarantee that at most M intersect.)

Step 7 tells us that there exists a market-clearing excess demand vector in the convex hull of (19). This convex hull can be written as

$$\begin{aligned} \{\mathbf{z}(\mathbf{p}^{\phi})\}_{\phi\in\Phi} &= \{z_{-L'}(\mathbf{p}^{*}) + \sum_{i=1}^{L'}\sum_{f=0}^{w_i} a_i^f d_i^f\} \\ &\quad \text{subject to} \\ a_i^f &\in [0,1] \text{ for all } i, f \\ \sum_{f=0}^{w_i} a_i^f &= 1 \text{ for all } i = 1, ..., L' \end{aligned} \tag{20}$$

The set (19) has a particularly interesting structure in case $L' = L$ (and so $w_i = 1$ for $i = 1, ..., L'$). Define $v_i = d_i^1 - d_i^0$. The vector v_i describes how i's demand changes as we raise price from $\mathbf{p}^*$ in a way that makes d_i^0 unaffordable. Observe that total excess demand at $\mathbf{p}^*$ satisfies $\mathbf{z}(\mathbf{p}^*) = z_{-L'}(\mathbf{p}^*) + \sum_{i=1}^{L'} d_i^0$. The set $\{\mathbf{z}(\mathbf{p}^{\phi})\}_{\phi\in\Phi}$ can be rewritten as

$$\begin{aligned} \{\mathbf{z}(\mathbf{p}^{\phi})\}_{\phi\in\Phi} &= \{\mathbf{z}(\mathbf{p}^{*}) + \sum_{i=1}^{L} b_i v_i\} \\ &\quad \text{subject to} \\ b_i &\in \{0,1\} \text{ for all } i \end{aligned} \tag{21}$$

The set (21) gives the vertices of a geometrical object called a zonotope. The zonotope itself is the convex hull of (21). A zonotope is the Minkowski sum of a set of generating vectors; here, the generating vectors are $v_1, ..., v_L$. If the vectors are linearly independent then the zonotope is a parallelotope, the multi-dimensional generalization of a parallelogram. (See Ziegler, 1995)

Step 9. *There exists a vertex of the geometric structure from Step 8, (19), that is within $\frac{\sqrt{M}\sigma}{2}$ distance of the market-clearing excess demand vector found in Step 7. That is, for some $\mathbf{z}(\mathbf{p}^{\phi'}) \in \{\mathbf{z}(\mathbf{p}^{\phi})\}_{\phi\in\Phi}$, $||\mathbf{z}(\mathbf{p}^{\phi'}) - \zeta||_2 \leq \frac{\sqrt{M}\sigma}{2}$.*

We are interested in bounding the distance between an element of (19) and an element of its convex hull (20), which we know contains ζ.[51]

Fix an arbitrary interior point of (20). That is, fix a set of $a_i^f \in [0,1]$ that satisfy the constraint $\sum_{f=0}^{w_i} a_i^f = 1$ for all $i = 1...L'$. For each i define a random vector $\Theta_i = (\Theta_i^0, ..., \Theta_i^{w_i})$ where the support of each Θ_i^f is $\{0,1\}$, $E(\Theta_i^f) = a_i^f$ for all i, f, and in any realization θ_i, $\sum_{f=0}^{w_i} \theta_i^f = 1$. Define the random matrix $\Theta = (\Theta_1, ..., \Theta_{L'})$, and suppose that the Θ_i's are independent. Let

$$\rho^2 = \left(\mathbb{E}_{\Theta}\left\|\sum_{i=1}^{L'}\sum_{f=0}^{w_i}[(a_i^f - \theta_i^f)d_i^f]\right\|_2\right)^2$$

By linearity of expectations, we have that

$$\begin{aligned}\rho^2 \quad &= \quad \sum_{i=1}^{L'}\left(\mathbb{E}_{\Theta_i}\left\|\sum_{f=0}^{w_i}[(a_i^f - \theta_i^f)]d_i^f\right\|_2\right)^2 \\ &\quad + \sum_{j\neq i}\sum_{f=0}^{w_i}\sum_{g=0}^{w_j}\mathbb{E}_{\Theta_i^f,\Theta_j^g}[(a_i^f - \theta_i^f)(a_j^g - \theta_j^g)](d_i^f \cdot d_j^g)\end{aligned} \tag{22}$$

And by independence we get that

$$\begin{aligned}&\mathbb{E}_{\Theta_i^f,\Theta_j^g}[(a_i^f - \theta_i^f)(a_j^g - \theta_j^g)] \\ = \quad &\mathbb{E}_{\Theta_i^f}[a_i^f - \theta_i^f]\mathbb{E}_{\Theta_j^g}[a_j^g - \theta_j^g] = 0\end{aligned} \tag{23}$$

since the random vectors are independent across agents and $E_{\Theta_i^f}\theta_i^f = a_i^f$ for all i, f.

