Dynamic Kidney Exchange*

M. Utku Ünver†
Boston College

First Draft: March 2007
Final Draft: March 2009

Abstract

We study how barter exchanges should be conducted through a centralized mechanism in a dynamically evolving agent pool with time- and compatibility-based preferences. We derive the dynamically efficient *two-way* and *multi-way* exchange mechanisms that maximize total discounted exchange surplus. Recently several live-donor kidney exchange programs were established to swap incompatible donors of end-stage kidney disease patients. Since kidney exchange can be modeled as a special instance of our more general model, dynamically efficient kidney exchange mechanisms are derived as corollaries. We make policy recommendations using simulations.

Keywords: Dynamic exchange, kidney exchange, matching, market design, dynamic optimization, Markov process.

JEL Classification Numbers: C78, C70, D78, C61

*I would like to thank especially Tayfun Sönmez, and also David Abraham, Murat Fadıloğlu, Fikri Karaesmen, David Kaufman, Onur Kesten, Fuhito Kojima, Alvin E. Roth, Andrew Schaefer, İnsan Tunalı, and Neşe Yıldız, participants at SAET Conference at Kos, Matching Workshops at Barcelona and Caltech, seminars at British Columbia, Boston College, Carnegie Mellon, Koç, and Bilkent for comments and suggestions. I would also like to thank the editor, an associate editor, and anonymous referees of the journal for their excellent comments. I am grateful to NSF for financial support.

†Address: Department of Economics, Boston College, 140 Commonwealth Ave., Chestnut Hill, MA 02467, USA; e-mail address: utku.unver@bc.edu; url: http://www2.bc.edu/~unver.

1 Introduction

There are about 79000 patients waiting for a kidney transplant in the United States as of March 2009. In 2008, about only 16500 transplants were conducted, about 10500 from deceased donors and 6000 from living donors, while about 32500 new patients joined the deceased donor waiting list and 4200 patients died while waiting for a kidney.[1] Although there is a substantial organ shortage, buying and selling an organ is illegal in many countries in the world, making donation the only source for kidney transplants. Especially in the last decade, the increase in the number of kidney transplants came from the utilization of live donors, who are typically relatives, friends, or spouses of the patients and are willing to donate one of their kidneys. However, many living donors still cannot be utilized, since the potential donor may not be able to donate to her loved one due to blood-type incompatibility or tissue rejection. The medical community has proposed innovative ways to utilize these living donors through *live-donor kidney exchanges* (Rapaport, 1986). In a live-donor kidney exchange, recipients with incompatible donors swap their donors if there is cross-compatibility. Since 1991, kidney exchanges have been done mostly in an ad-hoc manner in different countries around the world. Live-donor kidney exchanges accounted for at least 10% of all live donor transplants in Korea and the Netherlands in, 2004 (see Park et al., 2004 and de Klerk et al., 2005). The medical community has endorsed the practice of live-donor kidney exchanges as ethical (Abecassis et al., 2000). Unlike Korea and Netherlands, in the United States there is no national system to oversee kidney exchanges as of 2009. Different transplant centers around the country have recently started kidney exchange programs. For example, New England Program for Kidney Exchange (NEPKE) is an initiative of the transplant centers in New England together with economists (see Roth, Sönmez, and Ünver 2005b), and Alliance for Paired Donation (APD) is an initiative of Dr. Michael Rees at University of Toledo and the authors of the above studies. APD has already convinced a large number of transplant centers all around the US to participate. United Network for Organ Sharing is at the stage of launching the national kidney exchange program in the US.[2]

In many of these programs, a major objective has been to conduct as many transplants as possible. However, one question has frequently arisen in the implementation stage: How often and how exactly should the exchange be run? Roth, Sönmez, and Ünver (2004, 2005a) have recently proposed mechanisms to organize kidney exchanges in a Pareto-efficient and dominant strategy incentive-compatible fashion under different constraints on exchange sizes and preferences of the recipients for a static recipient population.

[1] According to SRTR/OPTN national data retrieved at `http://www.optn.org` on 3/17/2009.

[2] In Europe, other than the Netherlands, paired kidney exchange programs have not yet been well organized. The UK has only recently passed a law that makes kidney exchanges legal. France and Germany have stricter laws, and it is illegal to have a transplant from an unrelated and emotionally distant live-donor, making paired exchanges illegal. Spain has an excellent deceased donor program. Therefore, live donation is seen as of secondary importance, although there is overwhelming evidence that the long-run survival rates of live donor organs are far better than deceased-donor organs.

The above studies address the matching aspect of the problem. However, they do not consider the dynamic aspects of the exchange pool evolution.[3] From a more general perspective, in the *matching* literature in economics, although there is significant amount of work on mechanism design in static environments, there is virtually no study on mechanism design for dynamically changing populations, two recent exceptions withstanding.[4] A recent paper by Bloch and Cantala (2009) analyzes house allocation problems in an overlapping generations framework. In their model, they analyze assignment mechanisms that are fair, efficient, and independent. Seniority-based assignment rules characterize these properties when agents are homogeneous. However, when the types of agents are random, efficient, and fair rules only exist with two agent types. Independence and efficiency are incompatible in this case. Unlike our model, objects are not attached to agents in their model. Hence, they study assignment rather than exchange. In their model, objects remain in the problem after agents leave the problem. Thus, assignment of an object is not final. Moreover, their general preference structure is not compatibility-based, although they characterize Markovian assignment rules only for a dichotomous model. The second paper related to ours is by Kurino (2008). He studies an overlapping generations model like the Bloch and Cantala paper. However, he does not have random types in his model. Moroever, he introduces property rights. He finds extensions of well-known static mechanisms in the dynamic setting that are individually rational, strategy-proof, and efficient under restrictions of general preferences. Another closely related domain to ours is dynamic allocation setting with changing populations and monetary transfers, such as auctions. For example, Gershkov and Moldovanu (2007 and 2008) and Said (2009) introduce optimal dynamic mechanisms when agents arrive over time under Poisson processes in different environments under different objectives.[5]

We consider a general dynamic problem from the point of view of a central authority (e.g. a health

[3]In the operations research literature, Zenios (2002) considers a dynamic model with only two types of patient-donor pairs and different outside options. In this model pairs arrive continuously over time but not in a discrete process like ours. Moreover, since there are only two types of patient-donor pairs and the outside options are different, this model is substantially different from ours. Our model addresses the *matching* aspect of the dynamic kidney exchange problem.

[4]There is a vast economics literature on the allocation or exchange of indivisible goods, initiated by Shapley and Scarf (1974), Roth and Postlewaite (1977), Roth (1982), Abdulkadiroğlu and Sönmez (1999), Papai (2000), Ergin (2000), Ehlers (2002), Ehlers, Klaus, and Papai (2002), Kesten (2004), Sönmez and Ünver (2005, 2006). None of these works focuses on the stochastic dynamic problem, although Ergin (2000), Ehlers, Klaus and Papai (2002), Sönmez and Ünver (2006) inspect the problem with static exchange rules under varying populations.

[5]A recent paper by Abdulkadiroğlu and Loertscher (2006) inspects the dynamic preference formation in house allocation problems.

In other domains, there are several recent studies on optimal mechanisms in dynamic settings. For example, Jackson and Palfrey (1998) study optimal bargaining mechanisms in a dynamic setting, and Skreta (2006) studies optimal dynamic mechanism design when the designer cannot commit to a mechanism in the future. Another topic that is attracting recent attention is optimal auction design when valuation signals of agents evolve over time (e.g., cf. Bergemann and Välimaki 2006 and Athey and Segal 2007).

authority). Each agent (e.g. a recipient) arrives with an object to trade (e.g. a donor). Waiting in the pool for an exchange is costly. The agent has a need type and objects have object types. The desirability of an object is determined by its type and the need type of the agent. This compatibility relation is a partial order. That is: each agent finds an object with a type that is better than or the same as her own type (other than her own attached object) desirable. Thus, each pair is represented by a pair type determined by the need type of the agent and type of her paired object. Each pair type arrives with a stochastic Poisson arrival process.

The central authority's objective is to minimize the long-run total discounted waiting cost. We make an assumption in the derivation of the efficient *two-way* matching mechanism. We assume that in the long run, there is an arbitrarily large number of *underdemanded* types of pairs, whose object types are not compatible with the needs of recipients' need types. (Later, we show that this assumption is consistent with real-life arrival probabilities of different pairs for the case of kidney exchanges.) We show that an interesting characteristic of an efficient two-way matching mechanism is that it conducts the maximum number of exchanges as soon as they become available, that is: there is no need to sacrifice one or more currently feasible exchanges for the sake of conducting future exchanges (Theorem 1). However, this theorem no longer holds when larger exchanges are feasible, and we derive the efficient multi-way matching mechanism as a *threshold* matching mechanism under one additional assumption (Theorems 2 and 3, also see Remark 1 in Appendix B). In the simplified version of the model, when there are no self-demanded types participating exchange, i.e. types of pairs with the same agent need and object type, a threshold mechanism relies on a single threshold value. Suppose W_1 and W_2 are two object types that are not comparable under the compatibility partial order, that is: neither W_1 is better than W_2 nor W_2 is better than W_1. Then, the efficient mechanism considers the number of W_1-W_2 type pairs (W_1 type agents and W_2 type paired-objects) and reciprocal W_2-W_1 type pairs together. Depending on the frequencies of arrival of different pairs, one of the two types of threshold mechanisms is efficient. In the first possible solution, the efficient mechanism conducts the maximum size exchanges as soon as they become available as long as there are no W_2-W_1 type pairs. However, if there are some W_2-W_1 type pairs already available in the exchange pool, and their number does not exceed a threshold number, then the authority should not use the W_2-W_1 type pairs other than matching W_1-W_2 type pairs. In this case, it should avoid involving them in larger exchanges that do not have W_1-W_2 type pairs. Only when the stock of W_2-W_1 type pairs exceeds the threshold number should the authority conduct the largest possible exchanges as soon as they become available, and possibly use W_2-W_1 type pairs in exchanges without W_1-W_2 type of pairs. The second possible solution is just the symmetric version of the first solution, and instead treats W_1-W_2 type pairs as the stock variable. The mechanism takes into account all such non-comparable W_1 and W_2 types. We show that decisions regarding each incomparable object type pair W_1 and W_2 are independent from those regarding other incomparable object types (Propositions 2 and 6).

Our constraints in the general model are consistent with the medical and instituronal constraints of the kidney exchange problem.

First, in the medical literature, Gjertson and Cecka (2000) and Delmonico (2004) pointed out that a recipient is indifferent between two live donor kidneys as long as they are both compatible with the recipient. We adopt this assumption in our current study and assume that the preferences of the recipients fall into three indifference classes: being matched to a compatible kidney, being unmatched, and being matched to an incompatible kidney.[6] We also introduce time preferences of recipients into the dynamic setting. This is consistent with our general model's assumptions.

Second, the practice of kidney exchange has started by conducting exchanges that include only two recipients and their incompatible donors. More complicated exchanges that include three or more recipients (and their donors) will rarely take place at least initially, because all transplants of a single exchange cycle should take place simultaneously. Otherwise, some donors could potentially back out after their recipients receive transplants given the legal constraint that it is illegal to force a donor to sign a contract that would commit him to donation. Nevertheless, Roth, Sönmez, and Ünver (2007) have shown that larger exchanges, especially three-way exchanges including three recipient-donor pairs, would substantially increase the gains from exchange. In real life, larger size exchanges are occasionally also being conducted. In the current paper, we separately derive efficient dynamic matching mechanisms that conduct *two-way exchanges* and *multi-way exchanges*, which is consistent with our general derivation.

Since blood-type compatibility needed for kidney donations is a special case of the above compatibility partial order, the efficient kidney exchange mechanism is a special instance of the general mechanism. We compute the efficient mechanism under different pair arrival rates and time discount rates. Additionally, we conduct policy simulations and observe that the gains under the efficient multi-way kidney exchange mechanism are significantly higher than those under the efficient two-way mechanism.

2 The General Dynamic Exchange Model

2.1 Exchange Pool

We consider an exchange model in which each agent arrives at the exchange pool with an (indivisible) object to trade through barter exchange. A *pair* i consists of an *agent* a_i and her *object* o_i. There is a requirement type for each agent over the objects and each object belongs to one of these types. Let $\mathcal{T}$ be the finite set of requirement/object types. Since each agent's requirement and each object belongs to one of these types, there are $|\mathcal{T}|^2$ permutations for each pair. We call each of these permutations a *pair type*. The type of a pair is denoted as X-Y where X, Y $\in \mathcal{T}$ and X is the requirement type of the *agent* and Y is the type of her *object*. Let $\mathcal{P} = \mathcal{T} \times \mathcal{T}$ be the set of pair types. For any pair type X-Y $\in \mathcal{P}$, let $p_{\text{X-Y}}$ be the probability of a random pair being of type X-Y. We refer to $p_{\text{X-Y}}$ as

[6] We have the same assumption in Roth, Sönmez, and Ünver (2005a, 2007).

the *arrival probability* of pair type X-Y $\in \mathcal{P}$. We have $\sum_{\text{X-Y} \in \mathcal{P}} p_{\text{X-Y}} = 1$. For any X-Y$\in \mathcal{P}$, we refer to Y-X as the *reciprocal pair type* of X-Y.

