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STRATEGY-PROOFNESS AND ESSENTIALLY SINGLE-VALUED CORES

BY TAYFUN SÖNMEZ[1]

1. INTRODUCTION

IN THIS PAPER WE SEARCH for solutions to various classes of allocation problems. We require them to be *Pareto efficient* and *individually rational* (in the sense that no agent is ever worse off than he would be on his own). In addition to these minimal requirements we also would like agents not to be able to profitably misrepresent their preferences. This property is known as *strategy-proofness*.[2]

Two extensively analyzed classes of allocation problems are *marriage problems* (Gale and Shapley (1962)), and *housing markets* (Shapley and Scarf (1974)). Results concerning strategy-proofness in marriage problems are quite disappointing. Roth (1982a) shows that in the context of marriage problems there is no selection from the core correspondence that is strategy-proof. Moreover, Alcalde and Barberà (1994) show that there is no solution that is Pareto efficient, individually rational, and strategy-proof.[3] On the other hand results pertaining to housing markets are much more encouraging. Roth (1982b) shows that in the context of housing markets the core correspondence, which is shown to be single-valued by Roth and Postlewaite (1977), is strategy-proof. Moreover Ma (1994) shows that it is the only solution that is Pareto efficient, individually rational, and strategy-proof.

We search for foundations of the differences in these results. We do this by studying strategy-proofness on a general class of allocation problems that includes both models as subclasses. In addition to marriage problems and housing markets this class also includes

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[2] Strategy-proofness was first analyzed in abstract social choice models where there are few or no restrictions on preferences. Gibbard (1973) and Satterthwaite (1975) show that, under minor conditions strategy-proofness is equivalent to dictatorship. In models with more structure (such as economic models) some positive results are available. See, for example, Barberà, Gül, and Stacchetti (1993), Barberà and Jackson (1994), Ching (1994), Moulin (1980, 1994), Moulin and Shenker (1992), Roth (1982b), Sönmez (1996), Sprumont (1991).

[3] Kara and Sönmez (1996) weaken the incentive requirement and search for Nash implementable allocation rules. They show that any solution that is Pareto efficient, individually rational, and Nash implementable is a supersolution of the core correspondence.

roommate problems (Gale and Shapley (1962)), *indivisible goods exchange economies*, *coalition formation problems* (Banerjee, Konishi, and Sönmez (1997)), and *networks* (Jackson and Wolinsky (1996)). We show that for any model in this class, there exists a Pareto efficient, individually rational, and strategy-proof solution only if all allocations in the core are Pareto indifferent for all problems. In fact, any such solution selects an allocation in the core whenever the core is nonempty. Furthermore any selection from the core correspondence is strategy-proof as long as the core correspondence is essentially single-valued and the core of each problem is externally stable. We obtain the positive results of Roth (1982b), Ma (1994) for the housing markets, the negative results of Roth (1982a) and Alcalde and Barberà (1994), for marriage problems, and new negative results for indivisible goods exchange economies, coalition formation problems, and networks as direct applications of our general results.

An implication of our results is that, for a wide class of problems, the core is the key concept when one searches for strategy-proof solutions that are Pareto efficient and individually rational. We believe this conclusion provides important noncooperative support for the core correspondence, a cooperative solution.

2. THE GENERAL MODEL

A *generalized indivisible goods allocation problem*, or simply a *problem*, is a 4-tuple $(N, \omega, \mathscr{A}^f, R)$ where $N=\{1,\dots,n\}$ is a finite set of agents, $\omega=(\omega(1),\dots,\omega(n))$ is an *initial endowment*, $\mathscr{A}^f$ is a set of *feasible allocations*, and $R=(R_1,\dots,R_n)$ is a list of preference relations. Each agent i is endowed with a set of indivisible goods $\omega(i)$ that we refer to as agent i's initial endowment. For all $T \subseteq N$, let $\omega(T)=\bigcup_{i\in T}\omega(i)$ denote the set of indivisible goods owned by agents in coalition T. An allocation a is a mapping from N to $\omega(N)$ such that each good is assigned to one agent. Let $\mathscr{A}$ denote the set of all allocations. Formally,

$$\mathscr{A}=\{a: N \twoheadrightarrow \omega(N) \mid \forall x \in \omega(N), |a^{-1}(x)|=1\}.$$

We exogenously specify a subset $\mathscr{A}^f$ of the set of allocations as the set of feasible allocations. We require that $\omega \in \mathscr{A}^f$. The preference relation R_i of each agent $i \in N$ is a binary relation on $\mathscr{A}^f$ that is *complete* (for all $a, b \in \mathscr{A}^f$ we either have aR_ib or bR_ia) and *transitive* (for all $a,b,c \in \mathscr{A}^f$ if aR_ib and bR_ic then aR_ic). Let P_i denote the *strict preference relation* and I_i denote the *indifference relation* induced by R_i. Let $\mathscr{R}_i$ be the class of all preference relations for agent i. We require $\mathscr{R}_i$ to satisfy two conditions.

