

ON CORES AND INDIVISIBILITY*

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Received 11 September 1973

An economic model of trading in commodities that are inherently indivisible, like houses, is investigated from a game-theoretic point of view. The concepts of balanced game and core are developed, and a general theorem of Scarf's is applied to prove that the market in question has a nonempty core, that is, at least one outcome that no subset of traders can improve upon. A number of examples are discussed, and the final section reviews a series of other models involving indivisible commodities, with references to the literature.

1. Introduction

This paper has two purposes. To the reader interested in the mathematics of optimization, it offers an elementary introduction to n -person games, balanced sets, and the core, applying them to a simple but nontrivial trading model. To the reader interested in economics, it offers what may be a new way of looking at the difficulties that afflict the smooth functioning of an economy in the presence of commodities that come in large discrete units.

The core of an economic model, or of any multilateral competitive situation, may be described as the set of outcomes that are 'coalition optimal', in the sense that they cannot be profitably upset by the collusive action of any subset of the participants, acting by themselves. There is no reason, a priori, that such outcomes must exist; the core may well be empty. But it has been shown that important classes of economic models do have nonempty cores. In fact, whenever a system of *competitive prices* exists (prices under which individual optimization decisions will lead to a balance of supply and demand), the resulting outcome is in the core [see Debreu and Scarf (1963)]. The core may also exist in the absence of competitive prices. It is of some interest therefore, to relax

*This research was supported by the National Science Foundation, grant GS-31253. The authors are indebted to Professors David Gale and Bezalel Peleg for several helpful comments.

one or more of the classical 'regularity' assumptions that, taken together, ensure the existence of competitive prices – such as convexity of preferences, perfect divisibility of commodities, constant returns to scale in production, and absence of externalities – and to ask under what conditions the resulting system will have a core. There is already a considerable literature in this area.¹

We stress that the core is a general game-theoretic concept, definable without reference to any market model [see Aumann and Peleg (1960), Billera (1970), Scarf (1967), Shapley (1967 and 1973)]. Moreover, its existence for the classical exchange economies can be established, if we wish, without making use of the idea of competitive prices [see Scarf (1967), Shapley (1973), Shapley and Shubik (1969a)].

In this paper we consider the case of a commodity that is inherently indivisible, like a house. We formulate a class of markets in which a consumer never wants more than one item, but has ordinal preferences among the items available. We then prove that a core always exists for this model, making use of the concept of 'balanced sets'. The competitive prices for this model are next determined by a separate argument, providing an alternative and more constructive proof of the existence.

A counterexample is then considered, showing that if more complex schemes of preferences among the indivisible goods are allowed, the core may disappear even though all of the classical conditions except perfect divisibility are satisfied.

In the final section, for perspective, we review a series of other models involving indivisible commodities that have been discussed in the literature from the viewpoint of the core.

2. The model

Let there be n traders in the market, each with an indivisible good to offer in trade (e.g., a house). The goods are freely transferable, but we shall assume that a trader never has use for more than one item. There being no money or other medium of exchange, the only effect of the market activity is to redistribute the ownership of the indivisible goods, in accordance with the (purely ordinal) preferences of the traders. We shall describe these preferences with the aid of a square matrix: $A = (a_{ij})$, where $a_{ij} > a_{ik}$ means that trader i prefers item j to item k , and $a_{ij} = a_{ik}$ means that he is indifferent.² Owning no items, we assume, is ranked below all else, and owning several items is ranked only equal to the maximum of their separate ranks. Although only ordinal comparisons are involved in the model, it will be convenient to think of A as a matrix of real numbers.

¹For nonconvexity, see Shapley and Shubik (1966), Starr (1969), Henry (1972); for indivisibility, see Henry (1970 and 1972), Shapley and Shubik (1972), Shubik (1971); for non-constant returns, see Rader (1970), Scarf; for externalities, see Foley (1970), Shapley and Shubik (1969b). Not all of these refer directly to the core.

²By 'item j ' we mean the good brought to the market by trader j .

The final effect of any sequence of transfers can be described by another matrix $P = (p_{ij})$, called an *allocation*, in which $p_{ij} = 1$ if trader i holds item j at the end of trading, and $p_{ij} = 0$ otherwise. In the interesting cases, P will be a permutation matrix, i.e., a zero-one matrix with row-sums and column-sums all equal to 1. In any case, the column-sums of P will be equal to 1.