We define a universal binary relation $\blacktriangleright$ over $\mathcal{T}$ as follows: for any X,Y$\in \mathcal{T}$, X$\blacktriangleright$Y means that an object of type X can be consumed by an agent of requirement type Y, and we refer to this as *X is compatible with Y*.

We make some restrictions on the compatibility relation $\blacktriangleright$. We assume that $\blacktriangleright$ is a *partial order*, i.e., it is reflexive, transitive, and antisymmetric.

Though $\blacktriangleright$ is not a complete relation (i.e. not a linear order), for simplicity, we assume that for any type X$\in \mathcal{T}$, there exists at most one type in Y$\in \mathcal{T}$ that is not comparable with X. That is: for any X$\in \mathcal{T}$, there exists at most one Y$\in \mathcal{T}$ such that neither X$\blacktriangleright$Y nor Y$\blacktriangleright$X is true.[7]

Based on the compatibility relation, we can partially order types in levels. Let the level set $\mathcal{L} = \{1, 2, ..., |\mathcal{L}|\}$ be the partition of $\mathcal{T}$ such that for all $K, L \in \{1, 2, ... |\mathcal{L}| - 1\}$ with $K < L$, if X$\in K$ and Y$\in L$, then X$\blacktriangleright$Y. In this case, we say that *X is at a better compatibility level than Y*. Observe that for any $L \in \mathcal{L}$, X,Y$\in L$ with X $\neq$ Y imply X$\not\blacktriangleright$Y and Y$\not\blacktriangleright$X. For each compatibility type X$\in \mathcal{T}$, let $L_{\text{X}} \in \mathcal{L}$ be the compatibility level of X, i.e., X$\in L_{\text{X}}$. Though both notations, levels and binary relation $\blacktriangleright$, can be used interchangably, we will stick to the latter in most parts of the paper. We will use levels to quantify the magnitude of difference between levels of different types. See Figure 1 for two examples of feasible type sets and compatibility partial orders.

An agent cannot consume her own object. Each agent would like to consume another pair's object that is compatible with her. We assume that pairs arrive over time with a stochastic (discrete) Poisson arrival process in continuous time. Let λ be the arrival rate of the pairs, i.e. the expected number of pairs that arrive per unit time. Thus, each type X-Y arrives with a Poisson arrival process with rate $p_{\text{X-Y}}\lambda$. The *exchange pool* is the set of the pairs that arrived over time whose agent has not yet been assigned an object.

Each agent has preferences over objects and over time of waiting in the pool. Compatible objects are preferred to being unmatched. In turn, being unmatched is preferred to being matched to incompatible objects.[8] Moreover, time spent in the exchange pool is another dimension in the preferences of agents: waiting is costly. We will model the waiting cost through a fixed cost.

2.2 Exchange

An *exchange* is a list of pairs $(i_1, i_2, \ldots, i_k)$ for some $k \geq 2$ such that for any $\ell < k$, object o_{i_ℓ} is assigned to agent $a_{i_{\ell+1}}$, and object o_{i_k} is assigned to agent a_{i_1}. We will sometimes refer to an exchange by the types of the participating pairs, i.e. as (A_1-O_1, ..., A_k-O_k) where A_ℓ-O_ℓ is the pair type of i_ℓ. A *matching* is a set of exchanges such that each pair participates in at most one exchange.

[7]This assumption is the minimal structure needed to generate a result as in our Theorem 2.

[8]In the context of kidney transplantation, Gjertson and Cecka (2000) point out that each compatible live donor kidney will last approximately the same amount of time as long as the donor is not too old and in relatively in good health.

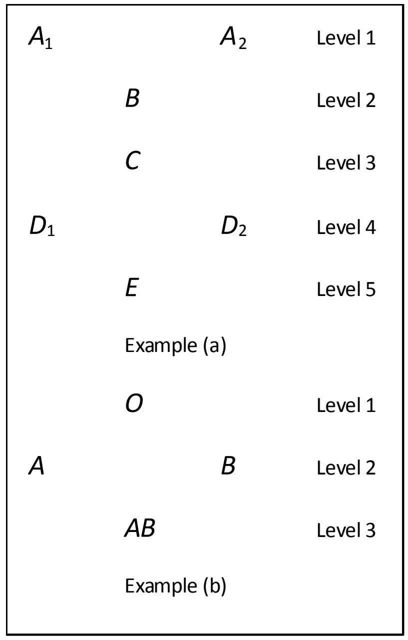


Figure 1: **Example (a)** $\mathcal{T} = \{A_1, A_2, B, C, D_1, D_2, E\}$ and $A_1, A_2 \blacktriangleright B \blacktriangleright C \blacktriangleright D_1, D_2 \blacktriangleright E$; $A_1 \not\blacktriangleright A_2; A_2 \not\blacktriangleright A_1; D_1 \not\blacktriangleright D_2$; and $D_2 \not\blacktriangleright D_1$. **Example (b)** $\mathcal{T} = \{O, A, B, AB\}$ and $O \blacktriangleright A, B \blacktriangleright AB$; $A \not\blacktriangleright B; B \not\blacktriangleright A$.

A matching or an exchange is *individually rational* if it never matches an agent with an incompatible object. From now on, when we talk about an exchange or a matching, it will be individually rational. A matching is *maximal* if it matches the maximum number of pairs possible at an instance of the pool.

A *(dynamic) matching mechanism* is a dynamic procedure such that at each time $t \geq 0$ it selects a (possibly empty) matching of the pairs available in the pool. Once a pair is matched at time t by a matching mechanism, it leaves the pool and its agent receives the assigned object.

Let $A(t)$ represent the set of pairs that arrived at the pool until time t. If a matching mechanism ϕ is executed (starting time 0), $\phi(t, A)$ is the set of pairs matched by mechanism ϕ under the flow A. There are $|A(t)| - |\phi(t, A)|$ pairs available at the pool at time t.

2.3 Dynamically Efficient Mechanisms

There is a central authority that oversees the exchanges. For each pair, we associate waiting in the pool with a monetary cost and we assume that there is a constant *unit time cost* $c > 0$ of waiting for an exchange.[9]

[9] In the context of kidney exchanges, the alternative option of a transplant is dialysis. A patient can undergo dialysis continuously. It is well known that receiving a transplant causes the patient to resume a better life (cf. Overbeck et al., 2005). Also health care costs for dialysis are more than those for transplantation in the long term (cf. Schweitzer et al., 1998). We model all the costs associated with undergoing continuous dialysis by the unit time cost c.

Suppose that the central authority implements a matching mechanism ϕ. For any time t, the current value of expected cost at time t under matching mechanism ϕ is given as[10]

$$E_t\left[\mathcal{C}^{\phi}(t,A)\right] = \int_t^{\infty} cE_t\left[|A(\tau)| - |\phi(\tau,A)|\right] e^{-\rho(\tau-t)}d\tau,$$

where ρ is the *discount rate*.

For any time τ, t such that $\tau > t$, we have $E_t\left[|A(\tau)|\right] = \lambda(\tau - t) + |A(t)|$, where the first term is the expected number of pairs to arrive at the exchange pool in the interval $[t,\tau]$ and the second term is the number of recipients that arrived at the pool until time t. Therefore, we can rewrite $E_t\left[\mathcal{C}^{\phi}(t,A)\right]$ as

$$E_t\left[\mathcal{C}^{\phi}(t,A)\right] = \int_t^{\infty} c\left(\lambda(\tau-t) + |A(t)| - E_t\left[|\phi(\tau,A)|\right] e^{-\rho(\tau-t)}\right)d\tau.$$

Since $\int_t^{\infty} e^{-\rho(\tau-t)}d\tau = \frac{1}{\rho}$ and $\int_t^{\infty}(\tau-t)e^{-\rho(\tau-t)}d\tau = \frac{1}{\rho^2}$, we can rewrite $E_t\left[\mathcal{C}^{\phi}(t,A)\right]$ as

$$E_t\left[\mathcal{C}^{\phi}(t,A)\right] = \frac{c\lambda}{\rho^2} + \frac{|A(t)|}{\rho} - \int_t^{\infty} cE_t\left[|\phi(\tau,A)|\right] e^{-\rho(\tau-t)}d\tau. \tag{1}$$

Only the last term in Equation 1 depends on the choice of mechanism ϕ. The previous terms cannot be controlled by the central authority, since they are the costs associated with the number of pairs arriving at the pool. We refer to this last term as the *exchange surplus at time t for mechanism ϕ* and denote it by

$$\mathcal{ES}^{\phi}(t,A) = \int_t^{\infty} cE_t\left[|\phi(\tau,A)|\right] e^{-\rho(\tau-t)}d\tau.$$

We can rewrite it as

$$\begin{aligned}\mathcal{ES}^{\phi}(t,A) &= \int_t^{\infty} c\left(E_t\left[|\phi(\tau,A)| - |\phi(t,A)|\right] + |\phi(t,A)|\right) e^{-\rho(\tau-t)}d\tau \\ &= \frac{c\,|\phi(t,A)|}{\rho} + \int_t^{\infty} c\left(E_t\left[|\phi(\tau,A)| - |\phi(t,A)|\right]\right) e^{-\rho(\tau-t)}d\tau.\end{aligned}$$

The first term above is the exchange surplus attributable to all exchanges that have been done until time t and at time t, and the second term is the *future exchange surplus* attributable to the exchanges to be done in the future. The central authority cannot control the number of past exchanges at time t either. Let $n^{\phi}(\tau,A)$ be the *number of matched recipients at time τ by mechanism ϕ*, and we have[11]

$$|\phi(t,A)| = \left(\sum_{\tau<t} n^{\phi}(\tau,A)\right) + n^{\phi}(t,A).$$

[10] E_t refers to the expected value at time t.

[11] For each finite time t, since with probability 1 the arrival interval between each arrival is finite and bounded from below, almost surely the total arrivals will be a finite number. However at the limit t=∞, this may not be correct. Thus, we will use a steady-state representation to handle the limit case.

We focus on the *present and future exchange surplus,* which is given as

$$\widetilde{\mathcal{ES}}^{\phi}(t) = \frac{cn^{\phi}(t, A)}{\rho} + \int_{t^+}^{\infty} c\left(E_t\left[|\phi(\tau, A)| - |\phi(t, A)|\right]\right) e^{-\rho(\tau - t)} d\tau. \tag{2}$$

A dynamic matching mechanism ϕ is *efficient* if for any t, it maximizes the *present and future exchange surplus* at time t given in Equation 2. We look for solutions of the problem independent of initial conditions and time t.[12]

3 Dynamically Efficient *Two-way* Matching Mechanisms

In this section, we derive the dynamically efficient *two-way* matching mechanism. A *two-way exchange* is an exchange involving only two pairs. A matching is a *two-way matching* if all exchanges in the matching are two-way exchanges. It will be useful to introduce the following concepts about two-way exchanges. We say that two pairs i and j are *mutually compatible* if object o_i is compatible with agent a_j and object o_j is compatible with agent a_i.

The following observation states the important individual rationality constraints that need to be respected in our derivation of an efficient matching mechanism:

Observation 1: A pair of type X-Y can participate in a two-way exchange only with a mutually compatible pair, i.e. a pair of type W-Z such that Y▶W and Z▶X.

We partition the set of pair types into four sets $\mathcal{P}^{\mathcal{O}}, \mathcal{P}^{\mathcal{U}}$, $\mathcal{P}^{\mathcal{S}}$, and $\mathcal{P}^{\mathcal{R}}$ as follows:

We refer to the set

$$\mathcal{P}^{\mathcal{O}} = \{\text{X-Y} \in \mathcal{P} : \text{Y} \blacktriangleright \text{X and X} \neq \text{Y}\}$$

as the set of *overdemanded* pair types. Since Y▶X, in two-way exchanges these pairs can save pairs of other W-Z types with Y▶W and Z▶X. These W-Z types can satisfy (a) W▶Z with W $\neq$ Z, (b) W⋫Z and Z⋫W, (c) W=Z, and (d) Z▶W with W $\neq$ Z. Class (d) W-Z types are also included in $\mathcal{P}^{\mathcal{O}}$. We will create other sets for W-Z types in classes (a), (b), and (c) as follows:

We refer to the set

$$\mathcal{P}^{\mathcal{U}} = \{\text{X-Y} \in \mathcal{P} : \text{X} \blacktriangleright \text{Y and X} \neq \text{Y}\}$$

as the set of *underdemanded* pair types X-Y, which can participate in two-way exchanges with pair types W-Z with Y▶W and Z▶X. Since W▶Z and W $\neq$ Z, by transitivity Y▶X and X $\neq$ Y. Hence, underdemanded types can only be matched with certain overdemanded pair types.