ASSUMPTION A: *For all $R_i \in \mathscr{R}_i$ and for all $a \in \mathscr{A}^f$,*

$$aI_i\omega \Leftrightarrow a(i)=\omega(i).$$

That is, an agent is indifferent between an allocation and the initial endowment if and only if he keeps his initial endowment.

ASSUMPTION B: *For all $R_i \in \mathscr{R}_i$ and for all $a \in \mathscr{A}^f$ with $aR_i\omega$, there exists $\tilde{R}_i \in \mathscr{R}_i$ such that*

1. $\forall b \in \mathscr{A}^f \setminus \{a\}, \quad bR_ia \Leftrightarrow b\tilde{R}_ia,$
2. $\forall b \in \mathscr{A}^f \setminus \{a\}, \quad aR_ib \Leftrightarrow a\tilde{R}_ib,$
3. $\forall b \in \mathscr{A}^f \setminus \{a\}, \quad aP_ib \Leftrightarrow a\tilde{P}_ib, \quad$ and $\quad a\tilde{R}_i\omega\tilde{R}_ib.$

That is, for any preference relation R_i and any allocation a that is at least as good as the initial endowment, there exists a preference relation $\tilde{R}_i$ such that, all allocations that are better than a under R_i are better than a under $\tilde{R}_i$, all that are worse remain worse, and the initial endowment ranks right after (or indifferent to) a.

An important class that satisfies these conditions is the class of preferences where an agent has strict preferences over own assignment and does not care for others' assignments (i.e., no consumption externalities). Let $\mathscr{R}_i^s$ be the class of all such preferences for agent i.

Let $\mathscr{R} = \prod_{i \in N} \mathscr{R}_i$ and $\mathscr{R}^s = \prod_{i \in N} \mathscr{R}_i^s$. For all $R \in \mathscr{R}$, and for all $T \subset N$, we denote the restriction of R to T by R_T, and the set $N \setminus T$ by $-T$. For all $i \in N$, we denote the set $N \setminus \{i\}$ by $-i$. Throughout the paper we fix $N, \omega, \mathscr{A}^f, \mathscr{R}$ and hence each preference profile $R \in \mathscr{R}$ defines a problem.

An allocation $a \in \mathscr{A}^f$ is *individually rational* under R if $aR_i\omega$ for all $i \in N$. We denote the set of all individually rational allocations under R by $\mathscr{I}(R)$.

An allocation $a \in \mathscr{A}^f$ *Pareto dominates* an allocation $b \in \mathscr{A}^f$ under R if aR_ib for all $i \in N$ and aP_jb for some $j \in N$. An allocation $a \in \mathscr{A}^f$ is *Pareto efficient* if it is not Pareto dominated by any allocation $b \in \mathscr{A}^f$. We denote the set of all Pareto efficient allocations under R by $\mathscr{P}(R)$. An allocation $a \in \mathscr{A}^f$ *strictly Pareto dominates* an allocation $b \in \mathscr{A}^f$ under R if aP_ib for all $i \in N$. Two allocations $a, b \in \mathscr{A}^f$ are *Pareto indifferent* under R if aI_ib for all $i \in N$.

An allocation $a \in \mathscr{A}^f$ *weakly dominates* the allocation $b \in \mathscr{A}^f$ via the coalition $T \subseteq N$ under R if

1. $\forall i \in T, \quad a(i) \subseteq \omega(T)$,
2. $\forall i \in T, \quad aR_ib$,
3. $\exists j \in T, \quad aP_jb$.

In that case we say that the coalition T *blocks* b under R via a. An allocation $a \in \mathscr{A}^f$ is in the *core*[4] of the problem $R \in \mathscr{R}$ if it is not weakly dominated by any feasible allocation. We denote the core of R by $\mathscr{C}(R)$. The *core correspondence* is the correspondence that assigns the set of allocations in the core for each problem. For a given 4-tuple $(N, \omega, \mathscr{A}^f, \mathscr{R})$, the core correspondence is *essentially single-valued* if the core of each problem is nonempty and any pair of allocations in the core are Pareto indifferent.

