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Strategy-proof allocation of indivisible goods

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Abstract. Strategy-proof allocation of a finite number of indivisible goods among a finite number of individuals is considered. The main result is that in a pure distributional case, a mechanism is strategy-proof, nonbossy and neutral if and only if it is serially dictatorial. If the indivisible goods are initially owned by the individuals, a mechanism is strategy-proof, individually rational and Pareto consistent if and only if it is the core mechanism.

1 Introduction

This study will determine the set of strategy-proof allocation mechanisms in a model with a finite number of indivisible goods to be allocated to a finite number of individuals. In the classical study of this type of model, Shapley and Scarf (1974), had houses in mind and the existence of a competitive equilibrium was established. Roth and Postlewaite (1977) showed that the core in this model contains only one allocation if preferences are strict rankings, while Roth (1982) proved that the core could be implemented by a strategy-proof allocation mechanism. Finally Ma (1994) showed that there is only one strategy-proof, individually rational and Pareto optimal[1] allocation mechanism in this model and that the outcome of the mechanism is the unique core allocation. This mechanism also has the property that no subset of individuals can improve the outcome of the mechanism for the group by coordinating and misrepresent their preferences (see Bird 1984).

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[1] *Pareto consistent* in our terminology.

Here the corresponding problem is examined but without an initial property right structure. For example, it is the government that has to distribute the indivisible goods among the individuals. We may still interpret the indivisible goods as houses or jobs, or as in Hylland and Zeckhauser (1979), as various positions with limited capacities. The basic assumption is that the individuals demand only one of the indivisible goods.

The main result of this paper is the complete characterization of the set of strategy-proof allocation mechanisms under the additional assumptions of neutrality and nonbossiness. In that case the allocation mechanisms must be serially dictatorial, i.e. every mechanism determines a queue order, an order in which the individuals have to choose an indivisible good from the set of "remaining" goods when individuals with higher rankings have made their choice[2]. The existence of a serial dictatorship is also found in Satterthwaite and Sonnenschein (1981) in a model with divisible goods.

Our strategy-proof result is very similar to the Gibbard-Satterthwaite theorem (Gibbard 1973; Satterthwaite 1975). If the indivisible goods are interpreted as public goods instead of private goods as in the present paper, the allocation mechanism is a "voting procedure" and serial dictatorship is replaced by dictatorship. In the case of public goods, nonbossiness is automatically satisfied, while neutrality in our model is there replaced by the voting procedure being "onto"[3].

We will also reconsider the model with property rights and give a simple and alternative proof of the result in Ma (1994); a mechanism is strategy-proof, individually rational and Pareto consistent if and only if it is the core mechanism. It is interesting that individual rationality and preferences together determines a queue order by which the individuals have to choose an indivisible good. The difference from the nonproperty rights case is that the queue order becomes endogenous.

Strategy-proof exchange in a neoclassical model with divisible goods is analyzed in Barberà and Jackson (1995). In that case strategy-proofness requires trade to occur according to a number of pre-specified proportions, but there is no obvious relationship with the indivisible goods case. Strategy-proof and efficient allotment rules are also characterized in Barberà, Jackson and Neme (1997), where the agents have to share a divisible good (or task). Their characterization leads to a class of mechanisms called "sequential allotment rules", which to some extent resemble a serial dictatorship.

More closely related works to the present study are Papai (1996, 1998a,b). In these studies, three different models with indivisible goods are considered and a complete characterization of the set of strategy-proof, Pareto-optimal

[2] In Svensson (1994) it is demonstrated that queue allocations may be implemented by a strategy-proof allocation mechanism also in the case when indifferences in the preferences are permitted.

[3] If a voting procedure is assumed to be neutral (or Pareto consistent) instead of onto, a very short and simple proof of the Gibbard-Satterthwaite theorem can be obtained, see Svensson (1997).

and nonbossy social choice functions is given. In the present study neutrality is an essential assumption and serial dictatorship is a consequence, while Papai (1998b) assumes Pareto optimality and receives a larger set of mechanisms called hierarchical exchange functions. Hence, within this framework our neutrality condition is a more restrictive assumption than Pareto optimality.