We refer to the set

$$\mathcal{P}^{\mathcal{S}} = \{\text{X-X} \in \mathcal{P}\}$$

as the set of *self-demanded* pair types. Self-demanded types X-X can only be matched with X-X type pairs or types W-Z with Z▶X▶W and Z $\neq$ W, which are certain overdemanded pairs.

[12] We will define a steady state formally. If such solutions exist, they depend only on the "current state of the pool" (defined appropriately) but not on time t or the initial conditions.

We refer to the set

$$\mathcal{P}^{\mathcal{R}} = \{\text{X-Y} \in \mathcal{P} : \text{X} \not\blacktriangleright \text{Y and Y} \not\blacktriangleright \text{X}\}$$

as the set of *reciprocally demanded* types. They can only be matched with their reciprocal types Y-X and certain pairs W-Z with Z▶X,Y▶W and Z $\neq$ W, which are certain overdemanded pairs.[13]

Throughout this section we will maintain one assumption:

Assumption 1 (Long-Run Assumption): Under any dynamic matching mechanism, in the long run, there is an arbitrarily large number of underdemanded pairs from each pair type in the exchange pool.[14]

Since we are interested in maximizing the exchange surplus and cannot control the inflow of the pairs, this assumption will not harm our objective, even though under cost minimization this assumption leads to ∞ cost for any matching mechanism.

We state the following Lemma directly using the construction of sets $\mathcal{P}^{\mathcal{O}}, \mathcal{P}^{\mathcal{U}}, \mathcal{P}^{\mathcal{S}}, \mathcal{P}^{\mathcal{R}}$ and their properties stated above (we skip its proof for brevity):

Lemma 1 *In a static population of pairs under Assumption 1, where $n_{X\text{-}Y}$ denotes the number of pairs of any type X-Y$\in \mathcal{P}$, the maximum number of pairs matched through two-way exchanges is given by*[15]

$$\sum_{X\text{-}Y \in \mathcal{P}^{\mathcal{O}}} 2n_{X\text{-}Y} + \sum_{V\text{-}V \in \mathcal{P}^{\mathcal{S}}} \lfloor \frac{n_{V\text{-}V}}{2} \rfloor + \sum_{W_1\text{-}W_2 \in \mathcal{P}^{\mathcal{R}}} \min\{n_{W_1\text{-}W_2}, n_{W_2\text{-}W_1}\}$$

We are ready to state our main result of this section:

Theorem 1 *Let dynamic matching mechanism ν be such that it matches only X-Y type pairs with their reciprocal Y-X type pairs immediately when such an exchange is feasible. Then, under Assumption 1,*

- *mechanism ν is a dynamically efficient two-way matching mechanism; and*
- *any dynamically efficient* two-way *matching mechanism conducts a two-way exchange whenever one becomes feasible.*

Let dynamic two-way matching mechanism ν be defined as in the hypothesis of Theorem 1, that is: for any arriving pair of any type X-Y $\in \mathcal{P}$, mechanism ν matches this pair immediately with an existing Y-X type pair if such a mutually compatible pair exists in the pool, and does not perform any exchanges, otherwise.

We will prove Theorem 1 using the following proposition:

[13] We will typically use the notations X-Y, Z_1-Z_2, V-V, and W_1-W_2 to denote a generic *overdemanded*, *underdemanded*, *self-demanded*, and *reciprocally demanded* pair type, respectively. When it is not ambiguous, we will use X-Y and W-Z also to denote generic pair types.

[14] When we discuss the example of kidney exchanges, we will show that regardless of the two-way matching mechanism used, this assumption will hold under realistic arrival probabilities for the pairs in our application for kidney exchange.

[15] For any real number x, $\lfloor x \rfloor$ represents the greatest integer less than or equal to x.

Proposition 1 *Under Assumption 1, within any time interval τ, mechanism ν matches the maximum number of pairs possible under any two-way matching mechanism.*

Proof of Proposition 1: Suppose that Assumption 1 holds. Suppose that mechanism ν is used for the exchange. Consider a time interval $\tau > 0$ in the long run. Let t_0 be the start of this time interval and $t_1 = t_0 + \tau$ be the end of this time interval. Since each type is matched with its reciprocal type under mechanism ν, in the long run we have,
(1) for any type Z$_1$-Z$_2$ $\in \mathcal{P}^{\mathcal{U}}$, by Assumption 1, there will be an arbitrarily large number of type Z$_1$-Z$_2$ pairs available;
(2) for any type X-Y $\in \mathcal{P}^{\mathcal{O}}$, since there is an arbitrarily large number of type Y-X pairs available (by Statement 1 above), for any incoming X-Y type pair i there will exist at least one Y-X type pair that is mutually compatible, and mechanism ν will immediately match these two pairs, implying that no type X-Y pairs will remain available;
(3) for any type V-V $\in \mathcal{P}^{\mathcal{S}}$, whenever x type V-V pairs are available in the exchange pool, mechanism ν will match $2\lfloor \frac{x}{2} \rfloor$ of these pairs with each other, implying that there will be 0 or 1 type V-V pair available,
(4) for any W$_1$-W$_2$ $\in \mathcal{P}^{\mathcal{R}}$, there is either no W$_2$-W$_1$ pair in the pool or there are finitely many. If there is a W$_2$-W$_1$ type pair, this pair and the incoming pair are mutually compatible and mechanism ν will immediately match these two pairs. In the pool, there will be either (1) no W$_1$-W$_2$ type pair and no W$_2$-W$_1$ type pair remaining, (2) no W$_2$-W$_1$ type pair and some W$_1$-W$_2$ type pairs remaining, or (3) no W$_1$-W$_2$ type pair and some W$_2$-W$_1$ type pairs remaining.

Clearly, the maximum number of exchanges in the interval $[t_0, t_1]$ is performed by not conducting any exchanges in interval $[t_0, t_1)$ and then conducting the maximal exchange at time t_1. Since time interval τ is finite, Statement 1 above is still valid at any time $t \in [t_0, t_1]$, regardless of which exchanges are conducted in $[t_0, t_1)$. Therefore, by Lemma 1, the maximum number of transplants that can be conducted in interval $[t_0, t_1]$ is

$$\sum_{\text{X-Y} \in \mathcal{P}^{\mathcal{O}}} 2n_{\text{X-Y}} + \sum_{\text{V-V} \in \mathcal{P}^{\mathcal{S}}} \lfloor \frac{n_{\text{V-V}}}{2} \rfloor + \sum_{\text{W}_1\text{-W}_2 \in \mathcal{P}^{\mathcal{R}}} \min\{n_{\text{W}_1\text{-W}_2}, n_{\text{W}_2\text{-W}_1}\} \tag{3}$$

where $n_{\text{X-Y}}$ is the number of type X-Y pairs that are available at time t_1, if no exchange has been conducted in interval $[t_0, t_1)$. Moreover for any X-Y $\in \mathcal{P}$, this number in Equation 3 can be achieved by matching each X-Y type pair with a reciprocal type pair, as long as it is possible.

Next consider the scenario, in which mechanism ν is used in interval $[t_0, t_1)$. Statements 1-4 above are valid for any time $t \in [t_0, t_1]$ under mechanism ν. Therefore, under mechanism ν, the number of matched pairs in interval $[t_0, t_1]$ is

- $\sum_{\text{X-Y} \in \mathcal{P}^{\mathcal{O}}} 2n_{\text{X-Y}}$ for the overdemanded and underdemanded pairs by Statements 1 and 2,
- $\sum_{\text{V-V} \in \mathcal{P}^{\mathcal{S}}} \lfloor \frac{n_{\text{V-V}}}{2} \rfloor$ for the self-demanded types by Statement 3, and
- $\sum_{\text{W}_1\text{-W}_2 \in \mathcal{P}^{\mathcal{R}}} \min\{n_{\text{W}_1\text{-W}_2}, n_{\text{W}_2\text{-W}_1}\}$ for the reciprocally demanded types by Statement 4,

and their sum is exactly equal to the expression given in Equation 3, completing the proof of Proposition 1. $\Diamond$

Theorem 1 can be proven using Proposition 1.

Proof of Theorem 1: Suppose Assumption 1 holds. Fix time τ in the long run. For any mechanism ϕ and any time $t > \tau$, $|\phi(t, A)| - |\phi(\tau, A)|$ is the total number of recipients matched between time τ and t under mechanism ϕ when the flow function is given by A, and is maximized by the mechanism ν by Proposition 1. Since $|\phi(t, A)| - |\phi(\tau, A)|$ is ex-post maximized for $\phi = \nu$ for any $t \geq \tau$, $E_\tau[|\phi(t, A)| - |\phi(\tau, A)|]$ is maximized by $\phi = \nu$, as well. Moreover, mechanism ν conducts the maximum possible number of exchanges at any given point in time as 0 or 2 (permitted by the two-way exchange restriction). Therefore, $n^\phi(\tau, A)$ is also maximized by $\phi = \nu$. These imply $\mathcal{ES}^{*\phi}(\tau) = \frac{cn^\phi(\tau, A)}{\rho} + \int_\tau^\infty cE_\tau[|\phi(t, A)| - |\phi(\tau, A)|]\, e^{-\rho(t-\tau)}dt$ is maximized for $\phi = \nu$, implying that ν is an efficient two-way matching mechanism. Moreover, it conducts the maximum number of transplants at each time, completing the proof of Theorem 1. $\Diamond$

Note that since we can roughly define an efficient mechanism ν independent of the state of the pool, we did not introduce an explicit state space for the pool. It turns out that under multi-way exchanges, an efficient mechanism explicitly depends on the state.

4 Dynamically Efficient *Multi-way* Matching Mechanisms

In this section, we consider matching mechanisms that allow for not only two-way exchanges, but larger exchanges as well. We maintain Assumption 1 throughout this section. Hence, there are arbitrarily many underdemanded pairs in the exchange pool in the long run.

We state Proposition 2 as follows:

Proposition 2 ***(Necessary and Sufficient Conditions for Matching Underdemanded, Self-Demanded, and Reciprocally Demanded Types in Multi-Way Exchanges)*** *Suppose that there exists exactly one overdemanded pair in the exchange pool, and it is of some type X-Y$\in \mathcal{P}^{\mathcal{O}}$*

- *An underdemanded pair of type Z_1-$Z_2 \in \mathcal{P}^{\mathcal{U}}$ can be matched in a multi-way exchange if and only if $X \not\blacktriangleright Z_2$ and $Z_1 \not\blacktriangleright Y$ and we use the overdemanded pair; if $Y \not\blacktriangleright Z_1$ then we use an additional reciprocally demanded pair of type Y-$Z_1 \in \mathcal{P}^{\mathcal{R}}$; and if $Z_2 \not\blacktriangleright X$ then we use an additional reciprocally demanded pair of type Z_2-$X \in \mathcal{P}^{\mathcal{R}}$.*

- *A self-demanded pair of type V-V$\in \mathcal{P}^{\mathcal{S}}$ can be matched in a multi-way exchange if and only if*

  - *we use another pair of type V-V,*

  or

- $X \not\blacktriangleright V$ *and* $V \not\blacktriangleright Y$ *and we use the overdemanded pair; if* $Y \not\blacktriangleright V$ *then we use an additional reciprocally demanded pair of type* Y*-*$V \in \mathcal{P}^{\mathcal{R}}$*; and if* $V \not\blacktriangleright X$ *then we use an additional reciprocally demanded pair of type* V*-*$X \in \mathcal{P}^{\mathcal{R}}$*.*

- *A reciprocally demanded pair of type* W_1*-*$W_2 \in \mathcal{P}^{\mathcal{R}}$ *can be matched in a multi-way exchange if and only if*

  - *we use a reciprocal* W_2*-*W_1 *type pair,*

    or

  - $Y \blacktriangleright Z_1$ *and* $Z_2 \blacktriangleright X$ *and we use the overdemanded pair of type* X*-*Y*.*

The proof of the above proposition is in Appendix A. We also state and prove Proposition 6, which is in regard to the sizes of maximal exchanges in Appendix A. These two propositions bring several simplifications to the optimization problem.