A set of allocations $\mathscr{B} \subseteq \mathscr{A}^f$ is *externally stable* under $R \in \mathscr{R}$ if every allocation in $\mathscr{A}^f \setminus \mathscr{B}$ is dominated by some allocation in $\mathscr{B}$. Note that any externally stable set under R is a superset of the core of R. An *allocation rule* is a function $\varphi : \mathscr{R} \to \mathscr{A}^f$. An allocation rule φ is *Pareto efficient* if $\varphi(R) \in \mathscr{P}(R)$ for all $R \in \mathscr{R}$, and *individually rational* if $\varphi(R) \in \mathscr{I}(R)$ for all $R \in \mathscr{R}$. An allocation rule φ is *strategy-proof* if for all $R \in \mathscr{R}$, for all $i \in N$, and for all $\tilde{R}_i \in \mathscr{R}_i$,

$$\varphi(R) R_i \varphi(R_{-i}, \tilde{R}_i),$$

and it is *weakly coalitionally strategy-proof* if for all $R \in \mathscr{R}$, for all $T \subseteq N$, and for all $\tilde{R}_T \in \mathscr{R}_T$ there exists $i \in T$ such that

$$\varphi(R) R_i \varphi(R_{-T}, \tilde{R}_T).$$

[4] This notion of core, which is defined by weak domination, is widely known as the *strict core*.

That is, an allocation rule is strategy-proof if no agent can ever benefit by unilaterally misrepresenting his preferences and it is weakly coalitionally strategy-proof if no group of agents can benefit by jointly misrepresenting their preferences. Note that weak coalitional strategy-proofness is stronger than strategy-proofness and weaker than the notion of coalitional strategy-proofness, which allows some agents to be indifferent when misrepresenting their preferences.

3. THE MAIN RESULTS

Our first result, which is a variant of a theorem in Demange (1987), concerns weak coalitional strategy-proofness of the core correspondence whenever it is essentially single-valued and the core of each problem is externally stable.

PROPOSITION 1: *Let $N, \omega, \mathscr{A}^f, \mathscr{R}$ be such that the core correspondence is essentially single-valued and the core of each problem is externally stable. Then any selection from the core correspondence is weakly coalitionally strategy-proof.*

PROOF: Let $N, \omega, \mathscr{A}^f, \mathscr{R}$ be such that:

1. $\forall R \in \mathscr{R}, \forall i \in N, \forall a, b \in \mathscr{C}(R), \quad aI_i b,$
2. $\forall R \in \mathscr{R}, \quad \mathscr{C}(R)$ is externally stable.

Let φ be a selection from the core correspondence. That is, $\varphi(R) \in \mathscr{C}(R)$ for all $R \in \mathscr{R}$. Suppose φ is not weakly coalitionally strategy-proof. Then, by definition there exist $R \in \mathscr{R}$, $T \subseteq N$, $\tilde{R}_T \in \prod_{i \in T} \mathscr{R}_i$ such that

$$\text{(1)} \qquad \forall i \in T, \qquad \varphi(R_{-T}, \tilde{R}_T) P_i \varphi(R).$$

Since $\varphi(R) \in \mathscr{C}(R)$ and the core is essentially single-valued, $\varphi(R_{-T}, \tilde{R}_T) \notin \mathscr{C}(R)$. Moreover $\mathscr{C}(R)$ is externally stable and therefore there exists $a \in \mathscr{C}(R)$ which dominates $\varphi(R_{-T}, \tilde{R}_T)$ under R. That is, there exists a coalition $U \subseteq N$ such that

$$\forall i \in U, \qquad a(i) \in \omega(U),$$

$$\forall i \in U, \qquad aR_i \varphi(R_{-T}, \tilde{R}_T),$$

$$\exists j \in U, \qquad aP_j \varphi(R_{-T}, \tilde{R}_T).$$

We have $aI_i \varphi(R)$ for all $i \in N$ and this together with relation (1) imply that $T \cap U = \emptyset$. But then a dominates $\varphi(R_{-T}, \tilde{R}_T)$ under $(R_{-T}, \tilde{R}_T)$, contradicting $\varphi(R_{-T}, \tilde{R}_T) \in \mathscr{C}(R_{-T}, \tilde{R}_T)$. *Q.E.D.*

REMARK 1: Demange (1987) introduces a notion of coalitional nonmanipulability for correspondences and shows that the core correspondence is coalitionally nonmanipulable as long as it is nonempty and satisfies the following weaker notion of external stability for all problems: A set of allocations $\mathscr{B}$ is weakly stable if every allocation outside $\mathscr{B}$ is blocked by a coalition all of whose members prefer an allocation in $\mathscr{B}$ to this allocation. Proposition 1 is still valid if external stability is replaced by weak stability. Demange's theorem reduces to this stronger version of Proposition 1 whenever the core correspondence is essentially single-valued.