There are also two other papers, Sönmez (1995) and Abdulkadiroglu and Sönmez (1997), analyzing models with indivisible goods like the present study. In the first paper the relationship between strategy-proofness and uniqueness of the core is examined (in a more general matching model than ours). As a corollary to his general result, Sönmez obtained Ma's (1994), and hence also our result (Theorem 2). The second paper proves the equivalence between two different ways of allocating the indivisible goods; serial dictatorship with a randomly determined order of choice or using the core correspondence after randomly distributing initial endowments. There are no directly overlapping results in that paper and in the present one, but the problems studied are closely related to each other[4].

2 The model and definitions

Let $N = \{1, 2, \dots n\}$ be a finite set of natural numbers, which also denotes a set of *individuals*, and let $A = \{a_1, a_2, \dots a_m\}$, $m \geq n$, be a finite set of *indivisible goods*. We may interpret the elements in A as houses, jobs, occupations, positions etc. *Preferences* over A are rankings of the various elements (i.e. complete, transitive and asymmetric binary relations). The set of all possible rankings of A represented by utility functions is denoted U, so for $u \in U$ and $a, b \in A$ with $a \neq b$, $u(a) > u(b)$ or $u(b) > u(a)$, but not both. *Preference profiles* are elements in $\mathscr{U} = U^n$. A preference profile $u = (u_1, u_2, \dots u_n)$ can also be denoted (u_i, u_{-i}) for $i \in N$, or (u_S, u_{-S}) for $S \subset N$, where u_S contains utility functions u_i with $i \in S$ and u_{-S} contains the rest of the utility functions.

An *allocation* is an injective mapping φ from N to A. For a given preference profile $u \in \mathscr{U}$, an allocation φ is *Pareto efficient* if there is no other allocation ψ such that for all $i \in N$, $u_i(\psi(i)) \geq u_i(\varphi(i))$, with strict inequality for some $i \in N$. Moreover, for a given preference profile $u \in \mathscr{U}$, φ is a *queue allocation* if there is a permutation π of N (a queue order) such that $\varphi(i_j)$ is the best element in the set $A - \{\varphi(i_1), \varphi(i_2), \dots \varphi(i_{j-1})\}$, where $\pi(i_j) = j$ for all $j \in N$, according to individual preferences u_{i_j}. Here $\pi(i)$ denotes the order in which the individuals have to choose an indivisible good from the set A; the individual i with $\pi(i) = 1$ has to choose first from A, then the individual with $\pi(i) = 2$ from the remaining part of A, and so forth.

An allocation mechanism, or for short a *mechanism*[5], is a mapping f from

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[4] The results in the present study and to some extent, the overlapping results in Papai (1998b) and Sönmez (1995), are independently obtained.
[5] An alternative name of "allocation mechanism" in this context is *social choice function*.

$\mathscr{U}$ to the set of allocations, i.e. every preference profile is mapped on an allocation of the indivisible goods. A mechanism f is *Pareto consistent* if $f(u)$ is Pareto efficient for all $u \in \mathscr{U}$. The mechanism is *manipulable* precisely when there is an individual $i \in N$, preferences $v_i \in U$, and a preference profile $u \in \mathscr{U}$, such that $u_i(f_i(v_i, u_{-i})) > u_i(f_i(u))$. If a mechanism is not manipulable, it is *strategy-proof*. If, for a given permutation π of N, the outcome allocation $f(u)$ of a mechanism f is always the corresponding queue allocation, the mechanism is called *serially dictatorial*. One easily shows that such a mechanism is strategy-proof, see e.g. Svensson (1994). However, the main object of this paper is to show that under some additional assumptions, nonbossiness and neutrality, *every* strategy-proof mechanism is serially dictatorial.

A mechanism f is *nonbossy*[6] if for all preferences $v_i \in U$ and preference profiles $u \in \mathscr{U}$, $f(v_i, u_{-i}) = f(u)$ when $f_i(v_i, u_{-i}) = f_i(u)$. Hence, nonbossiness entails that an individual cannot change the outcome of the mechanism without changing the outcome for himself at the same time.