Suppose that X-Y$\in \mathcal{P}^{\mathcal{O}}$ is the pair type of an arriving overdemanded pair, i.e., Y$\blacktriangleright$X and yet Y$\neq$X. Instead of matching this pair and serving one underdemanded pair (as in two-way exchanges, for example, by matching it with a pair of type Y-X), we can potentially use this pair in larger exchanges to serve more underdemanded pairs, if multi-way exchanges are allowed.

It could also be the case that there are multiple reciprocally demanded pairs of different compatibility levels existent in the exchange pool, when the X-Y type overdemanded pair arrives. Without loss of generality, we can assume that no two of these types are at the same compatibility level. If they were, we could have matched the pairs of these types with each other in two-way exchanges until one of them has no more pairs left. It could also be the case that there are multiple self-demanded pairs at different compatibility levels in the exchange pool. Without loss of generality, we can assume that no two of these pairs have the same pair type, (since otherwise, if there are k>1 pairs of type V-V, we could serve them in a k-way exchange as V-V,V-V,...,V-V).

We will refer to the pairs satisfying the conditions in Proposition 2 to be matched using an overdemanded pair as *matchable* pairs. We state the following corollary to Proposition 6 about matchable pairs:

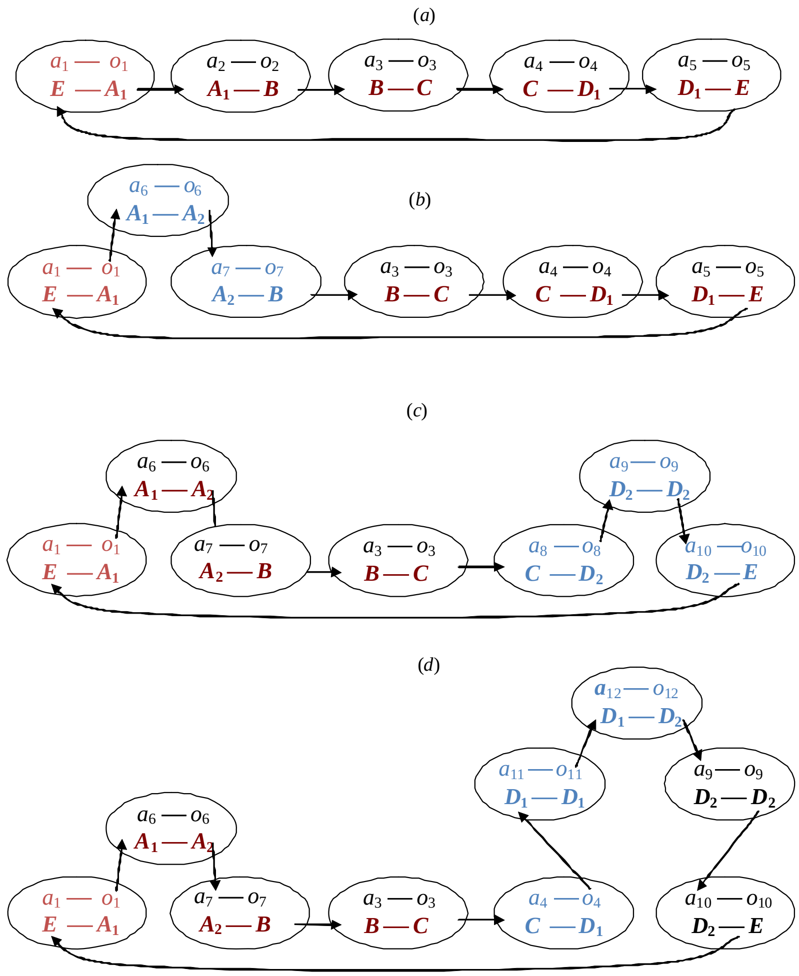
Corollary 1 *Under Assumption 1, when there is a single overdemanded type pair of type* X*-*Y*, under an efficient mechanism,*

- *we match* L_X*-*L_Y *underdemanded pairs; and*

- *decisions regarding* matchable self-demanded *and* reciprocally demanded pairs *with object and agent types located at* different compatibility levels *of the partial order* $\blacktriangleright$ *are* independent *from each other.*

That is: under an efficient mechanism, not only we match the maximum number of possible underdemanded pairs, but we can always enlarge an exchange by squeezing in a matchable reciprocally demanded or self-demanded pair belonging to a different compatibility level than the ones in the exchange, at the cost of replacing only underdemanded pairs in the exchange with different ones. Suppose an exchange E has been fixed with a single overdemanded pair. Then we can squeeze in a reciprocally demanded pair (if the conditions in Proposition 2 are satisfied) without affecting the rest of the overdemanded, reciprocally demanded, and self-demanded pairs in E, and match the same number of underdemanded pairs as E does. We can almost do the same thing using a self-demanded pair except if there is already another self-demanded pair of a compatibility type that is incomparable to this pair. In the latter case, a new reciprocally demanded pair belonging to this compatibility level is also needed (see Footnote 29). The new exchange may replace at most two underdemanded pairs of E with new ones. Since by Assumption 1 underdemanded pairs are abundant, this minimal change will have no effect on optimization. An example of such insertions are given below using the type sets in Figure 1(a).

An example regarding overdemanded type pairs helping out other pairs in multi-way exchanges: Consider the types in Figure 1(a). Suppose that an E-A_1 type overdemanded pair arrives under Assumption 1. E-A_1 can be used to match all underdemanded, self-, and reciprocally demanded pairs. Exchange (a) is conducted to match the maximum number of underdemanded pairs, which is $L_{A_1} - L_E = 4$. In Exchange (b), we *squeeze in* an A_1-A_2 type reciprocally demanded pair by replacing pair 3 in Exchange (a) with pair 7. In Exchange (c), we additionally *squeeze in* a D_2-D_2 type self-demanded pair by replacing pairs 4 and 5 in Exchange (b) with pairs 8 and 10 respectively. In Exchange (d), we *squeeze in* a D_1-D_1 type self-demanded pair. However, there is already another pair at the same compatibility level, namely pair 9 of type D_2-D_2. Thus, we need an additional D_1-D_2 or D_2-D_1 type reciprocally demanded pair to accommodate this new pair together with pair 9. In the figure, we use pair 12 of type D_1-D_2 as a "bridge pair" between D_1-D_1 and D_2-D_2 type pairs. In this case, we also replace underdemanded pair 8 with pair 4.



$\Diamond$

All of the above results assume that there exists at most one overdemanded pair in the pool. However, if we do not match an overdemanded pair immediately, there can be more than one. We state one other assumption, which will ensure that an overdemanded pair will never be kept in the pool.

Suppose that W$_1$-W$_2$ and W$_2$-W$_1$ are two reciprocally demanded pair types at the same compatibility level. We show that as long as the difference between W$_1$-W$_2$ and W$_2$-W$_1$ type arrival frequencies is not large, overdemanded type pairs will be matched immediately under the efficient mechanism. The proof of this proposition is given in Appendix A.

Proposition 3 *Suppose Assumption 1 holds. If for all W_1-W_2 and W_2-$W_1 \in \mathcal{P}^{\mathcal{R}}$, $2p_{W_1\text{-}W_2} > p_{W_2-W_1}$ and $2p_{W_2\text{-}W_1} > p_{W_1-W_2}$, then under any dynamically efficient multi-way matching mechanism, overdemanded type pairs are matched as soon as they arrive at the exchange pool.*

We state the hypothesis of this proposition as an assumption.

Assumption 2 (Assumption on Generic Arrival Rates of Reciprocally Demanded Types): For all W_1-W_2 and W_2-$W_1 \in \mathcal{P}^{\mathcal{R}}$, $2p_{W_1\text{-}W_2} > p_{W_2-W_1}$ and $2p_{W_2\text{-}W_1} > p_{W_1-W_2}$.

Under Assumptions 1 and 2, since an overdemanded pair will be matched immediately when it arrives, we will only need to make decisions in situations in which multiple exchanges of different sizes are feasible. Thus, using multi-way exchanges, we can benefit from not conducting the largest feasible exchange currently available and holding onto some of the pairs, which can currently participate in an exchange, in the expectation of saving more pairs sooner.

For example, consider a situation in which an overdemanded X-Y type pair arrives at the pool, while a reciprocally demanded W_1-W_2 type pair which can be matched using the X-Y type pair is also available.

By Corollary 1, the decisions regarding the W_1-W_2 type pair are independent from decisions regarding other reciprocally demanded pair types except type W_2-W_1.[16] Since by Assumption 1 there is an excess number of underdemanded type pairs in the long run, by Corollary 1 there are two candidates for an optimal exchange: for some number n

- an n-way kidney exchange without the W_1-W_2 type pair or
- an $(n + 1)$-way exchange including the W_1-W_2 type pair.[17]

Which exchange should the central authority choose?

We answer this question by converting the problem to an embedded *Markov decision process* with a state space consisting of $|\mathcal{P}|$ dimensional intege≤r vectors that show the number of pairs in the pool belonging to each pair-type.[18,19] There is additional structure to eliminate some of these state variables under Assumptions 1 and 2:

- *For the overdemanded types:* If an overdemanded pair i of type X-Y $\in \mathcal{P}^{\mathcal{O}}$ arrives, by Proposition 3, pair i will be matched immediately in some exchange. Hence, the number of overdemanded pairs remaining in the pool is always 0.

[16] When there is a W_2-W_1 type pair, we can immediately match this pair with the W_1-W_2 type pair. Thus suppose that there is no W_2-W_1 pair available in the pool.

[17] If there are also W_1-W_1 and W_2-W_2 type pairs existent in the pool, then the decision is between

- an n-way exchange without a W_1-W_2 type pair and with one of the two self-demanded pairs, and
- an $n + 2$-way exchange that additionally matches a W_1-W_2 type pair and the other self-demanded pair.

[18] See Puterman (1994) for an excellent survey of continuous time and discrete time Markov decision processes.

[19] Since the pairs arrive according to a Poisson process, which is memory-less, solving the problem only for this Markov decision process with $|\mathcal{P}|$ variables will also provide a solution of the original problem stated in Equation 2. Moreover, this solution will be independent of other characteristics such as the inflow and match history of the exchange, time, and initial conditions.

- *For the underdemanded types:* By Assumption 1, there will be an arbitrarily large number of underdemanded pairs. Hence, the number of underdemanded pairs is always ∞.

- For the self-demanded types: Whenever a self-demanded pair i of type V-V $\in \mathcal{P}^{\mathcal{S}}$ is available in the exchange pool, as an implication of Corollary 1, the decisions may be complicated by the existence of other self-demanded and reciprocally demanded type pairs in the same compatibility level. Selection of other pairs in an exchange may affect which self-demanded type pairs will be used if different type pairs are simultaneously available. A self-demanded type can never save an underdemanded or reciprocally demanded pair without the help of an overdemanded or reciprocally demanded pair(s) by Proposition 2. On the other hand, if there is more than 1 such type of a pair, then we can match all such pairs together in an exchange. This and the above observations imply that under an efficient matching mechanism, for any V-V $\in \mathcal{P}^{\mathcal{S}}$, at steady-state there will be either 0 or 1 V-V type pair.

  Therefore, self-demanded types' effect to the reduced state space will be reflected by 4 additional state variables, each getting values of either 0 or 1. We first derive the efficient dynamic matching mechanism by ignoring the self-demanded type pairs; then *we will reintroduce the self-demanded types to the problem* and comment on the dynamically efficient matching mechanism for our leading example of kidney exchanges in Appendix B.

Assumption 3 (No Self-Demanded Types Assumption): There are no self-demanded types available for exchange and $p_{\text{V-V}} = 0$ for all V-V $\in \mathcal{P}$.

- *For the reciprocally demanded types:* By the above analyses, there are no overdemanded and self-demanded type pairs available and there are infinitely many underdemanded type pairs. Therefore, the state of the exchange pool can simply be denoted by the number of reciprocally demanded pairs. On the other hand any reciprocally demanded pairs of types W_1-W_2 and W_2-$\text{W}_1 \in \mathcal{P}^{\mathcal{R}}$ can be matched in a two-way exchange. Moreover, by Proposition 2, a reciprocally demanded type pair cannot save an underdemanded pair in an exchange without the help of an overdemanded pair. Hence, the most efficient use of W_1-W_2 and W_2-W_1 type pairs is to be matched with each other in a two-way exchange. Therefore, under the efficient matching mechanism, a W_1-W_2 and W_2-W_1 type pair will never remain in the pool together, but will be matched via a two-way exchange.