The core correspondence being essentially single-valued is a very strong assumption.[5] However it is a necessary condition for the existence of a Pareto efficient, individually rational, and strategy-proof allocation rule on the classes of problems with a nonempty valued core correspondence. Furthermore if such an allocation rule exists, it is a selection from the core correspondence. We first prove a stronger version that asserts that if an allocation rule φ is Pareto efficient, individually rational, and strategy-proof, then all allocations in the core are Pareto indifferent and the allocation rule φ must select one of them whenever the core is nonempty.

Assumption B plays a crucial role in the proof of this result: For any true preference profile R and any allocation a in the core, it ensures the existence of a preference profile $\tilde{R}$ such that allocation a is Pareto indifferent to any allocation that is Pareto efficient and individually rational under $\tilde{R}$. Therefore the allocation $\varphi(\tilde{R})$ should be Pareto indifferent to a. Moreover if one of the agents reports his true preferences, then by strategy-proofness he should attain the same welfare level and by Pareto efficiency and individual rationality all other agents do too. A proof by induction on the cardinality of the set of truthful agents then shows that $\varphi(R)$ is Pareto indifferent to allocation a as well. Since this argument is valid for any allocation in the core, we cannot avoid a contradiction unless all allocations in the core are Pareto indifferent.

THEOREM 1: *If there exists an allocation rule $\varphi : \mathscr{R} \to \mathscr{A}^f$ that is Pareto efficient, individually rational, and strategy-proof, then N, $\omega, \mathscr{A}^f, \mathscr{R}$ are such that*

1. $\forall R \in \mathscr{R}, \forall i \in N, \forall a, b \in \mathscr{C}(R), \quad aI_ib,$
2. $\forall R \in \mathscr{R}$ *with* $\mathscr{C}(R) \neq \varnothing, \quad \varphi(R) \in \mathscr{C}(R).$

PROOF: Fix N, $\omega, \mathscr{A}^f, \mathscr{R}$. Let $\varphi : \mathscr{R} \to \mathscr{A}^f$ be Pareto efficient, individually rational, and strategy-proof. Let $R \in \mathscr{R}$ and $a \in \mathscr{C}(R)$. We will establish the proof by showing that $\varphi(R)$ is Pareto indifferent to a under R. Let $\tilde{R} \in \mathscr{R}$ be such that for all $i \in N$

$$\forall b \in \mathscr{A}^f \setminus \{a\}, \qquad bR_ia \Leftrightarrow b\tilde{R}_ia, \tag{2}$$

$$\forall b \in \mathscr{A}^f \setminus \{a\}, \qquad aR_ib \Leftrightarrow a\tilde{R}_ib, \tag{3}$$

$$\forall b \in \mathscr{A}^f \setminus \{a\}, \qquad aP_ib \Leftrightarrow a\tilde{P}_ib, \text{ and } a\tilde{R}_i\omega\tilde{R}_ib. \tag{4}$$

Assumption B ensures the existence of such a preference profile. Also note that $a \in \mathscr{C}(\tilde{R})$ for otherwise the blocking coalition would block it under R as well.

CLAIM 1: *The allocation $\varphi(\tilde{R})$ is Pareto indifferent to a under $\tilde{R}$. That is,*

$$\forall i \in N, \qquad \varphi(\tilde{R})\tilde{I}_ia.$$

[5] Nevertheless, because of the indivisibility of goods, it is satisfied in some interesting classes of problems such as housing markets.

Proof of Claim 1: We will prove Claim 1 by showing that a is Pareto indifferent to any allocation that is Pareto efficient and individually rational under $\tilde{R}$.

We have $a \in \mathscr{C}(\tilde{R}) \subseteq \mathscr{P}(\tilde{R}) \cap \mathscr{I}(\tilde{R})$. Let $b \in \mathscr{A}^f$ be such that $b \in \mathscr{P}(\tilde{R}) \cap \mathscr{I}(\tilde{R})$. Suppose that

$$\text{(5)} \qquad \exists i \in N, \qquad b\tilde{P}_i a.$$

We will show that this will lead to a contradiction. Consider the coalition

$$T = \{i \in N \mid b(i) \neq \omega(i)\}.$$

That is, $T \subseteq N$ is the set of agents who are not assigned their initial endowments at allocation b. We have two cases to consider:

Case 1: $T = \varnothing$. For this case we have $b = \omega$. Since $a \in \mathscr{C}(\tilde{R}) \subseteq \mathscr{I}(\tilde{R})$

$$\forall i \in N, \qquad a\tilde{R}_i \omega,$$

and therefore

$$\forall i \in N, \qquad a\tilde{R}_i b,$$

contradicting relation (5).