Let N denote the set of all traders, and let $S \subseteq N$. By an S -allocation P_S , we shall mean an n -by- n zero-one matrix containing one 1 in each column indexed by a member of S , but containing only zeros in the rows and columns indexed by members of $N - S$. An S -allocation describes a distribution of goods that the 'submarket' S could effect. An S -allocation with no row-sum greater than 1 is called an S -permutation. It is clear that for every S -allocation that is not an S -permutation, there is an S -permutation that is at least as desirable to every member of S , and more desirable to at least one member of S .

An allocation will be said to be a *core allocation* if there is no submarket S that could have done better for all its members. A core allocation, therefore, is one that cannot be improved upon by 'recontracting' in the sense of Edgeworth (1881). Our aim is to show that *every market of the kind described possesses at least one core allocation*.

3. Games and cores

First, let us recast the problem in a game-theoretic form. Let E^N denote the n -dimensional Euclidean space with coordinates indexed by the elements of the set N , and similarly E^S for $S \subset N$. For $x, y \in E^N$ and $S \subseteq N$, we define $y \geq_S x$ to mean that $y_i \geq x_i$, all $i \in S$. The notations $>_S$ and $=_S$ are defined similarly.

A 'cooperative game without side payments' [Aumann and Peleg (1960), Billera (1971), Scarf (1967), Shapley (1973)] will be identified with its 'characteristic function'. This is a function V from the nonempty subsets of the 'player space', N , to the subsets of the 'payoff space', E^N , satisfying the following conditions for each $S, \phi \subset S \subseteq N$:

- (a) $V(S)$ is closed.
- (b) If $x \in V(S)$ and $x \geq_S y$, then $y \in V(S)$.
- (c) $[V(S) - \bigcup_{i \in S} \text{int } V(\{i\})] \cap E^S$ is bounded and nonempty.³

Here 'int' denotes 'interior of'. Note that property (b) implies that each $V(S)$ is a *cylinder* (i.e., the Cartesian product of E^{N-S} with a subset of E^S). Intuitively, the projection of $V(S)$ on E^S is supposed to represent the pay-offs that the members of S , acting cooperatively, can achieve (or exceed) without outside help.

This interpretation suggests a fourth property, namely, *superadditivity*, which

³Condition (c) implies, in particular, that none of the $V(S)$ are either empty or equal to E^N .

may be expressed as

$$(d) \quad V(S \cup T) \supseteq V(T) \quad \text{if } S \cap T = \phi.$$

Although this condition often holds in practice, it will not be required in the definition of a game.

Of particular interest is a class of games in which the $V(S)$ are generated by finite sets $Y(S) \subseteq E^N$, as follows:

$$V(S) = \{x: y \geq_s x \quad \text{for some } y \in Y(S)\}.$$

Assuming that there are no superfluous generators, each payoff $y \in Y(S)$ identifies what might be called a 'corner' of the cylindrical set $V(S)$.⁴ Finitely generated games arise often in applications; they also figure prominently in the proof of the main theorem of Scarf (1967).⁵

The *core* of the game may be defined as the set

$$V(N) - \bigcup_{\phi \subset S \subseteq N} \text{int } V(S).$$

In other words, the core is the intersection of $V(N)$ with the closures of the complements of *all* of the $V(S)$, including $V(N)$ itself. The core is a closed subset of the boundary of $V(N)$, possibly empty but certainly bounded, and every point in the core is (weakly) Pareto optimal.

Intuitively, the core consists of those outcomes of the game that are *feasible* [i.e., are in $V(N)$], and that cannot be *improved upon* by any individual or coalition of individuals [i.e., are not interior to any $V(S)$]. It is of considerable interest in the analysis of any cooperative game to determine whether its core is nonempty.

Finally, we need the notion of a 'balanced' game. Let us call a family T of nonempty subsets of N *balanced* if the system of equations

$$\sum_{S: j \in S} \delta_S = 1, \quad j \in N,$$

has a nonnegative solution with $\delta_S = 0$ for all S not in T [see Shapley (1967 and 1973), Peleg (1965), Graver (1973)]. A partition of N is a simple example of a balanced family. The numbers δ_S are called *balancing weights* for T ; it has been shown [Shapley (1967)] that they are unique for T if and only if no proper subfamily of T is balanced.