  Let $\mathcal{P}^{\mathcal{R}^*} \subseteq \mathcal{P}^{\mathcal{R}}$ be fixed such that for all W_1-$\text{W}_2 \in \mathcal{P}^{\mathcal{R}}$

  $$\text{W}_1\text{-W}_2 \in \mathcal{P}^{\mathcal{R}^*} \iff \text{W}_2\text{-W}_1 \notin \mathcal{P}^{\mathcal{R}^*}.$$

  That is: For each compatibility level with two object types W_1 and W_2, only one pair type W_1-W_2 is in $\mathcal{P}^{\mathcal{R}^*}$.

  By the above observation, we can simply denote the state of the exchange pool by an integer vector $s = (s_{\text{W}_1\text{-W}_2})_{\text{W}_1\text{-W}_2 \in \mathcal{P}^{\mathcal{R}^*}}$, such that for all W_1-$\text{W}_2 \in \mathcal{P}^{\mathcal{R}^*}$, if $s_{\text{W}_1\text{-W}_2} > 0$, then $s_{\text{W}_1\text{-W}_2}$

refers to the number of W_1-W_2 type pairs in the exchange pool, and if $s_{\mathrm{W}_1\text{-}\mathrm{W}_2} < 0$, then $|s_{\mathrm{W}_1\text{-}\mathrm{W}_2}|$ refers to the number of W_2-W_1 type pairs in the exchange pool. Formally, $s_{\mathrm{W}_1\text{-}\mathrm{W}_2}$ is the difference between the number of W_1-W_2 type pairs and W_2-W_1 type pairs in the pool, and *only one of these two numbers can be non-zero.* Let $S = \mathbb{Z}^{|\mathcal{P}^{\mathcal{R}^*}|}$ be the state space.

In our running example explored in Figure 1(a), we can set $\mathcal{P}^{\mathcal{R}^*} = \{\mathrm{A}_1\text{-}\mathrm{A}_2, \mathrm{D}_1\text{-}\mathrm{D}_2\}$ and $S = \mathbb{Z}^2$.

4.1 Markov Chain Representation

In this subsection, we characterize the transition from one state to another under a dynamically efficient matching mechanism by a Markov chain when Assumptions 1, 2, and 3 hold:

Fix W_1-$\mathrm{W}_2 \in \mathcal{P}^{\mathcal{R}^*}$. By Corollary 1, the decisions regarding pairs of types W_1-W_2 and W_2-W_1 are independent from decisions regarding other reciprocal types. Therefore, we focus just on W_1-W_2 and W_2-W_1 type pairs. Hence, we consider the state (component) of the pool regarding W_1-W_2 and W_2-W_1 types, $s_{\mathrm{W}_1\text{-}\mathrm{W}_2}$. Suppose that $s_{\mathrm{W}_1\text{-}\mathrm{W}_2} > 0$, i.e. there are only W_1-W_2 type pairs in the pool, but no W_2-W_1 type pairs. We define the following partition of the overdemanded pairs:

$$\mathcal{P}^{\mathcal{O}}(\mathrm{W}_1\text{-}\mathrm{W}_2) = \left\{\mathrm{X}\text{-}\mathrm{Y} \in \mathcal{P}^{\mathcal{O}} : \mathrm{W}_2 \blacktriangleright \mathrm{X} \text{ and } \mathrm{Y} \blacktriangleright \mathrm{W}_1\right\},$$
$$\mathcal{P}^{\mathcal{O}}(\tilde{}\,\mathrm{W}_1\text{-}\mathrm{W}_2) = \mathcal{P}^{\mathcal{O}} \backslash \mathcal{P}^{\mathcal{O}}(\mathrm{W}_1\text{-}\mathrm{W}_2).$$

By Proposition 2, $\mathcal{P}^{\mathcal{O}}(\mathrm{W}_1\text{-}\mathrm{W}_2)$ is the set of overdemanded pairs that are required to match W_1-W_2 type pairs; and $\mathcal{P}^{\mathcal{O}}(\tilde{}\,\mathrm{W}_1\text{-}\mathrm{W}_2)$ is the set of remaining overdemanded pairs.

Next, we will analyze decisions regarding W_1-W_2 type pairs:

Assume that an X-Y type pair arrives. Three cases are possible regarding X-Y:

1. X-Y$\in \mathcal{P}^{\mathcal{O}}$: By Assumption 2 and Proposition 3, we need to match the X-Y type pair immediately. Moreover, we need to match L_{X}-L_{Y} underdemanded type pairs in such an exchange; otherwise such an exchange will not be efficient by Assumption 1 and Corollary 1. We isolate ourselves from all decisions regarding any type of reciprocally demanded pairs, but W_1-W_2 and W_2-W_1 type pairs. Suppose that E is a feasible n-way exchange for some $n \geq 2$ that can match L_{X}-L_{Y} underdemanded pairs, the X-Y type pair i, and possibly some reciprocally demanded pairs in an efficient way, but no W_1-W_2 type pairs. We have two cases regarding the decision for W_1-W_2 type pairs:

    - X-Y$\in \mathcal{P}^{\mathcal{O}}(\tilde{}\,\mathrm{W}_1\text{-}\mathrm{W}_2)$: By Proposition 2 and the fact that there are no W_2-W_1 pairs in the pool, there is no feasible exchange that can match a W_1-W_2 type pair. Thus, exchange E is an efficient exchange.
    - X-Y$\in \mathcal{P}^{\mathcal{O}}(\mathrm{W}_1\text{-}\mathrm{W}_2)$: By Corollary 1, instead of exchange E, we can conduct an $(n+1)$-way exchange, E', including the overdemanded and reciprocally demanded type pairs, the same number of underdemanded type pairs, and one more reciprocally demanded pair

from the W_1-W_2 type. In this case, we have two possible actions: conduct exchange E (smaller exchange) or conduct exchange E' (larger exchange). If E' is conducted, the state component for W_1-W_2 and W_2-W_1 type pairs decreases to $s_{\mathrm{W}_1\text{-}\mathrm{W}_2} - 1$, since one fewer W_1-W_2 type pair will remain in the exchange pool.

2. X-Y $\in \mathcal{P}^{\mathcal{U}}$: By Assumption 2 and Proposition 3, there are no overdemanded pairs available in the pool. And by Proposition 2, no exchanges are feasible.

3. X-Y $\in \mathcal{P}^{\mathcal{R}}$: Three cases are possible:

   - X-Y $=$ W_1-W_2: Since there are no W_2-W_1 type pairs, and no overdemanded pairs (by Assumption 2 and Proposition 3), by Proposition 2, there are no feasible exchanges, and the state component for W_1-W_2 and W_2-W_1 type pairs increases to $s_{\mathrm{W}_1\text{-}\mathrm{W}_2} + 1$.
   - X-Y $=$ W_2-W_1: A two-way exchange can be conducted using a W_1-W_2 type pair in the pool and the arriving X-Y type pair i. This is the only feasible type of exchange. Since matching a W_2-W_1 type pair with a W_1-W_2 type pair is the most efficient use of these types of pairs, we need to conduct such a two-way exchange. The state component for W_1-W_2 and W_2-W_1 type pairs decreases to $s_{\mathrm{W}_1\text{-}\mathrm{W}_2} - 1$.
   - X-Y $\notin \{\mathrm{W}_1\text{-}\mathrm{W}_2, \mathrm{W}_2\text{-}\mathrm{W}_1\}$: By Proposition 2, there is no feasible exchange regarding W_1-W_2 type pairs.

Figure 2 summarizes how the state of the pool regarding W_1-W_2 and W_2-W_1 type pairs evolves for $s_{\mathrm{W}_1\text{-}\mathrm{W}_2} > 0$.

For $s_{\mathrm{W}_1\text{-}\mathrm{W}_2} < 0$, i.e., when $|s_{\mathrm{W}_1\text{-}\mathrm{W}_2}|$ W_2-W_1 type pairs are available in the exchange pool, we observe the symmetric version of the above evolution. For $s_{\mathrm{W}_1\text{-}\mathrm{W}_2} = 0$, i.e., when there are no W_1-W_2 and W_2-W_1 type pairs available in the exchange pool, the evolution is somewhat simpler. Figure 3 summarizes the transitions from state component $s_{\mathrm{W}_1\text{-}\mathrm{W}_2} = 0$. The only state transition regarding $s_{\mathrm{W}_1\text{-}\mathrm{W}_2}$ occurs when a W_1-W_2 type pair arrives (to state component $s_{\mathrm{W}_1\text{-}\mathrm{W}_2} = 1$), or when a W_2-W_1 type pair arrives (to state component $s_{\mathrm{W}_1\text{-}\mathrm{W}_2} = -1$).

4.2 Bellman Equations

In this subsection, we derive the Bellman Equations that will be used to find the efficient matching mechanism. Let τ_1 be the time between two arrivals. Since the arrival process is a Poisson process with arrival rate λ, where λ is the expected number of arrivals in unit time, then τ_1 is distributed with an exponential distribution with parameter λ, that is: the probability density function of τ_1 is $\lambda e^{-\lambda \tau_1}$. Then, the expected total discounting that occurs until a new pair arrives is given by

$$E\left[e^{-\rho\tau_1}\right] = \int_0^\infty \lambda e^{-\rho\tau_1} e^{-\lambda\tau_1} d\tau_1 = \frac{\lambda}{\lambda + \rho}.$$

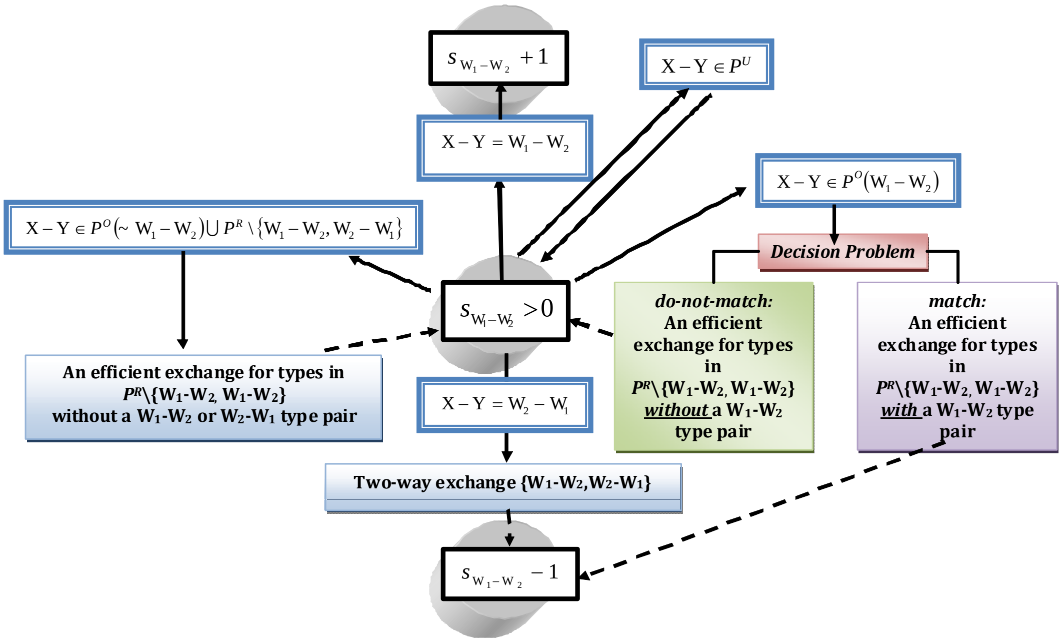


Figure 2: Transitions with arrivals when $s_{W_1\text{-}W_2}>0$

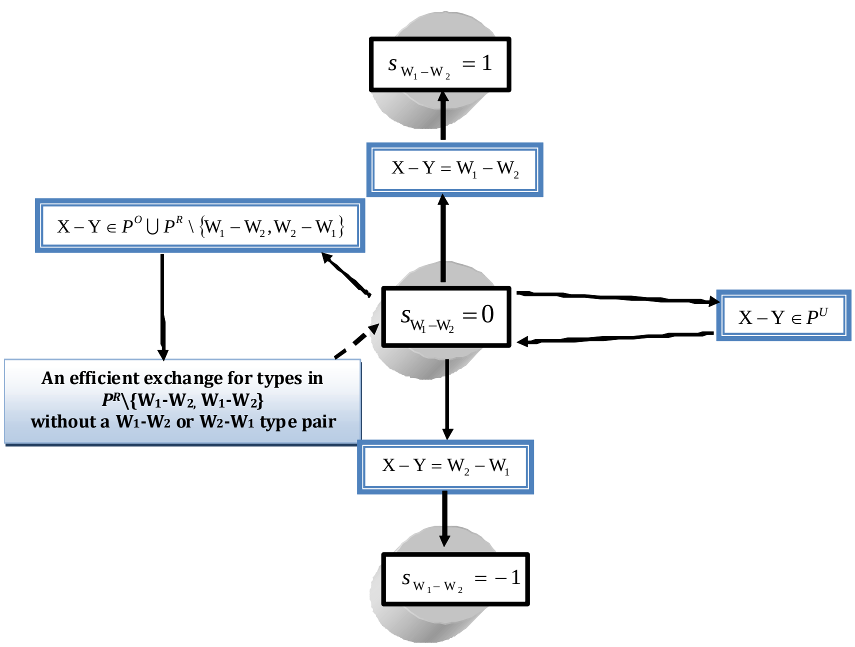


Figure 3: Transitions with arrivals when $s_{W_1\text{-}W_2}= 0$

When a pair is matched in the exchange pool, the surplus related to the pair is $\int_0^\infty ce^{-\rho\tau}d\tau = \frac{c}{\rho}$.[20]