Case 2: $T \neq \varnothing$. We have

$$\text{(6)} \qquad \forall i \in T, \qquad b(i) \subseteq \omega(T)$$

by the construction of the coalition T. Moreover since $b \in \mathscr{I}(\tilde{R})$

$$\forall i \in T, \qquad b\tilde{R}_i \omega.$$

Therefore construction of coalition T together with Assumption A imply

$$\forall i \in T, \qquad b\tilde{P}_i \omega,$$

which in turn implies

$$\text{(7)} \qquad \forall i \in T, \qquad b\tilde{R}_i a$$

by the construction of $\tilde{R}_i$. We also have

$$\text{(8)} \qquad \forall i \in N \setminus T, \qquad a\tilde{R}_i b$$

by $a \in \mathscr{I}(\tilde{R})$, $b(i) = \omega(i)$ for all $i \in N \setminus T$, and Assumption A. Suppose we have $a\tilde{R}_i b$ for all $i \in T$. This together with relation (8) imply either a Pareto dominates b under $\tilde{R}$ or a is Pareto indifferent to b under $\tilde{R}$, both of which contradict relation (5). Therefore we cannot have $a\tilde{R}_i b$ for all $i \in T$ and thus

$$\text{(9)} \qquad \exists j \in T, \qquad b\tilde{P}_j a.$$

But then coalition T blocks the allocation a under $\tilde{R}$ via b by relations (6), (7), (9), contradicting $a \in \mathscr{C}(\tilde{R})$, completing Case 2.

Therefore

$$\nexists\, i \in N, \qquad b\tilde{P}_i a,$$

and hence

$$\forall i \in N, \qquad a\tilde{R}_i b.$$

Moreover $b \in \mathscr{P}(\tilde{R}_i)$ and therefore a and b are Pareto indifferent under $\tilde{R}$. But $b \in \mathscr{P}(\tilde{R}) \cap \mathscr{I}(\tilde{R})$ is arbitrary and φ is Pareto efficient and individually rational. Therefore

$$\forall i \in N, \qquad \varphi(\tilde{R})\tilde{I}_i a.$$

CLAIM 2: *The allocation $\varphi(R)$ is Pareto indifferent to a under R. That is,*

$$\forall i \in N, \qquad \varphi(R) I_i a.$$

PROOF OF CLAIM 2: We will show that, for all $T \subseteq N$,

$$\forall i \in N \setminus T, \qquad \varphi(\tilde{R}_{-T}, R_T)\tilde{I}_i a,$$

$$\forall i \in T, \qquad \varphi(\tilde{R}_{-T}, R_T) I_i a,$$

by induction on the cardinality of T.

Let us first show this for $|T| = 1$. Let $j \in N$. First note that Claim 1 together with relations (2) and (3) imply

$$(10) \qquad \varphi(\tilde{R}) I_j a.$$

Consider the preference profile $(\tilde{R}_{-j}, R_j)$. By strategy-proofness we have

$$(11) \qquad \varphi(\tilde{R})\tilde{R}_j \varphi\big(\tilde{R}_{-j}, R_j\big),$$

$$(12) \qquad \varphi\big(\tilde{R}_{-j}, R_j\big) R_j \varphi(\tilde{R}).$$

Claim 1 together with relations (11) and (3) imply $aR_j\varphi(\tilde{R}_{-j}, R_j)$. Moreover we have $\varphi(\tilde{R}_{-j}, R_j)R_j a$ by relations (10) and (12), and therefore $\varphi(\tilde{R}_{-j}, R_j)I_j a$. But any allocation $b \in \mathscr{P}(\tilde{R}_{-j}, R_j) \cap \mathscr{I}(\tilde{R}_{-j}, R_j)$ with $bI_j a$ is Pareto indifferent to allocation a under $(\tilde{R}_{-j}, R_j)$ (as otherwise we contradict Claim 1). Therefore

$$\forall i \in N \setminus \{j\}, \qquad \varphi\big(\tilde{R}_{-j}, R_j\big)\tilde{I}_i a,$$

establishing the proof for $|T| = 1$. Next suppose that for all $T \subseteq N$ with $|T| = l < n$,

$$(13) \qquad \begin{aligned} &\forall i \in N \setminus T, \qquad \varphi(\tilde{R}_{-T}, R_T)\tilde{I}_i a, \\ &\forall i \in T, \qquad \varphi(\tilde{R}_{-T}, R_T) I_i a. \end{aligned}$$

We will show that the same holds for all $T \subseteq N$ with $|T| = l + 1 \leq n$.