Lemma 2. *For each $i = 1, ..., L'$, $\left(\mathbb{E}_{\Theta_i}\left\|\sum_{f=0}^{w_i}[(a_i^f - \theta_i^f)]d_i^f\right\|_2\right) \leq \frac{\sqrt{w_i\sigma}}{2}$*

Proof. Fix i. For any $d_i^f, d_i^{f'}$, we have $||d_i^f - d_i^{f'}||_2 \leq \sqrt{\sigma}$, where σ is the demand sensitivity of the economy. Let $\overline{d}_i = \sum_{f=0}^{w_i} a_i^f d_i^f$. In words, $\overline{d}_i$ is s_i's average demand as used in the convex

[51] The proof technique for this step closely follows that of Theorem 2.4.2 in Chapter 2 of Alon and Spencer, 2000. I am grateful to Michel Goemans for the pointer.

combination. Now rewrite

$$
\begin{aligned}
&\mathbb{E}_{\Theta_i}\left\|\sum_{f=0}^{w_i}[(a_i^f-\theta_i^f)]d_i^f\right\|_2 \\
&=\sum_{f=0}^{w_i} a_i^f\left\|\overline{d}_i-d_i^f\right\|_2 \qquad (24)
\end{aligned}
$$

If $w_i = 1$ then (24) is largest when $\left\|d_i^1 - d_i^0\right\|_2 = \sqrt{\sigma}$ and $a_i^0 = a_i^1 = \frac{1}{2}$; this maximum value is $\frac{\sqrt{\sigma}}{2}$ which is equal to the bound. If $w_i = 2$ then (24) is largest when $\{d_i^0, d_i^1, d_i^2\}$ forms an equilateral triangle of side length $\sqrt{\sigma}$ and $a_i^0 = a_i^1 = a_i^2 = \frac{1}{3}$; this maximum value is $\frac{\sqrt{3\sigma}}{3}$ which is strictly lower than the bound of $\frac{\sqrt{2\sigma}}{2}$. If $w_i = 3$ then (24) is largest when $\{d_i^0, d_i^1, d_i^2, d_i^3\}$ forms a triangular pyramid of side length $\sqrt{\sigma}$ and $a_i^0 = a_i^1 = a_i^2 = a_i^3 = \frac{1}{4}$; this maximum value is $\frac{\sqrt{6\sigma}}{4}$ which is strictly lower than the bound of $\frac{\sqrt{3\sigma}}{2}$. For $w_i \geq 4$ the bound can be obtained by observing that there exists some sphere of diameter $\sqrt{\sigma}$ that contains the convex hull of $\{d_i^f\}_{f=0}^{w_i}$, and the RHS of the bound $\frac{\sqrt{w_i\sigma}}{2} \geq \sqrt{\sigma}$. QED.

Combining Lemma 2, (22), and (23) yields

$$
\begin{aligned}
\rho^2 &= \sum_{i=1}^{L'}\left(\mathbb{E}_{\Theta}\left\|\sum_{f=0}^{w_i}[(a_i^f-\theta_i^f)d_i^f]\right\|_2\right)^2 \\
&\leq \sum_{i=1}^{L'}\frac{w_i\sigma}{4} \\
&= \frac{M\sigma}{4}
\end{aligned}
$$

This means that, for each interior point of (20) there must exist at least one realization of Θ such that $\left\|\sum_{i=1}^{L'}\sum_{f=0}^{w_i}[(a_i^f - \theta_i^f)]d_i^f\right\| \leq \frac{\sqrt{M\sigma}}{2}$. Call this realization $\widetilde{\theta}$. This realization points us to an element of $\{\mathbf{z}(\mathbf{p}^{\phi})\}_{\phi\in\Phi}$, namely $z_{-L'}(\mathbf{p}^*) + \sum_{i=1}^{L'}\sum_{f=0}^{w_i}\widetilde{\theta}_i^f d_i^f$, that is within $\frac{\sqrt{M\sigma}}{2}$ Euclidean distance of the interior point, namely $z_{-L'}(\mathbf{p}^*) + \sum_{i=1}^{L'}\sum_{f=0}^{w_i} a_i^f d_i^f$.