Using the analyses in the previous subsection, under Assumptions 1, 2, and 3, we can state Bellman equations for the reduced continuous time Markov process. By Proposition 2 and Corollary 1, $ES(s)$, the total surplus at state $s \in S$ under the efficient rule, can be written as the sum of surpluses regarding each type:

$$ES(s) = \sum_{\text{X-Y}\in\mathcal{P}^{\mathcal{O}}} ES_{\text{X-Y}} + \sum_{\text{V-V}\in\mathcal{P}^{\mathcal{S}}} ES_{\text{V-V}} + \sum_{\text{W}_1\text{-W}_2\in\mathcal{P}^{\mathcal{R}^*}} ES_{\text{W}_1\text{-W}_2}\left(s_{\text{W}_1\text{-W}_2}\right)$$

where

- $ES_{\text{X-Y}}$ for each type X-Y$\in \mathcal{P}^{\mathcal{O}}$ is equal to $\frac{\lambda}{\lambda+\rho} p_{\text{X-Y}}\left(L_{\text{X}} - L_{\text{Y}} + 1\right)\frac{c}{\rho}$ where $p_{\text{X-Y}}$ is the probability of next incoming pair being of type X-Y, $L_{\text{X}} - L_{\text{Y}}$ is the number of underdemanded type pairs and 1 is the number of the overdemanded pair that can be matched regardless of the reciprocally demanded pairs matched. Since underdemanded types cannot be matched without the help of overdemanded types (by Proposition 2), we incorporate their surplus to the surplus of overdemanded types.
- $ES_{\text{V-V}} = 0$ for all V-V$\in \mathcal{P}^{\mathcal{S}}$ by Assumption 3.
- $ES_{\text{W}_1\text{-W}_2}\left(s_{\text{W}_1\text{-W}_2}\right)$ is the surplus related to reciprocally demanded types W_1-W_2 and W_2-W_1, for each $\text{W}_1\text{-W}_2 \in \mathcal{P}^{\mathcal{R}^*}$ and $s_{\text{W}_1\text{-W}_2} \in \mathbb{Z}$. It can be maximized independently from other surpluses $ES_{\text{W}_3\text{-W}_4}\left(s_{\text{W}_3\text{-W}_4}\right)$ for all $\text{W}_3\text{-W}_4 \in \mathcal{P}^{\mathcal{R}^*}\setminus\{\text{W}_1\text{-W}_2\}$ and $s_{\text{W}_3\text{-W}_4} \in \mathbb{Z}$ by Corollary 1.

From now on, we focus on the exchange surplus related to reciprocally demanded types. Fix $\text{W}_1\text{-W}_2 \in \mathcal{P}^{\mathcal{R}^*}$. For $s_{\text{W}_1\text{-W}_2} > 0$, exchange surplus for types W_1-W_2 and W_2-W_1 is stated as

$$\begin{aligned} & ES_{\text{W}_1\text{-W}_2}\left(s_{\text{W}_1\text{-W}_2}\right) \qquad (4) \\ &= \frac{\lambda}{\lambda+\rho}\left[\begin{array}{c} \left(\sum_{\text{X-Y}\in\mathcal{P}^{\mathcal{O}}(\tilde{}\text{W}_1\text{-W}_2)\cup\mathcal{P}^{\mathcal{U}}\cup\mathcal{P}^{\mathcal{R}}\setminus\{\text{W}_1\text{-W}_2,\text{W}_2\text{-W}_1\}} p_{\text{X-Y}}\right) ES_{\text{W}_1\text{-W}_2}\left(s_{\text{W}_1\text{-W}_2}\right) \\ + \left(\sum_{\text{X-Y}\in\mathcal{P}^{\mathcal{O}}(\text{W}_1\text{-W}_2)} p_{\text{X-Y}}\right) \max\left\{ES_{\text{W}_1\text{-W}_2}\left(s_{\text{W}_1\text{-W}_2}\right), ES_{\text{W}_1\text{-W}_2}\left(s_{\text{W}_1\text{-W}_2}-1\right)+\frac{c}{\rho}\right\} \\ + p_{\text{W}_1\text{-W}_2} ES_{\text{W}_1\text{-W}_2}\left(s_{\text{W}_1\text{-W}_2}+1\right) + p_{\text{W}_2\text{-W}_1}\left(ES_{\text{W}_1\text{-W}_2}\left(s_{\text{W}_1\text{-W}_2}-1\right)+\frac{2c}{\rho}\right) \end{array}\right]. \end{aligned}$$

On the right-hand side of Equation 4, (1) the first row considers the case when an overdemanded pair that cannot be used in matching a W_1-W_2 type pair or an underdemanded pair or a reciprocally demanded pair of types other than W_1-W_2 and W_2-W_1 arrives; (2) the second row considers the case when an overdemanded pair that can be used in matching a W_1-W_2 type pair arrives, leading to a

[20] We will later observe that the optimal matching mechanism is independent of how this individual surplus is calculated. Thus, it is robust to the interpretation of surplus.

decision of either matching the W_1-W_2 type pair or not together with the efficient size exchanges decided for other types; and (3) the third row considers the case when a W_1-W_2 type pair arrives, leading to an increase in the state component, and when a W_2-W_1 type pair arrives, reducing the state component by conducting a two-way exchange and matching one W_2-W_1 and one W_1-W_2 type pairs.

We can rewrite Equation 4 by dividing both sides of the equation by $\frac{c}{\rho}$ and setting $ES^*_{W_1\text{-}W_2}(s_{W_1\text{-}W_2}) = \frac{\rho}{c} ES_{W_1\text{-}W_2}(s_{W_1\text{-}W_2})$ as follows:

$$
\begin{aligned}
&ES^*_{W_1\text{-}W_2}(s_{W_1\text{-}W_2}) \qquad (5)\\
&= \frac{\lambda}{\lambda+\rho}\left[\begin{array}{l}
\left(\sum_{X\text{-}Y \in \mathcal{P}^{\mathcal{O}}(\char`~W_1\text{-}W_2)\cup\mathcal{P}^{\mathcal{U}}\cup\mathcal{P}^{\mathcal{R}}\setminus\{W_1\text{-}W_2, W_2\text{-}W_1\}} p_{X\text{-}Y}\right) ES^*_{W_1\text{-}W_2}(s_{W_1\text{-}W_2}) \\
+\left(\sum_{X\text{-}Y \in \mathcal{P}^{\mathcal{O}}(W_1\text{-}W_2)} p_{X\text{-}Y}\right) \max\left\{ES^*_{W_1\text{-}W_2}(s_{W_1\text{-}W_2}), ES^*_{W_1\text{-}W_2}(s_{W_1\text{-}W_2}-1)+1\right\} \\
+p_{W_2\text{-}W_1} ES^*_{W_1\text{-}W_2}(s_{W_1\text{-}W_2}+1) + p_{W_1\text{-}W_2}\left(ES^*_{W_1\text{-}W_2}(s_{W_1\text{-}W_2}-1)+2\right)
\end{array}\right].
\end{aligned}
$$

Note that since by Assumption 3, $p_{V\text{-}V} = 0$ for all V-V$\in \mathcal{P}^{\mathcal{S}}$, we have

$$
\sum_{X\text{-}Y \in \mathcal{P}^{\mathcal{O}}(\char`~W_1\text{-}W_2)\cup\mathcal{P}^{\mathcal{U}}\cup\mathcal{P}^{\mathcal{R}}\setminus\{W_1\text{-}W_2, W_2\text{-}W_1\}} p_{X\text{-}Y} = 1 - \left(\sum_{X\text{-}Y \in \mathcal{P}^{\mathcal{O}}(W_1\text{-}W_2)} p_{X\text{-}Y}\right) - p_{W_1\text{-}W_2} - p_{W_2\text{-}W_1}
$$

Similarly, we can write the Bellman Equation for state components $s_{W_1\text{-}W_2} < 0$ as follows:

$$
\begin{aligned}
&ES^*_{W_1\text{-}W_2}(s_{W_1\text{-}W_2}) \qquad (6)\\
&= \frac{\lambda}{\lambda+\rho}\left[\begin{array}{l}
\left(\sum_{X\text{-}Y \in \mathcal{P}^{\mathcal{O}}(\char`~W_2\text{-}W_1)\cup\mathcal{P}^{\mathcal{U}}\cup\mathcal{P}^{\mathcal{R}}\setminus\{W_1\text{-}W_2, W_2\text{-}W_1\}} p_{X\text{-}Y}\right) ES^*_{W_1\text{-}W_2}(s_{W_1\text{-}W_2}) \\
+\left(\sum_{X\text{-}Y \in \mathcal{P}^{\mathcal{O}}(W_2\text{-}W_1)} p_{X\text{-}Y}\right) \max\left\{ES^*_{W_1\text{-}W_2}(s_{W_1\text{-}W_2}), ES^*_{W_1\text{-}W_2}(s_{W_1\text{-}W_2}+1)+1\right\} \\
+p_{W_2\text{-}W_1} ES^*_{W_1\text{-}W_2}(s_{W_1\text{-}W_2}-1) + p_{W_1\text{-}W_2}\left(ES^*_{W_1\text{-}W_2}(s_{W_1\text{-}W_2}+1)+2\right)
\end{array}\right].
\end{aligned}
$$

For state component 0, the Bellman equation is

$$
ES^*_{W_1\text{-}W_2}(0) = \frac{\lambda}{\lambda+\rho}\left[\begin{array}{l}
\left(\sum_{X\text{-}Y \in \mathcal{P}^{\mathcal{O}}\cup\mathcal{P}^{\mathcal{U}}\cup\mathcal{P}^{\mathcal{R}}\setminus\{W_1\text{-}W_2, W_2\text{-}W_1\}} p_{X\text{-}Y}\right) ES^*_{W_1\text{-}W_2}(0) \\
+p_{W_1\text{-}W_2} ES^*_{W_1\text{-}W_2}(1) + p_{W_2\text{-}W_1}\left(ES^*_{W_1\text{-}W_2}(-1)\right)
\end{array}\right]. \qquad (7)
$$

The following observations will be useful for our analysis in the next section. They follow from the formulation of the problem.

Observation 2: If an overdemanded pair that can be used to match a W_1-W_2 pair arrives and yet the smaller exchange without any W_1-W_2 type pair is chosen at a state component $s_{\mathrm{W}_1\text{-}\mathrm{W}_2} > 0$, then $ES^*_{\mathrm{W}_1\text{-}\mathrm{W}_2}(s_{\mathrm{W}_1\text{-}\mathrm{W}_2}) \geq ES^*_{\mathrm{W}_1\text{-}\mathrm{W}_2}(s_{\mathrm{W}_1\text{-}\mathrm{W}_2} - 1) + 1$. If the larger exchange with a W_1-W_2 type pair is chosen, then $ES^*_{\mathrm{W}_1\text{-}\mathrm{W}_2}(s_{\mathrm{W}_1\text{-}\mathrm{W}_2}) \leq ES^*_{\mathrm{W}_1\text{-}\mathrm{W}_2}(s_{\mathrm{W}_1\text{-}\mathrm{W}_2} - 1) + 1$.

Observation 3: If an overdemanded pair that can be used to match a W_2-W_1 pair arrives and yet the smaller exchange without any W_2-W_1 type pair is chosen at a state component $s_{\mathrm{W}_1\text{-}\mathrm{W}_2} < 0$, then $ES^*_{\mathrm{W}_1\text{-}\mathrm{W}_2}(s_{\mathrm{W}_1\text{-}\mathrm{W}_2}) \geq ES^*_{\mathrm{W}_1\text{-}\mathrm{W}_2}(s_{\mathrm{W}_1\text{-}\mathrm{W}_2} + 1) + 1$. If the larger exchange with a W_2-W_1 type pair is chosen, then $ES^*_{\mathrm{W}_1\text{-}\mathrm{W}_2}(s_{\mathrm{W}_1\text{-}\mathrm{W}_2}) \leq ES^*_{\mathrm{W}_1\text{-}\mathrm{W}_2}(s_{\mathrm{W}_1\text{-}\mathrm{W}_2} + 1) + 1$.