The game V is said to be *balanced* if the following inclusion statement:

$$(e) \quad \bigcap_{S \in T} V(S) \subseteq V(N),$$

holds for *all* balanced families T . Property (e) is obviously related to property (d) via the partitions of N , but neither condition directly implies the other. A

⁴That is, a vertex of the projection of $V(S)$ on E^S . A glance ahead to figs. 1, 2, or 4 may aid the reader in visualizing the definitions of this section.

⁵But not in the alternative proof given in Shapley (1973).

fundamental theorem states that the core of a balanced game is not empty [see Bondareva (1963), Scarf (1967), Shapley (1967 and 1973), or the survey article of Billera (1971)]. We shall apply this theorem to the market described in sect. 2.

4. The core of the market

We now return to the market model of sect. 2. First, we must determine the characteristic function. The sets $V(S)$ are finitely generated by the S -permutations, since all other S -allocations are dominated by S -permutations in the sense of (b) above. It will be convenient to express $V(S)$ with the aid of a zero-one matrix $B_S(x)$, defined for each $x \in E^N$ as follows:

$$b_{S|ij}(x) = \begin{cases} 1 & \text{if } a_{ij} \geq x_i \text{ and } i \in S, \\ 0 & \text{if } a_{ij} < x_i \text{ or } i \notin S. \end{cases}$$

Thus, $B_S(x)$ tells us, for each trader i in S , exactly which items he ranks *at or above* his utility level x_i . We can then define the game V as follows:

$$V(S) = \{x: B_S(x) \geq P_S \text{ for some } S\text{-permutation } P_S\}.$$

Theorem. V is a balanced game; hence the market in question has a non-empty core.

Proof. It is immediately evident that V satisfies conditions (a), (b), and (c) of sect. 3. [We could also show without difficulty that V satisfies (d).] It remains to be shown that V satisfies (e).

Let T be any balanced family of coalitions, and let $x \in \bigcap_{S \in T} V(S)$. Let $\{\delta_S\}$ be balancing weights for T . Then we have

$$B_N(x) = \sum_{S \in T} \delta_S B_S(x).$$

By definition of $V(S)$, there exists an S -permutation P_S , for each $S \in T$, such that $B_S(x) \geq P_S$, and so

$$\sum_{S \in T} \delta_S B_S(x) \geq \sum_{S \in T} \delta_S P_S.$$

Call the matrix on the right D ; then we have

$$B_N(x) \geq D.$$

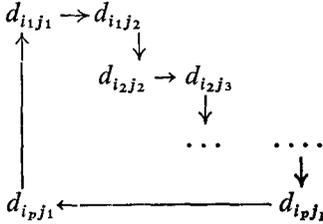
The crucial fact about D is that it is doubly stochastic; that is, it is nonnegative and has all row- and column-sums equal to 1. This follows directly from the definition of balancing weights; thus, the i th row sum is

$$\begin{aligned} \sum_{j \in N} \sum_{S \in T} \delta_S p_{S|ij} &= \sum_{S \in T} \delta_S \sum_{j \in N} p_{S|ij} \\ &= \sum_{S \in T} \delta_S \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases} = \sum_{S: i \in S} \delta_S = 1, \end{aligned}$$

and the argument for the column sums is the same.

The next step will be to change D into a permutation matrix P_N , that is, to eliminate any fractional entries without changing the row or column sums – and to do so without disturbing any entries that are already 0 or 1. Since all entries of $B_N(x)$ are 0 or 1, we will thereby ensure that $B_N(x) \geq P_N$.

Since a fraction cannot occur alone in a row or column, either D is already a permutation matrix or there is a closed loop of fractional entries:



Alternately adding and subtracting a fixed number ε to the elements of this loop will clearly preserve row and column sums. If ε is too large, then negative entries will be created, but making ε as large as possible consistent with non-negativity will produce a new doubly stochastic matrix D' that has at least one more zero than D , and hence fewer fractional entries. If D' is not yet doubly stochastic, we can repeat the operation. Eventually we must obtain what we want – a permutation matrix P_N such that $B_N(x) \geq P_N$. Hence x is in $V(N)$. Hence $\bigcap_{S \in T} V(S) \subseteq V(N)$. Hence the game is balanced. Q.E.D.