Now let π be a permutation of A – a change of names of the indivisible goods[7]. If φ is an allocation, an allocation $\pi\varphi$ is defined as $(\pi\varphi)(i) = \pi(\varphi(i))$, $i \in N$. If $u \in \mathscr{U}$ is a preference profile, a preference profile πu is defined as $(\pi u)_i(a) = u_i(\pi^{-1}(a))$ for $i \in N$ and $a \in A$. A mechanism f is *neutral* if, for all preference profiles $u \in \mathscr{U}$ and permutations π of A, $f(\pi u) = \pi f(u)$. This means that the "real" outcome of a neutral mechanism is independent of the names of the indivisible goods. Before a change of names an individual i receives $f_i(u)$ if the reported utility profile is u. After a change of names by π the same individuals will report the profile πu and the outcome for i is $f_i(\pi u)$. The original name of this good is $\pi^{-1}(f_i(\pi u))$ and therefore $\pi^{-1}(f_i(\pi u)) = f_i(u)$ for a neutral mechanism.

3 The set of strategy-proof mechanisms

Before characterizing the set of strategy-proof mechanisms we will give an example of a mechanism that is nondictatorial, strategy-proof, nonbossy and Pareto consistent. The mechanism is, however, not neutral. In the rest of the section neutrality is assumed and our main result becomes that strategy-proofness implies (serial) dictatorship. That result is proved with the help of two lemmas.

Example. Let $N = \{1, 2, 3\}$ and $A = \{a, b, c\}$. Let f be a mechanism defined so that if a is the best element in A according to preferences u_2, then $f_1(u)$ is the best element in $\{b, c\}$ according to preferences u_1, $f_2(u) = a$ and $f_3(u)$ is the remaining element. In all other cases $f_1(u)$ is the best element in A according to preferences u_1, $f_2(u)$ is the best element in $A - \{f_1(u)\}$ according to

[6] The concept of nonbossiness is due to Satterthwaite and Sonnenschein (1981).
[7] Strictly speaking, π is a permutation of the set $\{1, 2, \ldots m\}$ of indices of the elements in A. However, for $a_j \in A$ we will write $\pi(a_j)$ instead of $a_{\pi(j)}$.

preferences u_2 and $f_3(u)$ is the remaining element. Hence the mechanism f is serially dictatorial for all utility profiles except for those where individual 2 has a as the best element. (We can think of individual 2 as the owner of a.)

This mechanism is obviously not neutral – the element a has a special position – but it is strategy-proof (individual 2 cannot increase his utility by not telling the truth!). It is also nondictatorial since no individual always gets his best element, and it is nonbossy, since an individual cannot change the outcome in general without changing the outcome of the mechanism for the individual himself. Finally, the outcome of the mechanism is obviously always Pareto efficient.

So if we can accept nonneutral mechanisms we can also find nondictatorial mechanisms. But as is shown in the sequel, a neutrality condition reduces the set of possible mechanisms drastically. The main result in this section is Theorem 1 below. Two lemmas, which also have some interest of their own, will simplify the proof. The first lemma shows that a strategy-proof and nonbossy mechanism is constant in certain subsets of $\mathcal{U}$. A similar result can be found in Barberà and Jackson (1995) or Papai (1996) but in different models. In the second lemma it is proved that in the subset of $\mathcal{U}$ where all individuals' utility functions coincide the outcome of a strategy-proof, nonbossy and neutral mechanism is always Pareto efficient.

Lemma 1. *Let f be a strategy-proof and nonbossy mechanism, $u, v \in \mathcal{U}$ two preference profiles such that for $x \in A$ and $i \in N$, $v_i(x) \le v_i(f_i(u))$ if $u_i(x) \le u_i(f_i(u))$. Then $f(v) = f(u)$.*

Proof. We first prove that $f(v_i, u_{-i}) = f(u)$. From strategy-proofness it follows that $u_i(f_i(v_i, u_{-i})) \le u_i(f_i(u))$ and hence from the assumption of the lemma, $v_i(f_i(v_i, u_{-i})) \le v_i(f_i(u))$. But strategy-proofness also implies that $v_i(f_i(v_i, u_{-i})) \ge v_i(f_i(u))$ and then, because preferences are strict, $f_i(v_i, u_{-i}) = f_i(u)$. Finally, nonbossiness implies that $f(v_i, u_{-i}) = f(u)$.