The solution for $ES^*_{\mathrm{W}_1\text{-}\mathrm{W}_2}(s_{\mathrm{W}_1\text{-}\mathrm{W}_2})$ in Equations 5, 6, and 7 gives the normalized efficient exchange surplus regarding W_1-W_2 and W_2-W_1 type reciprocally demanded pairs. Does a solution exist to these equations, and if so, is it unique? The following proposition answers this question affirmatively. It is proven in Appendix A.

Proposition 4 *For any W_1-$W_2 \in \mathcal{P}^{\mathcal{R}^*}$, there exists a unique solution $ES^*_{W_1\text{-}W_2} : \mathbb{Z} \to \mathbb{R}_+$ to the Bellman Equations given in Equations 5, 6, and 7.*

4.3 The Efficient Matching Mechanism

A *(deterministic) Markov matching mechanism* ϕ is a matching mechanism that chooses the same action whenever the Markov chain is in the same state. In our reduced state and action problem, a Markov matching mechanism makes multiple decisions at a state depending on the type of an arriving pair and the number and the types of reciprocally demanded pairs in the pool. For each reciprocally demanded type W_1-W_2 existing in the pool, when an overdemanded type pair that can be used to match a W_1-W_2 type pair arrives, the two decisions are (a) conduct an exchange without a W_1-W_2 pair, but with the maximum possible number of underdemanded pairs (action *do-not-match*), or (b) conduct an exchange with a W_1-W_2 pair and the maximum possible number of underdemanded pairs (action *match*). The remaining choices of the Markov mechanism are straightforward: It chooses an exchange with the maximum number of underdemanded pairs when such an exchange becomes feasible as outlined in Figure 2 for positive states, Figure 3 for state zero, and the symmetric version of Figure 2 for negative states. Formally, $\phi : S \to \{\textit{do-not-match}, \textit{match}\}^{\left|\mathcal{P}^{\mathcal{R}^*}\right|}$ is a Markov matching mechanism.

A Markov matching mechanism $\phi^{\overline{s},\underline{s}} : S \to \{\textit{do-not-match}, \textit{match}\}^{\left|\mathcal{P}^{\mathcal{R}^*}\right|}$ is a *threshold matching mechanism with thresholds* $\overline{s}, \underline{s} \in S$ with $\underline{s} \leq 0$ and $\overline{s} \geq 0$, if for any W_1-$\mathrm{W}_2 \in \mathcal{P}^{\mathcal{R}^*}$,

$$\phi^{\overline{s},\underline{s}}_{\mathrm{W}_1\text{-}\mathrm{W}_2}(s) = \begin{cases} \textit{do-not-match} & \text{if } \underline{s}_{\mathrm{W}_1\text{-}\mathrm{W}_2} \leq s_{\mathrm{W}_1\text{-}\mathrm{W}_2} \leq \overline{s}_{\mathrm{W}_1\text{-}\mathrm{W}_2} \\ \textit{match} & \text{if } s_{\mathrm{W}_1\text{-}\mathrm{W}_2} < \underline{s}_{\mathrm{W}_1\text{-}\mathrm{W}_2} \text{ or } s_{\mathrm{W}_1\text{-}\mathrm{W}_2} > \overline{s}_{\mathrm{W}_1\text{-}\mathrm{W}_2} \end{cases},$$

where $\phi^{\overline{s},\underline{s}}(s) = \left(\phi^{\overline{s},\underline{s}}_{\mathrm{W}_1\text{-}\mathrm{W}_2}(s)\right)_{\mathrm{W}_1\text{-}\mathrm{W}_2 \in \mathcal{P}^{\mathcal{R}^*}}$.

When an overdemanded pair arrives, a threshold matching mechanism conducts the largest exchange that does not use existing W_1-W_2 or W_2-W_1 type pairs (*do-not-match* option) as long as the number of W_1-W_2 or W_2-W_1 type pairs is not greater than the threshold numbers, $\overline{s}_{\mathrm{W}_1\text{-}\mathrm{W}_2}$ and $\left|\underline{s}_{\mathrm{W}_1\text{-}\mathrm{W}_2}\right|$ respectively; otherwise it conducts the largest possible exchanges including the existing W_1-W_2 or W_2-W_1 type pairs (*match* option).

Our main theorem of this section is as follows:

Theorem 2 *Suppose Assumptions 1, 2, and 3 hold. There exist* $\overline{s}^*_{W_1\text{-}W_2} = 0$ *and* $\underline{s}^*_{W_1\text{-}W_2} \leq 0$, *or* $\overline{s}^*_{W_1\text{-}W_2} \geq 0$ *and* $\underline{s}^*_{W_1\text{-}W_2} = 0$ *for each* $W_1\text{-}W_2 \in \mathcal{P}^{\mathcal{R}^*}$ *such that* $\phi^{\overline{s}^*,\underline{s}^*}$ *is a dynamically efficient multi-way matching mechanism.*

The proof of Theorem 2 is in Appendix A. Through this theorem, we show that there exists a dynamically efficient matching mechanism, which is a special kind of a threshold mechanism. It stocks W_1-W_2 or W_2-W_1 type pairs, and does not use them in larger exchanges as long as the stock of the control group is less than or equal to $\overline{s}^*_{\mathrm{W}_1\text{-}\mathrm{W}_2}$ or $\left|\underline{s}^*_{\mathrm{W}_1\text{-}\mathrm{W}_2}\right|$, respectively. Either the number of W_1-W_2 type pairs or W_2-W_1 type pairs is the state variable, but not both. Under the first type of solution, the number of W_2-W_1 type pairs is the state variable. As long as the number of W_2-W_1 type pairs in the pool is zero, regardless of the number of W_1-W_2 type pairs, when the next arrival of an overdemanded pair occurs, the first type of efficient mechanism conducts the maximal size exchanges possible. If there are W_2-W_1 type pairs and their number does not exceed the threshold number $\left|\underline{s}^*_{\mathrm{W}_1\text{-}\mathrm{W}_2}\right|$, then these pairs are exclusively used to match incoming W_1-W_2 type pairs in two-way exchanges. On the other hand, if the number of W_2-W_1 type pairs exceeds the threshold number $\left|\underline{s}^*_{\mathrm{W}_1\text{-}\mathrm{W}_2}\right|$, they should be used in maximal exchanges which can be (1) a two-way exchange involving a W_1-W_2 type pair if the incoming pair type is W_1-W_2, or (2) a multi-way exchange involving an overdemanded pair in $\mathcal{P}^{\mathcal{O}}(\mathrm{W}_2\text{-}\mathrm{W}_1)$. The other types of maximal exchanges are conducted by the efficient mechanism as soon as they become feasible. The second possible solution is the symmetric version of the above mechanism taking the number of W_1-W_2 type pairs as a state variable.

Next, we specify the efficient mechanism more precisely.

Theorem 3 *Suppose Assumptions 1, 2, and 3 hold. Let* $W_1\text{-}W_2 \in \mathcal{P}^{\mathcal{R}^*}$. *Suppose that* $\phi^{\overline{s}^*,\underline{s}^*}$ *is an efficient multi-way matching mechanism.*

- *If* $p_{W_1\text{-}W_2} \geq p_{W_2\text{-}W_1}$, *that is: the* W_1-W_2 *type arrives at least as frequently as the* W_2-W_1 *type, and* $\sum_{X\text{-}Y \in \mathcal{P}^{\mathcal{O}}(W_1\text{-}W_2)} p_{X\text{-}Y} < \sum_{X\text{-}Y \in \mathcal{P}^{\mathcal{O}}(W_2\text{-}W_1)} p_{X\text{-}Y}$, *that is: the overdemanded types that can match* W_1-W_2 *type pairs in larger exchanges arrive less frequently than those for the* W_2-W_1 *type, then* $\underline{s}^*_{W_1\text{-}W_2} \leq 0$ *and* $\overline{s}^*_{W_1\text{-}W_2} = 0$.

- *If* $p_{W_1\text{-}W_2} = p_{W_2\text{-}W_1}$ *and* $\sum_{X\text{-}Y \in \mathcal{P}^{\mathcal{O}}(W_1\text{-}W_2)} p_{X\text{-}Y} = \sum_{X\text{-}Y \in \mathcal{P}^{\mathcal{O}}(W_2\text{-}W_1)} p_{X\text{-}Y}$, *then* $\underline{s}^*_{W_1\text{-}W_2} = 0$ *and* $\overline{s}^*_{W_1\text{-}W_2} = 0$, *i.e., maximal size exchanges are conducted whenever they become feasible.*

- *If* $p_{W_1\text{-}W_2} \leq p_{W_2\text{-}W_1}$, *and* $\sum\limits_{X\text{-}Y \in \mathcal{P}^{\mathcal{O}}(W_1\text{-}W_2)} p_{X\text{-}Y} > \sum\limits_{X\text{-}Y \in \mathcal{P}^{\mathcal{O}}(W_2\text{-}W_1)} p_{X\text{-}Y}$, *then* $\underline{s}^*_{W_1\text{-}W_2} = 0$ *and* $\overline{s}^*_{W_1\text{-}W_2} \geq 0$.

Proof of Theorem 3: Let Assumptions 1, 2, and 3 hold. Let $\mathrm{W_1\text{-}W_2} \in \mathcal{P}^{\mathcal{R}^*}$.

- Let $p_{\mathrm{W_1\text{-}W_2}} \geq p_{\mathrm{W_2\text{-}W_1}}$ and $\sum\limits_{\mathrm{X\text{-}Y} \in \mathcal{P}^{\mathcal{O}}(\mathrm{W_1\text{-}W_2})} p_{\mathrm{X\text{-}Y}} < \sum\limits_{\mathrm{X\text{-}Y} \in \mathcal{P}^{\mathcal{O}}(\mathrm{W_2\text{-}W_1})} p_{\mathrm{X\text{-}Y}}$. By Theorem 2, threshold mechanism $\phi^{\overline{s}^*,\underline{s}^*}$ is efficient. Hence, we have $\overline{s}^*_{\mathrm{W_1\text{-}W_2}} = 0$ and $\underline{s}^*_{\mathrm{W_1\text{-}W_2}} \leq 0$, or $\overline{s}^*_{\mathrm{W_1\text{-}W_2}} \geq 0$ and $\underline{s}^*_{\mathrm{W_1\text{-}W_2}} = 0$ such that $\phi^{\overline{s}^*,\underline{s}^*}$ is a dynamically efficient multi-way matching mechanism. If we conducted maximal number of exchanges at every state, there will be excess $\mathrm{W_1\text{-}W_2}$ type pairs on average remaining at the pool. Suppose $\underline{s}^*_{\mathrm{W_1\text{-}W_2}} > 0$. We will have even more excess $\mathrm{W_1\text{-}W_2}$ type pairs on average, since we do not always match them in larger exchanges. Therefore, the expected surplus under mechanism $\phi^{\left(\overline{s}^*_{-(\mathrm{W_1\text{-}W_2})},0\right),\left(\underline{s}^*_{-(\mathrm{W_1\text{-}W_2})},0\right)}$ is higher than under mechanism $\phi^{\overline{s}^*,\underline{s}^*}$, contradicting the claim that the latter one is efficient. Thus, $\overline{s}^*_{\mathrm{W_1\text{-}W_2}} = 0$. By Theorem 2, $\underline{s}^*_{\mathrm{W_1\text{-}W_2}} \leq 0$.

- Let $p_{\mathrm{W_1\text{-}W_2}} = p_{\mathrm{W_2\text{-}W_1}}$ and $\sum\limits_{\mathrm{X\text{-}Y} \in \mathcal{P}^{\mathcal{O}}(\mathrm{W_1\text{-}W_2})} p_{\mathrm{X\text{-}Y}} = \sum\limits_{\mathrm{X\text{-}Y} \in \mathcal{P}^{\mathcal{O}}(\mathrm{W_2\text{-}W_1})} p_{\mathrm{X\text{-}Y}}$. Then, the Bellman Equations stated in Equation 5 for positive states and in Equation 6 for negative states are completely symmetric, implying that $ES^*_{\mathrm{W_1\text{-}W_2}}(s_{\mathrm{W_1\text{-}W_2}}) = ES^*_{\mathrm{W_1\text{-}W_2}}(-s_{\mathrm{W_1\text{-}W_2}})$ for any $s_{\mathrm{W_1\text{-}W_2}} \in \mathbb{Z}$. Suppose that $\underline{s}^*_{\mathrm{W_1\text{-}W_2}} < 0$. Then at state component -1 regarding $\mathrm{W_1\text{–}W_2}$ types, the *do-not-match* option is executed. By Observation 3, we have $ES^*_{\mathrm{W_1\text{-}W_2}}(-1) \geq ES^*_{\mathrm{W_1\text{-}W_2}}(0)+1$. Then, $ES^*_{\mathrm{W_1\text{-}W_2}}(1) = ES^*_{\mathrm{W_1\text{-}W_2}}(-1) \geq ES^*_{\mathrm{W_1\text{-}W_2}}(0) + 1$ as well, implying together with Observation 3 that the *do-not-match* option is executed for $\mathrm{W_1\text{-}W_2}$ type pairs at state component 1, and that $\overline{s}^*_{\mathrm{W_1\text{-}W_2}} > 0$. However, $\underline{s}^*_{\mathrm{W_1\text{-}W_2}} < 0$ and $\overline{s}^*_{\mathrm{W_1\text{-}W_2}} > 0$ contradict Theorem 2. Therefore, $\underline{s}^*_{\mathrm{W_1\text{-}W_2}} = 0$. With the symmetric argument, we show that $\overline{s}^*_{\mathrm{W_1\text{-}W_2}} = 0$.