5. An example to illustrate the theorem

Let $n = 3$, and let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

This means that the first and third traders want only item 2, while the second trader wants either item 1 or item 3, indifferently.

The characteristic function, being finitely generated, can be described in terms of its ‘corners’ as follows:⁶

- $V(\{1\})$: (0, —, —),
- $V(\{2\})$: (—, 0, —),
- $V(\{3\})$: (—, —, 0),
- $V(\{1, 2\})$: (1, 1, —),
- $V(\{1, 3\})$: (0, —, 0),
- $V(\{2, 3\})$: (—, 1, 1),
- $V(\{1, 2, 3\})$: (1, 1, 0) and (0, 1, 1).

⁶See sect. 3. Here, for example, (1, 1, —) means the set $\{(1, 1, x_3): x_3 \text{ arbitrary}\}$. Note that since $V(N)$ happens to require more than one generator in this example it is a nonconvex set.

As shown in fig. 1, this game has an L-shaped core, with successive vertices $(1, 1, 0)$, $(0, 1, 0)$, $(0, 1, 1)$.

It is a curious fact that *all* outcomes in the core of this example are 'weakly' improvable, in the sense that one member of an effective coalition can do better while the other member does no worse. The reader may verify that $\{2, 3\}$ can weakly improve upon any point in the core except $(0, 1, 1)$, while $\{1, 2\}$ can weakly improve upon that point as well as any other point in the core except $(1, 1, 0)$. This illustrates the fact that the 'strict' improvability implicit in the

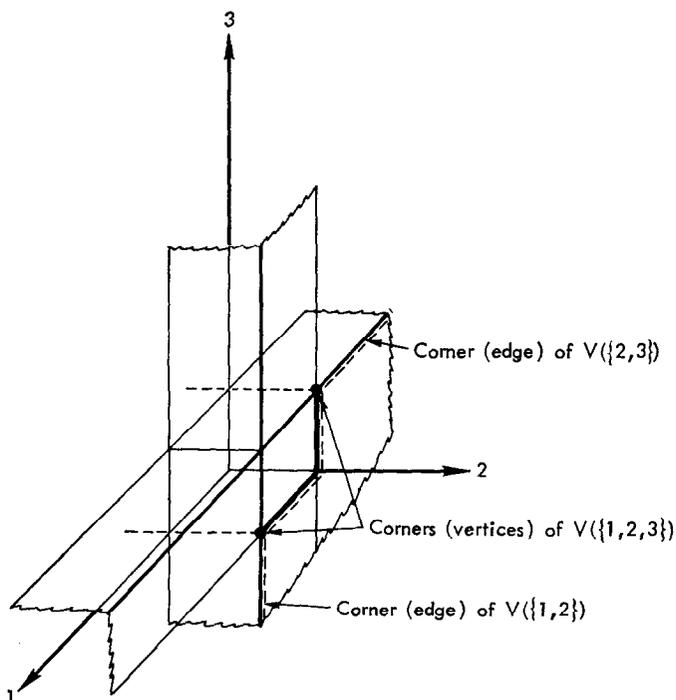


Fig. 1

definition of the core cannot be dispensed with, unless we are willing to give up the existence theorem for balanced games.

It should not be overlooked that what we are calling 'outcomes' of the game do not always correspond to actual trades in the market. Indeed, there are only finitely many ways in which the goods can be reallocated. Only if one allowed some sort of free disposability, permitting the traders to diminish at will the value of the goods,⁷ would it be possible to realize an arbitrary payoff

⁷The reader with some experience in the paradoxes of bargaining will recognize that it is not inconceivable that deliberately damaging one's goods might change the overall pattern of trade in a way that returns an advantage to the damager.

vector in the 'feasible' set $V(N)$. This observation does not affect the *existence* of core allocations in the market when a core exists for the game, since every 'unrealizable' point in $V(N)$ is majorized by a 'realizable' point. In the present example, of course, only the two tips of the 'L' represent actual trades in the market.