Step 10. *Use the vertex found in Step 9 to produce prices, budgets and an allocation that satisfy the statement of Theorem 1.*

In Step 9 we showed that the excess demand vector $\mathbf{z}(\mathbf{p}^{\phi'})$ approximately clears the market at prices $\mathbf{p}^*$. There is no guarantee that $\mathbf{p}^{\phi'} \in P$; in particular if $p_j^* = 0$ it is possible that $p_j^{\phi'}$ is strictly negative. So we will use the prices $\mathbf{p}^*$, which are guaranteed to be in P, and perturb budgets in a way that generates excess demand at $\mathbf{p}^*$ equal to $\mathbf{z}(\mathbf{p}^{\phi'})$ from Step 9.

If agent s_i is not named on any of the L budget-constraint hyperplanes of step 5, then his consumption is $x_i^* = d_i(\mathbf{p}^*, b_i', \tau_i^*)$ and we set $b_i^* = b_i' + \tau^*_{ix_i^*}$. Observe that requirement (i) of Step 2 implies that any bundle he prefers to x_i^* costs strictly more than $b_i' + \tau^*_{ix_i^*}$, else he would demand it at prices $\mathbf{p}^*$, budget b_i', and taxes of τ_i^*.

If agent s_i is named on some of the budget-constraint hyperplanes, then we will use the information in ϕ' to perturb his taxes and ultimately his budget. The label ϕ' tells us which side $\mathbf{p}^{\phi'}$ is on of each of s_i's hyperplanes $H(i, x^1), ..., H(i, x^{w_i})$. If $\mathbf{p}^{\phi'} \in H^1(i, x^f)$, i.e., it is on the unaffordable side of bundle x^f, then set $\tau^{**}_{ix^f} = \tau^*_{ix^f} - \delta_2$ for $\delta_2 > 0$ but tiny. At the price vector $\mathbf{p}^*$ and initial taxes τ^* agent s_i could exactly afford bundle x^f, i.e., $\mathbf{p}^*$ was on the budget-constraint hyperplane $H(i, x^f)$. This tiny perturbation ensures that at taxes τ^{**} he can no longer afford x^f.[52] Choose δ_2 small enough that conditions (i)-(iii) of Step 1 still obtain. For all other bundles, including bundles not named on any hyperplane, set $\tau^{**}_{ix} = \tau^*_{ix}$. Consider $d_i(\mathbf{p}^*, b_i', \tau_i^{**})$: this is simply s_i's demand at the original budget and taxes but at prices $\mathbf{p}^{\phi'}$, i.e., $d_i(\mathbf{p}^*, b_i', \tau_i^{**}) = d_i(\mathbf{p}^{\phi'}, b_i', \tau_i^*)$.

Set $x_i^* = d_i(\mathbf{p}^*, b_i', \tau_i^{**})$ and set $b_i^* = b_i' + \tau^*_{ix_i^*}$. Since we set δ_2 small enough to ensure requirement (ii) of Step 1 still obtains, we preserve optimality. Similarly, we have preserved the original level of budget inequality and the ε bounds, by requirements (iii) and (i), respectively, of Step 1. Finally, approximate market clearing is ensured by Step 9. So budgets of $\mathbf{b}^*$, prices of $\mathbf{p}^*$, and the allocation $\mathbf{x}^*$ satisfy all of the requirements of Theorem 1. QED.