Let $T \subseteq N$ be such that $|T| = l + 1$. Let $j \in T$. Consider the preference profile $(\tilde{R}_{-T}, R_T)$. By strategy-proofness we have

$$\varphi\left(\tilde{R}_{(N \setminus T) \cup \{j\}}, R_{T \setminus \{j\}}\right) \tilde{R}_j \varphi(\tilde{R}_{-T}, R_T),$$

$$\varphi(\tilde{R}_{-T}, R_T) R_j \varphi\left(\tilde{R}_{(N \setminus T) \cup \{j\}}, R_{T \setminus \{j\}}\right),$$

and therefore

$$\varphi(\tilde{R}_{-T}, R_T) I_j \varphi\left(\tilde{R}_{(N \setminus T) \cup \{j\}}, R_{T \setminus \{j\}}\right) I_j a$$

by relation (13) and the construction of $\tilde{R}_j$. (Note that $|T \setminus \{j\}| = l$.) That is, we have

$$\forall i \in T, \qquad \varphi_i(\tilde{R}_{-T}, R_T) I_i a.$$

But any allocation $b \in \mathscr{P}(\tilde{R}_{-T}, R_T) \cap \mathscr{I}(\tilde{R}_{-T}, R_T)$ with $bI_i a$ for all $i \in T$ is Pareto indifferent to allocation a under $(\tilde{R}_{-T}, R_T)$ (as otherwise we contradict Claim 1). Therefore

$$\forall i \in N \setminus T, \qquad \varphi(\tilde{R}_{-T}, R_T) \tilde{I}_i a, \tag{14}$$

establishing the proof for $|T| = l + 1$. Hence by induction

$$\forall i \in N, \qquad \varphi(R) I_i a,$$

proving Claim 2.

Now we are ready to complete the proof. Suppose we also have $b \in \mathscr{C}(R)$ and yet b is not Pareto indifferent to a under R. Then by similar arguments we have $\varphi(R) I_i b$ for all $i \in N$, contradicting relation (14). Therefore $N, \omega, \mathscr{A}^f, \mathscr{R}$ are such that

$$\forall R \in \mathscr{R}, \forall a, b \in \mathscr{C}(R), \forall i \in N, \qquad a I_i b.$$

Moreover $a \in \mathscr{C}(R)$ together with Claim 2 imply $\varphi(R) \in \mathscr{C}(R)$ and therefore $\varphi(R) \in \mathscr{C}(R)$ for all $R \in \mathscr{R}$ such that $\mathscr{C}(R) \neq \emptyset$. *Q.E.D.*

The following corollary follows immediately from Theorem 1.

COROLLARY 1: *Let $N, \omega, \mathscr{A}^f, \mathscr{R}$ be such that the core correspondence is non-empty valued. Then whenever there exists an allocation rule φ that is Pareto efficient, individually rational, and strategy-proof, the core correspondence is essentially single-valued and the allocation rule φ is a selection from the corecorrespondence.*

4. APPLICATIONS

In this section we have several applications of our results. We omit proofs of Corollaries 2, 4, 5, 6, and 7, which consist of simple examples with two or more core allocations that are not Pareto indifferent.

4.1. *Indivisible Goods Exchange*

Consider the following class of *indivisible goods exchange economies*. There is a group of agents each of whom owns a set of indivisible goods. Each agent has strict preferences over bundles of indivisible goods and a feasible allocation is a re-allocation of the goods among the agents.

We can represent indivisible goods exchange economies as a subclass of our general model by simply letting $\mathscr{A}^f = \mathscr{A}$ and $\mathscr{R} = \mathscr{R}^s$. Let's first consider the case where there is at least one agent who is endowed with more than one indivisible good. For this case one can easily find an example where the core has at least two allocations that are not Pareto indifferent. Hence we have the following corollary.

COROLLARY 2: *Consider any subclass of indivisible goods exchange economies where at least one agent owns more than one good. There is no allocation rule in this context that is Pareto efficient, individually rational, and strategy-proof.*

What if each agent owns one indivisible good? That case is extensively studied and it is our next application.

4.2. *Housing Markets*

Shapley and Scarf (1974) introduce and study the following class of problems known as *housing markets*. Each agent owns one indivisible good (say a house), and has strict preferences over all houses. An allocation is a permutation of the houses among the agents.[6] We can represent housing markets as a subclass of our general model as follows: Let $|\omega(i)| = 1$ for all $i \in N$, $\mathscr{A}^f = \{a \in \mathscr{A} \mid \forall i \in N, |a(i)| = 1\}$, and $\mathscr{R} = \mathscr{R}^s$. We have the following corollary.