6. Competitive prices

After the proof in sect. 4 had been discovered, David Gale pointed out to the authors a simple constructive method for finding competitive prices in this market, and hence a point in its core. The following is based on his idea.

Let $R \subseteq N$, and define a *top trading cycle* for R to be any set S , $\phi \subset S \subseteq R$, whose s members can be indexed in a cyclic order:

$$S = \{i_1, i_2, \dots, i_s = i_0\},$$

in such a way that each trader i_v likes the i_{v+1} 'st good at least as well as any other good in R . It is evident that every nonempty $R \subseteq N$ has at least one top trading cycle, for we may start with any trader in R and construct a chain of best-liked goods that eventually must come back to some earlier element.⁸

Using this idea, we can partition N into a sequence of one or more disjoint sets:

$$N = S^1 \cup S^2 \cup \dots \cup S^p,$$

by taking S^1 to be any top trading cycle for N , then taking S^2 to be any top trading cycle for $N - S^1$, then taking S^3 to be any top trading cycle for $N - (S^1 \cup S^2)$, and so on until N has been exhausted. We can now construct a payoff vector x by carrying out the indicated trades within each cycle. That is, if $i = i_v^j \in S^j$, then x_i is i 's utility for the good of trader i_{v+1}^j . We assert (1) that x , so constructed, is in the core, and (2) that a set of competitive prices exists for x .

To establish (1), let S be any coalition. Consider the first j such that $S \cap S^j \neq \phi$. Then we have

$$S \subseteq S^j \cup S^{j+1} \cup \dots \cup S^p = N - (S^1 \cup \dots \cup S^{j-1}).$$

Let $i \in S \cap S^j$. Then i is already getting in x the highest possible payoff available to him in S . No improvement is possible for him, unless he deals outside of S . Hence S cannot strictly improve, and it follows that x is in the core.

To establish (2), we merely assign arbitrary prices

$$\pi^1 > \pi^2 > \dots > \pi^p > 0$$

to the goods belonging to the respective cycles S^1, S^2, \dots, S^p . Then trader i in

⁸A top trading cycle may consist of a single trader!

S^j can sell his own item for π^j 'dollars'. He cannot afford any items from S^1, \dots, S^{j-1} , and so his utility is maximized if he buys the item of his cyclic successor in S^j . This, of course, costs him precisely π^j 'dollars' and yields him the payoff x_i .

Not surprisingly, given the discreteness of the model, the conditions on the prices are purely ordinal. To the extent that there may be different ways of constructing top trading cycles, nonuniqueness may occur in the final outcome as well as in the price ordering. It is easily seen, however, that there are no other competitive prices beyond those constructed in the above fashion, except that when two or more *disjoint* top trading cycles exist, at any stage of the construction, they may be assigned equal prices.

In the example of sect. 5, either $\{1, 2\}$ or $\{2, 3\}$ will serve as the first top trading cycle S^1 , so that we can use either $\pi_1 = \pi_2 > \pi_3$ or $\pi_3 = \pi_2 > \pi_1$ for the competitive prices. The corresponding competitive outcomes are the two tips of the L-shaped core (fig. 1). Note that $\pi_1 = \pi_2 = \pi_3$ would *not* be competitive, as the demand for item 2 would then exceed the supply. Thus, we see from this example that the set of competitive prices is not necessarily closed.

7. Another example

It may be wondered in models of this type whether the core is really more general than the competitive solution. In the previous example the core of the *game* contained many 'outcomes' in addition to the competitive outcome, but none of them were realizable in the *market*, given the indivisibility and undamageability of the goods. One might suspect that every *core allocation* is necessarily competitive, after all.

A counterexample is provided, however, by the following preference matrix:

	Goods		
	0	1	2
Traders	1	0	-1
	-1	1	0

The characteristic function is depicted in fig. 2; the point marked ■ is the unique 'corner' of $V(N)$. Here there is a unique top trading cycle, namely N itself, so the competitive prices are all equal and the unique competitive payoff is $(2, 1, 1)$, resulting from the N -permutation

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

This is of course the point ■. The core of the game evidently consists of the three shaded rectangles grouped around that point, and we see that it contains

the point $\bullet = (1, 1, 0)$ that results from the N -permutation

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which is *not* a competitive allocation.