B Proof of Lemma 1 and Theorem 2

Proof of Lemma 1

In step 7 of the proof of Theorem 1 we showed that $\sum_{\phi \in \Phi} \lambda^\phi \mathbf{p}^* \cdot \mathbf{z}(\mathbf{p}^\phi) = 0$ for a set of prices $(\mathbf{p}^\phi)_{\phi \in \Phi}$ arbitrarily close to $\mathbf{p}^*$. Recall that $\varepsilon > 0$ is the maximum discrepancy between the target budgets of $\mathbf{b}$ and the final budgets of $\mathbf{b}^*$. At each price $\mathbf{p}^\phi$ we have the simple identity that $\mathbf{p}^\phi \cdot d_i(\mathbf{p}^\phi, b_i', \tau_i^*) \le b_i^* + \varepsilon$, and so $\sum_{i=1}^N \mathbf{p}^\phi \cdot d_i(\mathbf{p}^\phi, b_i', \tau_i^*) \le \sum\limits_{i=1}^N b_i^* + N\varepsilon$. Let $\varepsilon_2 > 0$ denote the maximum distance in the L_1 norm between $\mathbf{p}^*$ and any of the $\mathbf{p}^\phi$'s. Note that for any bundle x

[52] In Step 5 we indicated that if p^* is on the boundary of P then it is possible that some of the combinations of half spaces $\phi \in \Phi$ are entirely disjoint from P. By perturbing budgets rather than prices and keeping prices at p^* we avoid the worry that we end up with an illegal price vector.

and any ϕ this means $\mathbf{p}^{\phi} \cdot x \geq \mathbf{p}^{*} \cdot x - \varepsilon_2$. Adding up over all $\mathbf{p}^{\phi}$ we have

$$\begin{aligned}
& \sum_{\phi \in \Phi} \lambda^{\phi} \mathbf{p}^{\phi} \sum_{i=1}^{N} d_i(\mathbf{p}^{\phi}, b'_i, \tau_i^{*}) \\
\geq & \sum_{\phi \in \Phi} \lambda^{\phi} \mathbf{p}^{*} \sum_{i=1}^{N} d_i(\mathbf{p}^{\phi}, b'_i, \tau_i^{*}) - N\varepsilon_2 \\
= & \sum_{\phi \in \Phi} \lambda^{\phi} \mathbf{p}^{*} \cdot (\mathbf{z}(\mathbf{p}^{\phi}) + \mathbf{q}) - N\varepsilon_2 \\
= & \sum_{\phi \in \Phi} \lambda^{\phi} \mathbf{p}^{*} \cdot \mathbf{q} - N\varepsilon_2 \\
= & \mathbf{p}^{*} \cdot \mathbf{q} - N\varepsilon_2
\end{aligned}$$

So $\sum_{i=1}^{N} b_i + N\varepsilon \geq \mathbf{p}^{*} \cdot \mathbf{q} - N\varepsilon_2$. In the proof of Theorem 1 we are free to choose $\varepsilon, \varepsilon_2$. Choose them sufficiently small such that $N(\varepsilon + \varepsilon_2) < \delta(\sum_{i=1}^{N} b_i + N\varepsilon)$. Then $\mathbf{p}^{*} \cdot \mathbf{q} \leq \sum_{i=1}^{N} b_i^{*}(1+\delta)$, as required. QED.

Proof of Theorem 2.

Since $\mathbf{b}^{*}$ and $\mathbf{p}^{*}$ are part of an (α, β)-approximate CEEI with $\mathbf{p}^{*} \in P(\delta, \mathbf{b}^{*})$, $N(1+\beta)(1+\delta) \geq \sum_{i=1}^{N} b_i^{*}(1+\delta) \geq \mathbf{p}^{*} \cdot \mathbf{q}$. Suppose that there is some agent s_i who cannot afford any bundle in her $N+1$ maximin split $\widetilde{\mathbf{x}}$ at $\mathbf{p}^{*}$. Then $\mathbf{p}^{*} \cdot \widetilde{x}_l > b_i^{*} \geq 1$ for all $l = 1, ..., N, N+1$. By the definition of a maximin split we have $\sum_l \mathbf{p}^{*} \cdot \widetilde{x}_l \leq \mathbf{p}^{*} \cdot \mathbf{q}$. Putting this all together gives

$$N(1+\beta)(1+\delta) \geq \mathbf{p}^{*} \cdot \mathbf{q} \geq \sum_{l} \mathbf{p}^{*} \cdot \widetilde{x}_l > (N+1)$$

So set β sufficiently small that $N(1+\beta)(1+\delta) > (N+1)$ is a contradiction, i.e., set $\beta < \frac{1-N\delta}{N(1+\delta)}$. QED.