For the rest of the proof let $u^p = (v_1, \dots v_{p-1}, u_p, \dots u_n) \in \mathcal{U}$, i.e. $u^{p+1} = (v_p, u^p_{-p})$, be a sequence of preference profiles. From the first part of the proof it then follows that $f(u^p) = f(v_p, u^p_{-p}) = f(u^{p+1})$. But then $f(v) = f(u)$ because $f(v) = f(u^{n+1})$ and $f(u) = f(u^1)$. ■

Lemma 2. *Let f be a strategy-proof, nonbossy and neutral mechanism, and $u \in \mathcal{U}$ a preference profile where all individual preferences coincide, $u_i = u_j$ for all $i, j \in N$. The outcome of the mechanism is then Pareto efficient.*

Proof. Assume that $f(u)$ is not Pareto efficient. There is then $a \in A - \{f_i(u); i \in N\}$ such that for some $k \in N$, $u_i(f_k(u)) < u_i(a)$. Also let k be chosen so that

$$u_i(f_k(u)) = \max_j \{u_i(f_j(u)); u_i(f_j(u)) < u_i(a)\}$$

and let $b = f_k(u)$.

Now define individual preferences $v_i \in U$, equal for all $i \in N$, such that $v_i(x) = u_i(x)$ for all $x \in A - \{a, b\}$, while $v_i(a) = u_i(b)$ and $v_i(b) = u_i(a)$.

Hence $v_i(a) < u_i(a)$ and $v_i(b) > u_i(b)$. We can then show that the preference profiles u and v satisfy the assumptions in Lemma 1. Let for $i \in N$, $x \in A$ be such that $u_i(x) \le u_i(f_i(u))$.

First consider $x \neq b$. The inequality $v_i(x) \le u_i(x)$ follows from the definition of v_i. Hence, $v_i(x) \le u_i(x) \le u_i(f_i(u)) \le v_i(f_i(u))$, where the last inequality follows because $a \neq f_i(u)$.

Next assume that $x = b$. If $i = k$ we have directly that $v_i(b) = v_i(f_i(u))$. If $i \neq k$, the inequality $u_i(b) < u_i(f_i(u))$ implies that $u_i(a) < u_i(f_i(u))$ because of the maximal choice of $f_k(u))$. But then $v_i(f_i(u)) = u_i(f_i(u))$. Thus we have $v_i(b) = u_i(a) < u_i(f_i(u)) = v_i(f_i(u))$.

From Lemma 1 it now follows that $f(u) = f(v)$.

Finally let π be a permutation of A such that $\pi(x) = x$ if $x \in A - \{a, b\}$ while $\pi(a) = b$ and $\pi(b) = a$. Thus $u = \pi v$, and hence by neutrality, $\pi f(v) = f(\pi v) = f(u)$. This is a contradiction to $f(u) = f(v)$, so the statement in the lemma is true. ■

To prove the theorem below we first consider the outcome of f when all individuals have the same preferences. In that way the conflict in the allocation situation is large and it is possible to find directly the order in which the individuals have to choose elements from A. Next we apply Lemma 1 to prove that the order of choices remains the same in the entire set $\mathcal{U}$ of preference profiles.

Theorem 1. *A strategy-proof, nonbossy and neutral mechanism f is serially dictatorial.*

Proof. Let $u \in \mathcal{U}$ be a preference profile where all individual preferences coincide and $u_i(a_j) > u_i(a_k)$ when $k > j$ for all $i \in N$. From Lemma 2 follows then that $\{f_i(u); i \in N\} = \{a_1, \dots a_n\}$. Now there is a permutation σ of N such that $f_{\sigma(j)}(u)$, $j \in N$, is the best element in $A - \{f_{\sigma(1)}(u), \dots f_{\sigma(j-1)}(u)\}$ according to the common preferences. With no loss of generality we may assume that $\sigma(i) = i$ and $f_i(u) = a_i$ for all $i \in N$. Hence, for this particular preference profile, individual 1 is the dictator and chooses from the entire set A, individual 2 chooses from the remaining alternatives, and so forth, and the mechanism is serially dictatorial.

From neutrality it easily follows that this order of choice is the same for all preference profiles where all individual preferences coincide. To see this let $v \in \mathcal{U}$ and $v_i = v_j$ for all $i, j \in N$. There is then a permutation π of A such that $u_i(a) = v_i(\pi(a))$ for all $a \in A$ and $i \in N$. But then $\pi u = v$ and hence by neutrality, $\pi f(u) = f(\pi u) = f(v)$. This entails that the utility distributions are the same for all preference profiles where all individual preferences coincide because

$$v_i(f_i(v)) = v_i(f_i(\pi u)) = v_i(\pi(f_i(u))) = u_i(f_i(u)).$$

Thus individual 1 is the dictator, individual 2 is the second one to choose, and so forth. We will finally show that this order of choice remains for all preference profiles.