This example is not completely satisfying, since the noncompetitive outcome \bullet in the core is weakly majorized by the competitive outcome \blacksquare . Perhaps a

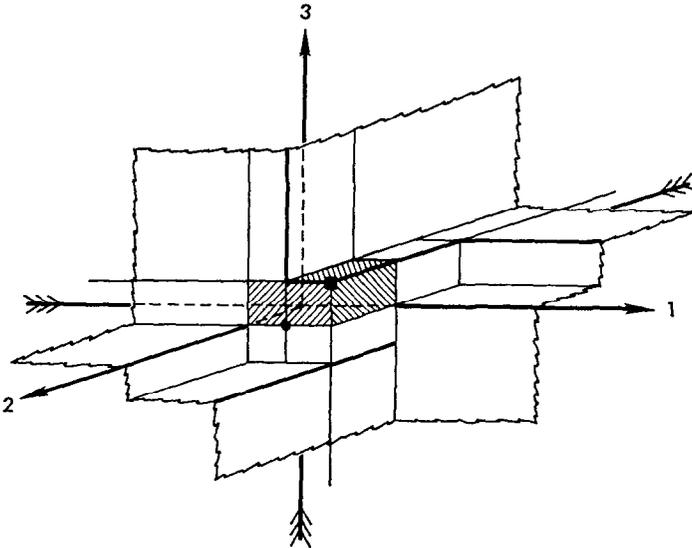


Fig. 2

better example exists, with more traders. However, as we have already remarked, we cannot base the definition of the core on weak majorization without losing the fundamental existence theorem.

8. More complex preferences: A counterexample

Let three traders have symmetric holdings in a tract of nine houses, as shown in fig. 3. (Thus, trader 1 owns houses 1, 1', and 1'') For reasons inscrutable, each trader wants to acquire three houses in a row including exactly one of his original set. Moreover, each prefers the long row that meets this condition to the short row.

We shall show that this example of a slightly more general trading game than the preceding is not balanced, and indeed has no core – thereby dispelling any

idea that the core might prove a universal remedy for market failure due to indivisibility.

The configuration of the tract is such that any two traders can make a profitable exchange. For example, a swap of 1' and 1'' for 2 and 2' gives trader 1 his long row and trader 2 his short row. Let us assign numerical values 2, 1, and 0 to the possession of the long row, the short row without the long row, and neither

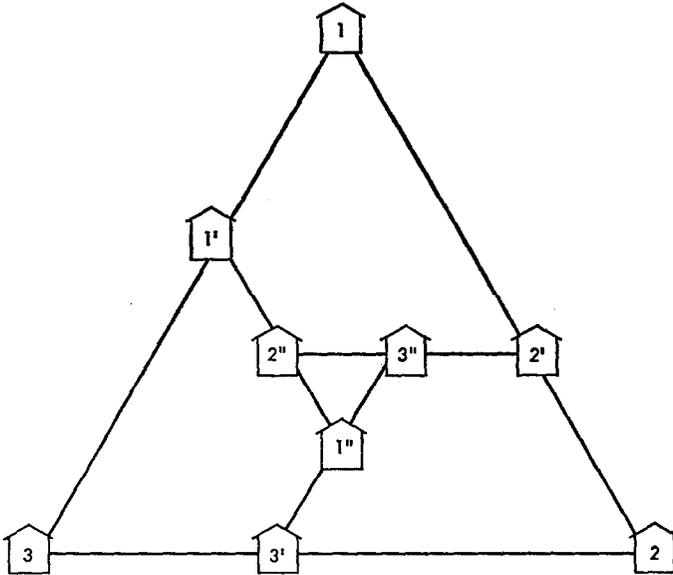


Fig. 3

row, respectively. Then the two-person coalitions have single 'corners':

$$V(\{1, 2\}): (2, 1, -),$$

$$V(\{1, 3\}): (1, -, 2),$$

$$V(\{2, 3\}): (-, 2, 1),$$

as shown in fig. 4. The three-person coalition cannot improve upon these pairwise exchanges; so its characteristic function is generated by three 'corners':

$$V(\{1, 2, 3\}): \begin{cases} (2, 1, 0), \\ (1, 0, 2), \\ (0, 2, 1), \end{cases}$$

as shown in the inset. Of course, the three singleton coalitions can achieve only 0.