Proof of Proposition 6.

Suppose $\alpha = 0$. Then condition (ii) of the definition of an Approximate CEEI implies that $N(1+\beta) \geq \mathbf{p}^{*} \cdot \mathbf{q}$. Let $\widetilde{\mathbf{x}} = (\widetilde{x}_1, ..., \widetilde{x}_{N+1})$ be agent s_i's $N+1-$maximin split, ordered such that $u_i(\widetilde{x}_l) \geq u_i(\widetilde{x}_{l+1})$ for $l = 1, ..., N$. We know from Theorem 2 that we can at least guarantee $u_i(x_i^{*}) \geq u_i(\widetilde{x}_{N+1})$. Suppose that s_i gets exactly his $N+1$ maximin share, i.e., $u_i(x_i^{*}) = u_i(\widetilde{x}_{N+1})$. If $u_i(\widetilde{x}_N) = u_i(\widetilde{x}_{N+1})$ then (a) is satisfied and we are done. Suppose $u_i(\widetilde{x}_N) > u_i(\widetilde{x}_{N+1})$. Condition (i) of the definition of Approximate CEEI implies

$$b_i^{*} < \mathbf{p}^{*} \cdot \widetilde{x}_l \text{ for } l = 1, ..., N \tag{25}$$

Since $\alpha = 0$, condition (ii) of the definition of Approximate CEEI implies that the other $N-1$ agents can collectively afford the endowment but for $\widetilde{x}_{N+1}$, i.e.,

$$\sum_{l=1}^{N} \mathbf{p}^* \cdot \widetilde{x}_l \leq \sum_{l \neq i} b_l^* \tag{26}$$

The β inequality bound implies

$$\sum_{l \neq i} b_l^* \leq (N-1)(1+\beta) b_i^* \tag{27}$$

Combining (25) and (27) the previous two inequalities gives $N b_i^* \leq (N-1)(1+\beta) b_i^*$. So if $\beta < \frac{1}{N-1}$ we have a contradiction, and so $u_i(x_i^*) > u_i(\widetilde{x}_{N+1})$, as required for (b). QED

References

[1] Abdukadiroğlu, Atila, Parag Pathak, and Alvin E. Roth (2009). "Strategyproofness versus Efficiency in Matching with Indifferences: Redesigning the NYC High School Match." Forthcoming, *American Economic Review*.

[2] Abdukadiroğlu, Atila, Parag Pathak, Alvin E. Roth, and Tayfun Sonmez (2006). "Changing the Boston School Choice Mechanism." Mimeo.

[3] Abdukadiroğlu, Atila and Tayfun Sönmez (1999). "House Allocation with Existing Tenants." *Journal of Economic Theory*, 88: 233-60.

[4] Abdukadiroğlu, Atila and Tayfun Sönmez (1998). "Random Serial Dictatorship and the Core from Random Endowments in House Allocation Problems." *Econometrica*, 66(3), 689-702.

[5] Abdukadiroğlu, Atila and Kyle Bagwell (2007). "Trust, Reciprocity and Favors in Cooperative Relationships." Mimeo.

[6] Alkan, Ahmet, Gabrielle Demange and David Gale (1991). "Fair Allocation of Indivisible Goods and Criteria of Justice." *Econometrica*, 59(4), 1023-1039.

[7] Alon, Noga and Joel Spencer (2000). The Probabilistic Method. Wiley. Second Edition.

[8] Athey, Susan and David A. Miller (2006). "Efficiency in Repeated Trade with Hidden Valuations." Mimeo.

[9] Aumann, Robert J., 1964. "Existence of Competitive Equilibria in Markets with a Continuum of Traders." *Econometrica*, 32: 39-50.

[10] Ausubel, Lawrence M. and Peter Cramton (2008). "Auction Design for the Rescue Plan." October 5, Powerpoint Presentation.

[11] Bartlett, Thomas (2008). "Class Warfare: When Getting in is the Hardest Part." *Chronicle of Higher Education* 54(23), Feb 15, Page A1.

[12] Bikhchandani, S and JW Mamer, 1997. "Competitive Equilibrium in Exchange Economy with Indivisibilities." *Journal of Economic Theory*, 74(2), 385-413.