Let $v \in \mathscr{U}$ be an arbitrary preference profile and define recursively a subset $\{a_{i_j}\}_{j=1}^{n}$ of A according to:

a_{i_j} is the best element in $A - \{a_{i_1}, a_{i_2}, \ldots a_{i_{j-1}}\}$ according to v_j,

i.e. the elements a_{i_j} are the outcome of a serial dictatorship where individual 1 makes the first choice, individual 2 is the second one to choose, and so forth.

Let $u \in \mathscr{U}$ be a preference profile where all individual preferences coincide, $u_i = u_j$ for all $i, j \in N$, and satisfy:

$$u_i(a_{i_j}) > u_i(a_{i_k}) \quad \text{if } j < k \leq n,$$

$$u_i(a_{i_n}) > u_i(a) \quad \text{for all } a \in A - \{a_{i_1}, a_{i_2}, \ldots a_{i_n}\}.$$

From the first part of the proof it follows that $f_j(u) = a_{i_j}$ for all $j \in N$, i.e. individual 1 is the dictator etc. It remains to show (by Lemma 1) that $f(u) = f(v)$.

Let $a \in A$ and $u_j(a) \leq u_j(f_j(u))$. Then $a \in A - \{a_{i_1}, a_{i_2}, \ldots a_{i_{j-1}}\}$. From the definition of $\{a_{i_j}\}_{j=1}^{n}$ follows then that $v_j(a) \leq v_j(a_{i_j}) = v_j(f_j(u))$ and hence by Lemma 1, $f(u) = f(v)$. ■

Theorem 1 shows that if there is a strategy-proof, nonbossy and neutral mechanism it must be serially dictatorial. But one easily shows that the serially dictatorial mechanism also satisfies those conditions, so the set of such mechanisms is nonempty.

In the example in the beginning of this section we demonstrated that neutrality was important for the result in Theorem 1; strategy-proof, nonbossy and Pareto consistent mechanisms that are not dictatorial do exist. A related problem is whether the nonbossy requirement is necessary for the result. However, without nonbossiness a strategy-proof and neutral mechanism must not necessarily be serially dictatorial. For instance, the "pure" dictatorial dictatorship where $f_i(u)$ is the best element in $A - \{f_1(u), f_2(u), \ldots f_{i-1}(u)\}$ according to individual 1's preferences u_1 is strategy-proof and neutral but not Pareto consistent.

Suppose then that nonbossiness in Theorem 1 is replaced by Pareto consistency. The following example shows that serial dictatorship is not the implication. Let $\#N = \#A = 4$ and consider a mechanism that is "almost" serially dictatorial; the order of choice is individual 1, 2, 3 and 4 in all cases except when individual 1 and 2 report the same preferences. In that case individual 4 makes his choice before individual 3. It is easy to show that this mechanism is strategy-proof, neutral and Pareto consistent, but it is also bossy. I do not have, however, a complete characterization of mechanisms that are strategy-proof, neutral and Pareto consistent[8].

In next section we will assume that each alternative in A is owned by precisely one individual and hence we cannot assume neutrality. Nonbossiness will also be replaced by Pareto consistency.

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[8] The set of strategy-proof, *nonbossy* and Pareto consistent mechanisms are characterized in Papai (1998b).

4 Strategy-proofness with property rights

Now suppose that $m = n$ and that each individual owns precisely one indivisible good. With no loss of generality we assume that individual $i \in N$ owns $a_i \in A$. Let $S \subset N$ be a nonempty set of individuals (a coalition), $u_i \in U, i \in S$, utility functions, and φ an allocation. Then the coalition S *blocks* the allocation φ if there is another allocation ψ such that $\psi(i) = a_{\pi(i)}$ for all $i \in S$ for some permutation π of S, and such that $u_i(\psi(i)) \geq u_i(\varphi(i))$ for all $i \in S$ with strict inequality for some $i \in S$.

An allocation φ is *individually rational* if no coalition with one member blocks φ, *Pareto efficient* if the entire coalition N does not block φ, and in the *core*[9] if no coalition blocks φ.