COROLLARY 3 (Roth (1982b), Bird (1984), Ma (1994)): *The core correspondence in the context of the housing markets is weakly coalitionally strategy-proof. Furthermore it is the only allocation rule that is Pareto efficient, individually rational, and strategy-proof.*

PROOF: Roth and Postlewaite (1977) show that the core is a singleton and is externally stable for each problem. Therefore the core correspondence is weakly coalitionally strategy-proof, and hence strategy-proof due to Proposition 1. Uniqueness follows from Theorem 1.

4.3. *Marriage and Roommate Problems*

Gale and Shapley (1962) introduce and study the following class of two-sided matching problems[7] known as *marriage problems*: There are two sets of agents interpreted as a set of men and a set of women. Each man has strict preferences over the set of women and

[6] See Moulin (1995) for a comprehensive survey of results concerning the housing markets.

[7] For an exposition of game theoretic modelling and analysis of such problems, see Roth and Sotomayor (1990).

staying single. Similarly each woman has strict preferences over the set of men and staying single. An allocation is a matching of men and women.

We can obtain marriage problems as a special case of our general model as follows: We partition N into two nonempty disjoint sets M and W. That is, $M \cup W = N$, $M \neq \emptyset$, $W \neq \emptyset$, and $M \cap W = \emptyset$. Let $\omega(i) = \{i\}$ for all $i \in N$,

$$\mathscr{A}^f = \{a \in \mathscr{A} \mid \forall i \in N, |a(i)| = 1; \forall m \in M, a(m) \subset W \cup \{m\};$$
$$\forall w \in W, a(w) \subseteq M \cup \{w\}; \text{ and } \forall i, j \in N, a(i) = \{j\} \Leftrightarrow a(j) = \{i\}\},$$

and $\mathscr{R} = \mathscr{R}^s$.

COROLLARY 4 (Alcalde and Barberà (1994), Roth (1982a)): *Consider any subclass of marriage problems with at least two men and two women. There is no allocation rule in this context that is Pareto efficient, individually rational, and strategy-proof (and hence there is no strategy-proof selection from the core correspondence).*

Gale and Shapley (1962) also consider a generalization of marriage problems that is known as *roommate problems*. There is a group of agents each of whom has strict preferences over all agents. An allocation is a partition of the set of agents into groups of size one and two. Here we are assigning either one or two persons to a room.

We can also obtain roommate problems as a special case of our model: Let $\omega(i) = \{i\}$ for all $i \in N$,

$$\mathscr{A}^f = \{a \in \mathscr{A} \mid \forall i \in N, |a(i)| = 1 \text{ and } \forall i, j \in N, a(i) = \{j\} \Leftrightarrow a(j) = \{i\}\},$$

and $\mathscr{R} = \mathscr{R}^s$.

COROLLARY 5: *Consider any subclass of roommate problems with at least four agents. There is no allocation rule in this context that is Pareto efficient, individually rational, and strategy-proof.*

4.4. *Coalition Formation Problems*

Consider the following class of *coalition formation problems* (Banerjee, Konishi, and Sönmez (1997)): There is a group of agents and each agent has strict preferences over coalitions that include him. A feasible allocation is a partition of the set of agents.

We can represent coalition formation problems as a subclass of our general model as follows: For all $i \in N$, let $\omega(i) = \{\omega_{ij} \mid j \in N, j \neq i\}$. That is, each agent i is endowed with $|N| - 1$ indivisible goods where good ω_{ij} is interpreted as a permit for agent j to join a coalition with agent i. Let

$$\mathscr{A}^f = \{a \in \mathscr{A} \mid \forall i, j \in N, \omega_{ij} \in a(j) \Rightarrow \omega_{ji} \in a(i), \text{and}$$
$$\forall i, j, k \in N, \omega_{ij} \in a(j) \text{ and } \omega_{jk} \in a(k) \Rightarrow \omega_{ik} \in a(k)\}.$$

The first feasibility condition requires that if agent j is in the same coalition with agent i, then agent i should be in the same coalition with agent j. The second one requires that if agent j is in the same coalition with agent i, and agent k is in the same coalition with agent j, then agent k should be in the same coalition with agent i. Let $\mathscr{R} = \mathscr{R}^s$.

COROLLARY 6: *Consider any subclass of coalition formation problems with at least three agents. There is no allocation rule in this context that is Pareto efficient, individually rational, and strategy-proof.*

4.5. *Networks*

The following *network* model is due to Jackson and Wolinsky (1996): Let N be the set of agents. The network relations among these agents are represented by graphs whose nodes are identified with agents and whose arcs capture the pairwise relations. For any $S \subseteq N$, let g^S denote the set of all subsets of S of size 2. The *complete graph* is the pair (N, g^N). Any pair (N, g) with $g \subseteq g^N$ is a feasible allocation and it is referred to as a *graph*. Any pair (S, g) with $S \subseteq N$, $g \subseteq g^S$ is referred to as a *subgraph*. A subgraph (S, g) is *connected* if for all $i, j \in S$, there exists $\{i_1, i_2, \ldots, i_k\} \subseteq S$ such that $\{\{i, i_1\}, \{i_1, i_2\}, \ldots, \{i_{k-1}, i_k\}, \{i_k, j\}\} \subseteq g$.