[13] Bikhchandani, S and J Ostroy, 2002. "The Package Assignment Model." *Journal of Economic Theory*, 107(2), 377-406.

[14] Brainard, William C. and Herbert E. Scarf (2005). "How to Compute Equilibrium Prices in 1891". *American Journal of Economics and Sociology*, 64(1), 57-83.

[15] Brams, Steven J., Paul H. Edelman and Peter C. Fishburn (2003). "Fair Division of Indivisible Items." *Theory and Decision* 55: 147-80.

[16] Brams, Steven J. and Philip D. Straffin, Jr. (1979). "Prisoners' Dilemma and Professional Sports Drafts." *The American Mathematical Monthly*, 86(2), 80-88.

[17] Brams, Steven J and Alan D. Taylor (1996). Fair Division: From Cake Cutting to Dispute Resolution. Cambridge University Press

[18] Brams, Steven J and Alan D. Taylor (1999). "The Win-Win Solution: Guaranteeing Fair Shares to Everybody." W.W. Norton and Company.

[19] Budish, Eric and Estelle Cantillon (2008). "Strategic Behavior in Multi-Unit Assignment Problems: Theory and Evidence from Course Allocation." Mimeo.

[20] Budish, Eric, Yeon-Koo Che, Fuhito Kojima, and Paul Milgrom (2008). "Implementing Random Assignments: A Generalization of the Birkhoff-von Neumann Theorem." In Preparation.

[21] Cramton, Peter, Yoav Shoham and Richard Steinberg eds. (2006). Combinatorial Auctions. MIT Press.

[22] Crawford, Vincent P. (1977). "A Game of Fair Division." *The Review of Economic Studies*, 44(2), 235-47.

[23] Cripps, Martin W. and Jeroen M. Swinkels (2006). "Efficiency of Large Double Auctions." *Econometrica*, 74(1), 47-92.

[24] Cromme, Ludwig J. and Immo Diener (1991). "Fixed Point Theorems for Discontinuous Mapping." *Mathematical Programming*, 51, 257-67.

[25] Debreu, George and Herbert Scarf (1963). "A Limit Theorem on the Core of an Economy." *Econometrica.*

[26] Dierker, Egbert (1971). "Equilibrium Analysis of Exchange Economies with Indivisible Commodities." *Econometrica*, 39(6), 997-1008.

[27] Dubins, L.E. and E.H. Spanier (1961). "How to Cut a Cake Fairly." *American Mathematical Monthly*, 68, 1-17.

[28] Dworkin, Ronald (1981). "What is Equality? Part 2: Equality of Resources." *Philosophy and Public Affairs*, 10(4), 283-345.

[29] Dworkin, Ronald (2000). Sovereign Virtue: The Theory and Practice of Equality. Harvard University Press.

[30] Ehlers, Lars and Bettina Klaus (2003). "Coalitional Strategy-proof and Resource-Monotonic Solutions for Multiple Assignment Problems." *Social Choice and Welfare*, 21: 265-80.

[31] Foley, Duncan (1967). "Resource Allocation and the Public Sector." *Yale Economic Essays*, 7, 45-98.

[32] Gale, David and Lloyd Shapley (1962). "College Admissions and the Stability of Marriage." *American Mathematical Monthly*, 69(1), 9-15.

[33] Graves, Robert L., Linus Schrage and Jayaram Sankaran (1993). "An Auction Method for Course Registration." *Interfaces*, 23(5), 81-92.

[34] Guernsey, Lisa (1999). "Business School Puts Courses in Hands of an On-Line Market." *New York Times*. Sept 9.

[35] Gul, Faruk and Ennio Stachetti (2000). "The English Auction with Differentiated Commodities." *Journal of Economic Theory*, 92(1), 66-95.

[36] Harsanyi, John C. (1953). "Cardinal Utility in Welfare Economics and in the Theory of Risk-taking." *Journal of Political Economy*, 61(5), 434-435.

[37] Hatfield, John William (2005). "Strategy-proof and Nonbossy Quota Allocations." Mimeo.

[38] Herreiner, Dorothea and Clemens Puppe (2002). "A Simple Procedure for Finding Equitable Allocations of Indivisible Goods." *Social Choice and Welfare*, 19, 415-30.