Moreover, a mechanism f is *individually rational* if the allocation $f(u)$ is individually rational for all preference profiles $u \in \mathcal{U}$, *Pareto consistent* if $f(u)$ is Pareto efficient for all preference profiles $u \in \mathcal{U}$, and a *core mechanism* if $f(u)$ is in the core for all preference profiles $u \in \mathcal{U}$.

Let φ be an allocation and $S \subset N, S \neq \emptyset$ a nonempty subset of individuals. The set $\varphi(S) = \{\varphi(i); i \in S\}$ is a *cycle* if $S = \{\pi^k(j); k = 1, \dots n\}$ for some $j \in S$, where π is the (unique) permutation of N given by $\varphi(\pi(i)) = a_i$ for all $i \in N$. Hence, the element a_j is allotted to individual $\pi(j) \in S$, the element $a_{\pi(j)}$ is allotted to individual $\pi^2(j) = \pi(\pi(j)) \in S$, the element $a_{\pi^2(j)}$ is allotted to individual $\pi(\pi^2(j)) = \pi^3(j) \in S$ etc. Note that $\varphi(i) \neq a_i$ for all $i \in S$ if $\# S > 1$.

Lemma 3. *For each $u \in \mathcal{U}$ there is precisely one core allocation φ. There is also a corresponding partition*[10] $\{N_j\}_{j=1}^r$, $r \leq n$, *of the set N such that*

$$A_j = \varphi(N_j) \quad \text{are cycles for all } j, \quad 1 \leq j \leq r, \quad \text{and}$$

$$\text{if } i \in N_j \quad \text{then } u_i(\varphi(i)) \geq u_i(a) \quad \text{for all } a \in A - \bigcup_{k=1}^{j-1} A_k.$$

Proof. The uniqueness of the core allocation φ follows from Roth and Postlewaite (1977). The sets $\{N_j\}_{j=1}^r$ are defined sequentially. First define N_1. Let $\rho : N \to N$ be a mapping such that for $i \in N$, $a_{\rho(i)}$ is the best element in A according to u_i. Because N is finite there is a set $\{i_1, i_2, \dots i_k\} \subset N$ such that $i_j \neq i_l$ if $j \neq l, i_{j+1} = \rho(i_j)$ and $i_1 = \rho(i_k)$. Let $N_1 = \{i_1, i_2, \dots i_k\}$. Hence $N_1 = \{\rho^q(i_1); q = 1, ... n\}$. Since φ is the unique core allocation $\varphi(N_1) = \{a_{\rho(i)}; i \in N_1\}$. Obviously $\varphi(N_1)$ is a cycle with $\rho^{-1}(i) = \pi(i)$ for $i \in N_1$ and $\varphi(\pi(i)) = a_i$. Denote by A_1 the set $\varphi(N_1)$. From the construction of N_1 also follows that $u_i(\varphi(i)) \geq u_i(a)$ for all $a \in A$ if $i \in N_1$. To define N_2 and A_2 we repeat the procedure in the sets $N - N_1$ and $A - A_1$, and so forth for the rest of the sets. ■

[9] The core defined by weak domination like here is also called the strict core.
[10] One can also show that the partition is unique as far as the numeration of the sets N_j, but that is not needed for the proof of the next theorem.

The existence of a strategy-proof, individually rational and Pareto consistent mechanism is proved in Roth (1982). Here we give an alternative proof of Ma's (1994) result that the core mechanism is the only one. A more simple proof is obtained if in addition nonbossiness is assumed. That proof is given in an appendix. From Lemma 3 we may also note that a core allocation is a queue allocation where the queue order is endogenously determined by the preference profile. This is in contrast to the pure distributional case in the preceding section where strategy-proof mechanisms were characterized by an exogenously given queue order.

Theorem 2. *A strategy-proof, individually rational and Pareto consistent mechanism f is the core mechanism.*

Proof. Let $u \in \mathcal{U}$ and let φ be the unique corresponding core allocation. Also let $\{N_j\}_{j=1}^r$ and $\{A_j\}_{j=1}^r, r \leq n$, be partitions of the sets N and A in accordance with Lemma 3. To simplify notation, we may assume with no loss of generality that $i > k$ if $i \in N_j$, $k \in N_l$ and $j > l$.