Depending on application one may consider various restrictions on the preferences. We consider the case where each agent has strict preferences over all connected subgraphs that include him. That is, an agent is indifferent between two graphs if and only if the maximal subgraph to which he belongs is the same in two graphs.

We can represent this network model as a specific case of our general model as follows: For all $i \in N$, let $\omega(i) = \{\omega_{ij} \mid j \in N, j \neq i\}$. We interpret good ω_{ij} as a permit for agent j to form a link with agent i. Let

$$\mathscr{A}^f = \{a \in A \mid \forall i, j \in N, \omega_{ij} \in a(j) \Rightarrow \omega_{ji} \in a(i)\}.$$

That is, a link between two agents can be formed only with mutual consent.

To specify the preference domain we need some additional notation. Let agents i and j be *connected* at allocation a if there exists $\{i_1, i_2, \ldots, i_k\} \subseteq N$ such that $w_{ii_1} \in a(i_1)$, $w_{i_1 i_2} \in a(i_2), \ldots, w_{i_{k-1} i_k} \in a(i_k)$, and $w_{i_k j} \in a(j)$. Let $N_i(a)$ denote the set of agents that are connected to agent i at allocation a. Let $\mathscr{R}_i^{net}$ be the class of all complete and transitive binary relations on $\mathscr{A}^f$ that satisfy the following condition: For all $i \in N$,

$$\forall a, b \in \mathscr{A}^f, \qquad a I_i b \Leftrightarrow \begin{cases} a(i) = b(i), \\ N_i(a) = N_i(b), \\ \forall j \in N_i(a), \qquad a(j) = b(j). \end{cases}$$

Let $\mathscr{R} = \mathscr{R}^{net} = \prod_{i \in N} \mathscr{R}_i^{net}$. Note that $\mathscr{R}^{net}$ satisfies Assumptions A and B.

COROLLARY 7: *Consider any subclass of networks with at least three agents. There is no allocation rule in this context that is Pareto efficient, individually rational, and strategy-proof.*

5. CONCLUDING REMARKS AND RELATED LITERATURE

Strategy-proofness is a property motivated by the presumption that agents will manipulate their preferences whenever they can gain by doing so. It's motivation is noncooperative. On the other hand, the core correspondence is one of the major solution concepts in cooperative game theory. Our results show that for a wide class of allocation problems with indivisibilities, it is possible to achieve strategy-proofness together with Pareto efficiency and individual rationality only by means of the core correspondence. Hence this result provides a link between cooperative game theory and noncooperative game theory.

In addition to those cited in Section 4, there are two papers that are closely related to this paper. Sönmez (1996) studies strategy-proofness in the context of *college admissions problems* (Gale and Shapley (1962)) and shows that there exists an allocation rule that is Pareto efficient, individually rational, and strategy-proof if and only if each college has an unlimited number of slots. In this case the core correspondence is single-valued and it is strategy-proof. Therefore in this model too, there exists an allocation rule that is Pareto efficient, individually rational, and strategy-proof only if the core correspondence is essentially single-valued. One natural question is whether we can obtain this result as a corollary to Theorem 1. The answer is negative. In Sönmez (1996) college preferences are *separable*[8] (Barberà, Sonnenschein, and Zhou (1991)) and they do not satisfy Assumption B. On the positive side, this shows that even if Assumptions A or B are not satisfied, it may still be possible to obtain parallel results using the specific features of the model under consideration.

Ledyard (1977) searches for *strategy-proof selections from the core correspondence* in a rich model. On domains where every allocation in the core strictly Pareto dominates the initial allocation he obtains the following result: If a selection from the core correspondence is strategy-proof, then the core correspondence is essentially single-valued. Ledyard's domain restriction does not hold in any of our applications. Indeed this restriction and Assumption B are incompatible, unless there is a fixed allocation a that is one of the best allocations for all agents, for all preference profiles. In this very special case a is always Pareto indifferent to all Pareto efficient allocations and hence the conclusion is trivial.

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[8] A preference relation R_i is *separable* if

$$\forall x \in \omega(N), \forall X \subseteq \omega(N) \setminus \{x\}, \qquad \{x\} R_i \varnothing \Leftrightarrow X \cup \{x\} R_i X.$$

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