[39] Hylland, Aanund and Richard Zeckhauser (1979). "The Efficient Allocation of Individuals to Positions." *Journal of Political Economy*, 87(2), 293-314.

[40] Immorlica, Nicole and Mohammad Mahdian (2005). "Marriage, Honesty, and Stability." *SODA* 2005, 53-62.

[41] Klaus, Bettina and Eiichi Miyagawa (2001). "Strategy-proofness, Solidarity, and Consistency for Multiple Assignment Problems." *International Journal of Game Theory*, 30: 421-435.

[42] Konishi, Hideo, Thomas Quint and Jun Wako (2001). "On the Shapley-Scarf economy: the case of multiple types of indivisible goods." *Journal of Mathematical Economics*, 35: 1-15.

[43] Kojima, Fuhito (2007). "Random Assignment of Multiple Indivisible Objects." Mimeo.

[44] Kojima, Fuhito and Mihai Manea (2007). "Strategy-Proofness of the Probabilistic Serial Mechanism in Large Random Assignment Problems." Mimeo.

[45] Kojima, Fuhito and Parag Pathak (2007). "Incentives and Stability in Large Two-Sided Matching Markets." Forthcoming, *American Economic Review*.

[46] Kolm, Serge-Cristophe., 1972. Justice et Equité. Paris: Editions du CNRS. Translated 2002 by Harold F. See. MIT Press.

[47] Konishi, Hideo, Thomas Quint, and Jun Wako (2001). "On the Shapley-Scarf Economy: the Case of Multiple Types of Indivisible Goods." *Journal of Mathematical Economics*, 35(1), 1-15.

[48] Krishna, Aradhna and Utku Unver (2008). "Improving the Efficiency of Course Bidding at Business Schools: Field and Laboratory Studies." *Management Science*, 27(March/April), 262-82.

[49] Levitt, Steven (2008a). "What Do I Have in Common with Hannah Montana?" Freakonomics Blog on the *New York Times*. January 8.

[50] Levitt, Steven (2008b). "The Coase Theorem Rules at NYU Law." Freakonomics Blog on the *New York Times*. August 11.

[51] McKesson (2008). Website description of eShift. Accessed October 2008 at http://www.mckesson.com/en_us/McKesson.com/For%2BHealthcare%2BProviders/Hospitals/Workforce%2BManagement%2BSolutions/eShift.html

[52] Milgrom, Paul (2004). Putting Auction Theory to Work. Cambridge University Press.

[53] Milgrom, Paul (2007). "Package Auctions and Exchanges." *Econometrica*, 75(4), 935-65.

[54] Milgrom, Paul (2008). "Assignment Exchanges." Mimeo.

[55] Moulin, Hervé (1990). "Welfare Bounds in the Cooperative Production Problem." *Games and Economic Behavior*, 4, 373-401.

[56] Moulin, Hervé (1991). "Welfare Bounds in the Fair Division Problem." *Journal of Economic Theory*, 54, 321-37.

[57] Moulin, Hervé (1995). Cooperative Microeconomics. Prentice Hall.

[58] Moulin, Hervé (2004). Fair Division and Collective Welfare. MIT Press.

[59] Neil, Martha (2008). "NYU Students Seek Coveted Law School Classes, Will Pay Cash." *ABA Journal*. Jul 28.

[60] Nisan, Noam and Ilya Segal (2006). "The Communication Requirements of Efficient Allocations and Supporting Prices." *Journal of Economic Theory*, 129, 192-224.

[61] Othman, Abe, Eric Budish and Tuomas Sandholm (2008). "ApproxCEEI: A Computational Algorithm for Finding Approximate Competitive Equilibria from Approximately Equal Incomes." In Preparation.

[62] Pápai, Szilvia (2001). "Strategyproof and Nonbossy Multiple Assignments." *Journal of Public Economic Theory*, 3(3): 257-71.

[63] Parkes, David (2006). "Iterative Combinatorial Auctions." In Peter Cramton et al, Combinatorial Auctions. MIT Press.

[64] Pathak, Parag (2006). "Lotteries in Student Assignment." Mimeo.

[65] Pathak, Parag and Tayfun Sonmez (2008). "Comparing Mechanisms by their Vulnerability to Manipulation." Mimeo.