We first prove that for all $i \in N_1$, $f_i(u_{N_1}, u'_{-N_1}) = \varphi(i)$ for *all* preference profiles u'_{-N_1} for the set $N - N_1$ of individuals. If $\# N_1 = 1$ then by individual rationality, $f_1(u_{N_1}, u'_{-N_1}) = \varphi(1) = a_1$. On the other hand, if $\# N_1 > 1$ then for $i \in N_1$, $\varphi(i) \neq a_i$ because $\varphi(N_1)$ is a cycle. In this case, let $N_1 = \{1, 2, \ldots k\}$. For $i \in N_1$ define utility functions $v_i \in U$ according to

$$v_i(a) = u_i(a) \quad \text{for all } a \in A - \{a_i\}, \quad \text{while for } a_i,$$

$$u_i(\varphi(i)) > v_i(a_i) > u_i(a) \quad \text{for all } a \in A - \{a_i, \varphi(i)\}.$$

In other words, with preferences v_i, initial resources are always second best while the ranking of other elements are the same in v_i and u_i. Also denote by $v_{N_1}^p$ preference profiles for the set N_1 such that there are p individuals having the original utility functions u_i while the rest of the individuals in N_1 have utility functions v_i, e.g. $v_{N_1}^p = (v_1, u_2, \ldots u_{p+1}, v_{p+2}, \ldots v_k)$. Then by Pareto consistency and individual rationality, $f_i(v_{N_1}^0, u'_{-N_1}) = \varphi(i)$ for all $i \in N_1$.

Suppose now that for $i \in N_1, f_i(v_{N_1}^p, u'_{-N_1}) = \varphi(i)$ for all $v_{N_1}^p$ for a fixed number p, $0 \leq p < k$. Then $f_i(v_{N_1}^{p+1}, u'_{-N_1}) = \varphi(i)$ for all $i \in N_1$ having utility functions u_i by strategy-proofness and hence by individual rationality, $f_i(v_{N_1}^{p+1}, u'_{-N_1}) = \varphi(i)$ also for the rest of the individuals in N_1. Then by induction, $f_i(u_{N_1}, u'_{-N_1}) = \varphi(i)$ for all $i \in N_1$ because $u_{N_1} = v_{N_1}^k$.

To complete the proof we consider the sets $N - N_1$ and $A - A_1$ and repeat the procedure above to prove that for $i \in N_2$, $f_i(u_{N_1}, u_{N_2}, u'_{-(N_1 \cup N_2)}) = \varphi(i)$ for all u', and so forth for the rest of the sets N_j and A_j. ■

5 Appendix

Theorem 3. *A strategy-proof, individually rational, nonbossy and Pareto consistent mechanism f is the core mechanism.*

Proof. Let $u \in \mathcal{U}$ and let φ be the unique corresponding core allocation. Also let $\{N_j\}_j$ and $\{A_j\}_j$ be partitions of the sets N and A in accordance with Lemma 3 in section 4. To simplify notation, we may assume with no loss of generality that $i > k$ if $i \in N_j$, $k \in N_l$ and $j > l$. Now define recursively a preference profile $v \in \mathcal{U}$ by:

For $i \in N_j$ define utility functions $v_i \in U$ according to $v_i = u_i$ if $\varphi(i) = a_i$, while if $\varphi(i) \neq a_i$,

$$v_i(a) = u_i(a) \quad \text{for all } a \in A - \{a_i\} \quad \text{while for } a_i,$$

$$u_i(\varphi(i)) > v_i(a_i) > u_i(a) \quad \text{for all } a \in \left(A - \bigcup_{k=1}^{j-1} A_k\right) - \{a_i, \varphi(i)\}.$$

But then by Pareto consistency and individual rationality, $f(v) = \varphi$ according to the construction of v. To determine $f(u)$ first consider $f(v^p)$, where $v^p = (u_1, u_2, \ldots u_{p-1}, v_p, \ldots v_n) \in \mathcal{U}$. Obviously $v^1 = v$ and $v^{n+1} = u$. Suppose that $f(v^p) = \varphi$. Then $f_p(v^{p+1}) = \varphi(p)$ by strategy-proofness and then by nonbossiness, $f(v^{p+1}) = \varphi$. Hence $f(u) = \varphi$ by induction. ■

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