It is easy to see that this game is *not* balanced [condition (e) in sect. 3], since the point (1, 1, 1) is in $V(\{1, 2\}) \cap V(\{1, 3\}) \cap V(\{2, 3\})$, but not $V(\{1, 2, 3\})$.

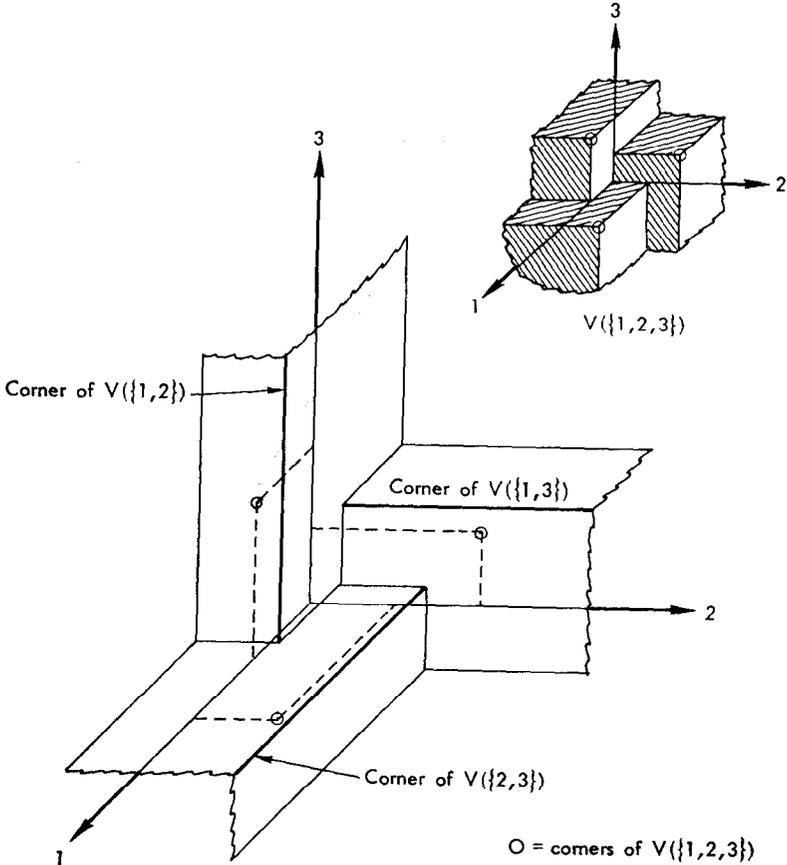


Fig. 4

It is equally easy to see that the game has no core,⁹ since each of the generators of $V(\{1, 2, 3\})$ is interior to one of the $V(\{i, j\})$. In other words, the set $V(\{1, 2, 3\})$ is completely hidden by the union of the $V(\{i, j\})$, as suggested by the broken lines in the main figure.

It might be asked whether the absence of a core in this example might not arise from some intrinsic nonconvexity in the preference sets themselves that has nothing to do with indivisibility. The answer is no; to see this, define

$$U^1(x) = \min[2, \min(x_1, x_2, x_2) + 2 \min(x_3, x_1, x_3)],$$

with $U^2(x)$ and $U^3(x)$ similar. These utility functions are concave in the nine real variables x_1, \dots, x_3 (since taking a 'min' preserves concavity), and so gener-

⁹We remind the reader that balancedness is sufficient, but not necessary, for a nonempty core.

ate convex preference sets. Moreover, if we restrict the variables to be 0 or 1 we obtain just the payoffs used above. Thus, the actual preference sets are nonconvex *only* because of indivisibility.

9. Other indivisibility models

Several other trading models with indivisible goods have been considered in the literature from the point of view of the core. They range from fairly general situations for which positive results can be obtained to specific illustrative counterexamples.