[66] Pratt, John W. (2007). "Fair (and Not So Fair) Division." *Journal of Risk and Uncertainty*, 35(3), 203-36.

[67] Rawls, John (1971). A Theory of Justice. Revised Edition. Harvard University Press.

[68] Roberts, John and Andrew Postlewaite (1976). "Incentives for Price-Taking Behavior in Large Exchange Economies." *Econometrica*, 44(1), 115-27.

[69] Robertson, Jack and William Webb (1998). Cake-Cutting Algorithms: Be Fair if You Can. AK Peters.

[70] Roth, Alvin E. (1982). "Incentive Compatibility in a Market with Indivisible Goods." *Economics Letters*, 9, 127-132.

[71] Roth, Alvin E. (1984). "The Evolution of the Labor Market for Medical Interns and Residents: A Case Study in Game Theory." *Journal of Political Economy*, 92, 991-1016.

[72] Roth, Alvin E. (2007). "Repugnance as a Constraint on Markets." *Journal of Economic Perspectives*, 21(3), 37-58.

[73] Roth, Alvin E. (2002). "The Economist as Engineer: Game Theory, Experimentation, and Computation as Tools for Design Economics." *Econometrica*, 70(4): 1341-1378.

[74] Roth, Alvin E. and Marilda Sotomayor (1990). Two-Sided Matching. Cambridge University Press.

[75] Roth, Alvin E. and Elliot Peranson (1999). "The Redesign of the Matching Market for American Physicians: Some Engineering Aspects of Economic Design." *American Economic Review*, 89, 748-80.

[76] Russell, Stuart and Peter Norvig (2002). Artificial Intelligence: A Modern Approach. Prentice Hall, 2nd Edition.

[77] Rustichini, Aldo, Mark A. Satterthwaite and Steven R. Williams (1994). "Convergence to Efficiency in a Simple Market with Incomplete Information." *Econometrica*, 62(5), 1041-63.

[78] Sandel, Michael, 1998. "What Money Can't Buy: the Moral Limits of Markets." The Tanner Lecture on Human Values, Oxford.

[79] Scarf, Herbert. (1960). "Some examples of global instability of competitive equilibria." *International Economic Review*, 1, 157–172.

[80] Sen, Amartya (1979). "Equality of What?" Tanner Lecture on Human Values. Reprinted in Choice, Welfare and Measurement, 1982. Harvard University Press.

[81] Shapley, Lloyd and Herbert Scarf (1974). "On Cores and Indivisibility." *Journal of Mathematical Economics*, 1(1) 23-37.

[82] Sönmez, Tayfun (1999). "Strategy-Proofness and Essentially Single-Valued Cores." *Econometrica*, 67(3): 677-89.

[83] Sönmez, Tayfun and Utku Unver (2008). "Course Bidding at Business Schools." Forthcoming, *International Economic Review*.

[84] Starr, Ross M. (1969). "Quasi-Equilibria in Markets with Non-Convex Preferences." *Econometrica*, 37(1), 25-38.

[85] Steinhaus, Hugo (1948). "The Problem of Fair Division." *Econometrica*. 16(1), 101-4.

[86] Thomson, WIlliam and Hal R. Varian (1985). "Theories of Justice Based on Symmetry." In Hurwicz, Schmeidler and Sonnenschein (eds), Social Goals and Social Organiztion: Essays in Memory of Elisha Pazner. Cambridge University Press.

[87] Varian, Hal, 1974. "Equity, Envy and Efficiency." *Journal of Economic Theory*, 29(2), 217-244.

[88] Wallis, W. D.; Street, Anne Penfold; Wallis, Jennifer Seberry (1972). Combinatorics: Room Squares, Sum-free Sets, Hadamard Matrices. Lecture Notes in Mathematics, Vol. 292. Springer-Verlag, Berlin-New York.

[89] Wharton (2007). "The Course Registration Auction for MBA Electives." Version 15.0, Graduate Division, MBA Program Office, The Wharton School, University of Pennsylvania.

[90] Ziegler, Gunter M. (1995). Lectures on Polytopes. Springer-Verlag.

[91] Zhou, Lin (1990). "On a Conjecture by Gale About One-Sided Matching Problems." *Journal of Economic Theory*, 52(1), 123-135.