In the 'marriage market' or 'dance floor' of Gale and Shapley (1962), there are two *types* of traders. The members of each type rank those of the other type in order of preference as partners; then they pair off. There are generally many core allocations for this situation, that is, arrangements into pairs such that no two individuals of opposite type could do better. Curiously enough, it may be that none of these core allocations gives anyone his (or her) first choice. (In contrast the 'top trading cycle' construction of sect. 6 obviously ensures that at least one trader gets his first choice.) A simple 'courtship' algorithm is described in Gale and Shapley (1962) for reaching a point in the core.¹⁰ In fact, two extremal allocations are reached: one gives every member of the first type the best outcome possible within the core; the other does the same for the second type. These two allocations coincide only in the case of a one-point core.¹¹

It does not appear to be possible to set up a conventional market for this model, in such a way that a competitive price equilibrium will exist and lead to an allocation in the core.

A very similar model is the 'problem of the roommates', also discussed in Gale and Shapley (1962). The difference is that there are no types; each trader ranks *all* the others as potential partners before they pair off. In this case, a very simple four-person example shows that there need not be a core.¹²

In the 'treasure hunt' of Shapley and Shubik,¹³ a party of explorers finds

¹⁰The deferred acceptance algorithm of Gale and Shapley (1962) is easily described: (1) Let each boy propose to his favorite girl; (2) let each proposed-to girl keep the best suitor waiting and reject all others; (3) let each rejected boy propose to his next-best girl; (4) repeat steps (2) and (3) until either there are no rejected boys left or every rejected boy has exhausted the list of girls. The resulting pairing-off corresponds to a point in the core, and no boy can do better at any other point in the core. See also: *The New Yorker*, 11 Sept. 1971, p. 94.

¹¹Similar results hold for the more general 'college admissions market', in which each trader of the first type can accommodate a large number of traders of the second type; see Gale and Shapley (1962).

¹²Let A rank $B > C > D$; let B rank $C > A > D$; let C rank $A > B > D$; and let D rank arbitrarily. Then no pairing is stable, in the sense of the core. For example, $(AB)(CD)$ can be improved upon by the coalition $\{B, C\}$.

¹³Inspired by the Huston-Traven classic 'Treasure of the Sierra Madre', whose plot turns on the coalitional instability of a party of *three* prospectors in a rather similar predicament.

a big cache of treasure chests in the desert, momentarily exposed during a sandstorm, but so heavy that it takes two men to carry each chest to high ground. If we consider the bargaining game that determines how the profits are to be divided, it is not difficult to see that the existence of a core depends on whether the size of the party is even or odd. Similarly, in the 'Bridge game economy' of Shubik (1971), an exact multiple of four players is required, if the card party is to have a core.

In most of these examples, the real issue is the indivisibility of the participants themselves, rather than the indivisibility of some more or less tangible economic commodity that is owned and is transferable. The individual is required to participate fully and exclusively in a single activity in order to have any effect. Thus, these examples could also be regarded as instances of increasing returns to scale in the labor inputs to certain production processes, or even as non-convexities in the preferences for certain forms of consumption.¹⁴

In the 'assignment game' of Shapley and Shubik (1972),¹⁵ there are again two types of traders, namely, *sellers* and *buyers*. The first have houses, say, and the second have money. Preferences are not merely ordinal, but are expressed as monetary evaluations of the different houses. Since money is fully transferable and infinitely divisible, the sets $V(S)$ are not finitely generated; instead they are half-spaces, and the market reduces to a 'game with side payments'. Competitive prices exist and are closely related to the linear-programming solution to the problem of maximizing the total monetary value of the allocation, which turns out to be the familiar 'optimal assignment' problem. These prices are usually not unique. The core in this case is exactly the set of competitive allocations, in contrast to what we found in sect. 7 above. As in the marriage market, two extremal allocations can be distinguished in the core: a 'high-price' corner, which is best possible for every seller, and a 'low-price' corner, which is best possible for every buyer. If the houses happen to be all alike, then the core reduces to a line segment and at any given point in the core all houses have the same price.

It would be interesting if a general framework could be found that would unify some or all of these scattered results.

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¹⁴Compare the 'gin and tonic' example in Shapley and Shubik (1966), where again the existence of a core depends on the parity of the number of traders.

¹⁵See also Gale (1960, pp. 160–162).

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