- Let $p_{\mathrm{W_1\text{-}W_2}} \leq p_{\mathrm{W_2\text{-}W_1}}$ and $\sum\limits_{\mathrm{X\text{-}Y} \in \mathcal{P}^{\mathcal{O}}(\mathrm{W_1\text{-}W_2})} p_{\mathrm{X\text{-}Y}} > \sum\limits_{\mathrm{X\text{-}Y} \in \mathcal{P}^{\mathcal{O}}(\mathrm{W_2\text{-}W_1})} p_{\mathrm{X\text{-}Y}}$. The symmetric argument of the first part of the proof holds. $\Diamond$

The intuition behind Theorem 3 can be stated as follows: $\mathrm{W_1\text{-}W_2}$ ($\mathrm{W_2\text{-}W_1}$) type pairs are most efficiently used in matching $\mathrm{W_2\text{-}W_1}$ ($\mathrm{W_1\text{-}W_2}$) type pairs in two-way exchanges. This is true because, by Proposition 2, these two reciprocally demanded pair types cannot be used to save any underdemanded type pairs. Moreover, the use of overdemanded pairs exclusively to save $\mathrm{W_1\text{-}W_2}$ and $\mathrm{W_2\text{-}W_1}$ type pairs is costly, since they can instead be used to save underdemanded pairs which are abundant. Consider a situation in which $p_{\mathrm{W_1\text{-}W_2}} \geq p_{\mathrm{W_2\text{-}W_1}}$ and $\sum\limits_{\mathrm{X\text{-}Y} \in \mathcal{P}^{\mathcal{O}}(\mathrm{W_1\text{-}W_2})} p_{\mathrm{X\text{-}Y}} < \sum\limits_{\mathrm{X\text{-}Y} \in \mathcal{P}^{\mathcal{O}}(\mathrm{W_2\text{-}W_1})} p_{\mathrm{X\text{-}Y}}$, that is: the types that can be used to serve $\mathrm{W_1\text{-}W_2}$ type pairs arrive less frequently than the types that can be used to serve $\mathrm{W_2\text{-}W_1}$ type pairs. Under these two conditions, *W_2-W_1 type pairs do not arrive as frequently as W_1-W_2 type pairs, and W_2-W_1 type pairs can be used more frequently in larger* *exchanges*. Consider a positive state component of the pool regarding W_1-W_2 and W_2-W_1 types, i.e., there are W_1-W_2 type pairs. Since W_2-W_1 type pairs arrive *on average* less frequently than W_1-W_2 type pairs for two-way exchanges, whenever an overdemanded pair that can serve a W_1-W_2 type pair arrives, the W_1-W_2 type pairs can safely be used in larger exchanges. On the other hand, if the state component of the pool regarding W_1-W_2 and W_2-W_1 types is negative, i.e., there are W_2-W_1 type pairs, since W_1-W_2 type pairs arrive *on average* more frequently for two-way exchanges, this means that the average arrival process is interrupted, and we end up with some excess W_2-W_1 type pairs. So, we have a higher option value for keeping W_2-W_1 type pairs than matching them in larger exchanges. We should have a positive stock of W_2-W_1 type pairs in hand to match them exclusively with future coming W_1-W_2 type pairs.[21] On the other hand, if $p_{W_1\text{-}W_2} = p_{W_2\text{-}W_1}$ and $\sum_{X\text{-}Y \in \mathcal{P}^{\mathcal{O}}(W_1\text{-}W_2)} p_{X\text{-}Y} = \sum_{X\text{-}Y \in \mathcal{P}^{\mathcal{O}}(W_2\text{-}W_1)} p_{X\text{-}Y}$, on average W_1-W_2 and W_2-W_1 types arrive at the same rate exclusively for two-way exchanges. Therefore, using existing W_1-W_2 or W_2-W_1 type pairs in larger exchanges instead of matching with incoming reciprocal pairs has no expected future costs. Thus, we do not need to worry about carrying a positive stock of W_1-W_2 or W_2-W_1 type pairs.

5 Dynamically Efficient Kidney Exchange

The kidney exchange problem is a special case of the general model that we considered above. In this problem, we refer to an object-agent pair as a recipient-donor pair. The type space for kidney needs is defined through blood- and tissue-type compatibility.

Before a donor is deemed compatible with a recipient, two tests are required: a blood-type compatibility test and a tissue-type compatibility test (or crossmatch test). There are four blood types, O, A, B, and AB. An O blood type recipient can only receive a transplant from an O donor, an A blood type recipient can only receive a transplant from an O or an A donor, a B blood type recipient can only receive a transplant from an O or a B donor, and an AB blood type recipient can receive a transplant from all donors. A recipient and a donor are *blood-type compatible* if the donor can feasibly donate a kidney to the recipient based on their blood types.

Observe that the blood-type compatibility relation forms a partial order with three levels of compatibility. The O blood type is located at the highest level at level 1, the A and B blood types are located at level 2, and the AB blood type is at level 3 (see Figure 4).

Yet there is another type of incompatibility for kidney recipients. Sometimes a recipient cannot receive a kidney from a blood-type compatible donor due to tissue-type incompatibility. There are 6 proteins on human DNA that determine the tissue type of a person. Some tissue types can be rejected by a recipient's immunological system. A formal test is done by mixing the blood of the donor and the recipient for testing tissue-type incompatibility prior to the transplant. If antibodies form in

[21] Since there is discounting and the number of W_2-W_1 type pairs can only be an integer, sometimes the threshold can be 0 instead of a positive number.

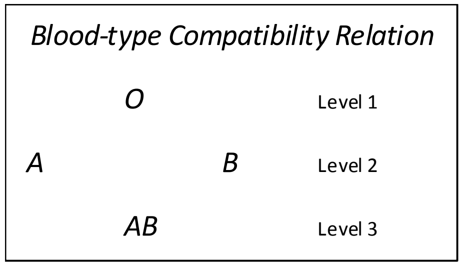


Figure 4: Blood-type Compatibility Relation (also see Figure 1(b))

the recipient's blood against the donor's tissue antigens then there is *positive crossmatch* between the recipient and the donor, meaning that the donor and the recipient are tissue-type incompatible. A donor is *tissue-type compatible* with a recipient if there is negative crossmatch between them. A donor is *compatible* with a recipient if he is both blood- and tissue-type compatible with the recipient. A recipient can receive a kidney only from compatible donors.

Usually, when a donor is compatible with his paired recipient, such a pair does not participate in exchange since the donor directly donates his kidney to the recipient. Hence, a blood-type compatible pair becomes available for exchange if and only if the pair is tissue-type incompatible. For tissue-type incompatibility between the donors and patients of different pairs, we will make an additional assumption. This will give us an idea on limits of kidney exchange (see Roth, Sönmez, and Ünver 2007):

Assumption 4 (Limit Assumption): No recipient is tissue-type incompatible with the donor of *another* pair.

Recipients can be tissue-type incompatible with their own donors, and we assume that $p_c > 0$ is the probability for that to happen. This ensures that blood-type compatible pairs arrive at the pool. On the other hand, recipients will never be tissue-type incompatible with donors of other pairs under Assumption 4; thus, two pairs will be mutually compatible if and only if they are mutually blood-type compatible. Note that average tissue-type incompatibility (positive crossmatch) probability is reported as $p_c = 0.11$ by Zenios, Woodle, and Ross (2001).

We will continue to maintain Assumptions 1,2, and 3 for kidney exchanges as well. In Section 5.2.1, we show that these assumptions are plausible for kidney exchange. We also comment on what happens when these assumptions are relaxed in Section 5.2.1 and in Appendix B.

5.1 The Efficient Kidney Exchange Mechanism

We are ready to present the pair type space for kidney exchanges. The blood types give the compatibility type of each recipient and donor (under Assumption 4). Therefore, the type of a pair is represented by the blood type of the recipient and the blood type of the donor in the pair. There

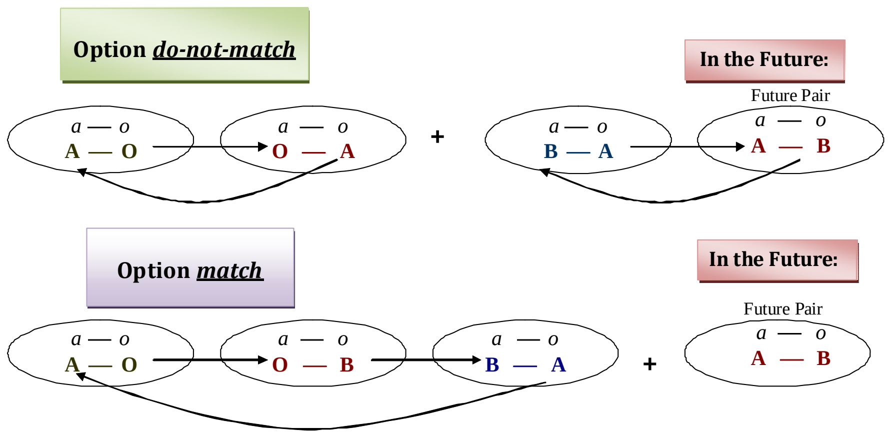


Figure 5: Kidney exchange options when an A-O$\in \mathcal{P}^{\mathcal{O}}$ (B-A) type pair arrives and there are B-A type pairs in the exchange pool. When an AB-B$\in \mathcal{P}^{\mathcal{O}}$ (B-A) type pair arrives, option *do-not-match* will involve a two-way exchange between AB-B and B-AB type pairs, and option *match* will involve a three-way exchange among AB-B, B-A, and A-AB type pairs.

are 16 pair types. We state the overdemanded, underdemanded, self-demanded, and reciprocally demanded types as follows:

$$\mathcal{P}^{\mathcal{O}} = \{\text{A-O, B-O, AB-O, AB-A, AB-B}\}$$

$$\mathcal{P}^{\mathcal{U}} = \{\text{O-A, O-B, O-AB, A-AB, B-AB}\}$$

$$\mathcal{P}^{\mathcal{S}} = \{\text{O-O, A-A, B-B, AB-AB}\}$$

$$\mathcal{P}^{\mathcal{R}} = \{\text{A-B, B-A}\}$$

Since there are only two reciprocally demanded types, we can represent the reduced Markov chain using a single integer s representing the number of A-B type pairs if $s > 0$ and the number of B-A type pairs if $s < 0$. Hence, the state space is the set of integers, i.e., $S = \mathbb{Z}$. In Figures 5 and 6, the exchange options and their trade-offs in the decision problem are depicted when there are B-A type pairs in the pool and an overdemanded pair in $\mathcal{P}^{\mathcal{O}}$ (B-A) $= \{$A-O,AB-B,AB-O$\}$ arrives. Also note that $\mathcal{P}^{\mathcal{O}}$ (A-B) $= \{$B-O,AB-A,AB-O$\}$. The options regarding the case with A-B type pairs are just the symmetric versions of the options with B-A type pairs.

By Theorem 2, under Assumptions 1, 2, 3, and 4, the efficient mechanism is given by a threshold Markov mechanism, $\phi^{\overline{s}^*,\underline{s}^*} : \mathbb{Z} \to \{\textit{do-not-match}, \ \textit{match}\}$, with $\overline{s}^* \geq 0$ and $\underline{s}^* = 0$ or $\overline{s}^* = 0$ and $\underline{s}^* \leq 0$.

The real-life arrival probabilities derived from unrelated recipient-donor matching for a pair (also used in the simulations section below) dictate $p_{\text{B-A}} \leq p_{\text{A-B}}$ and $\sum_{\text{X-Y} \in \mathcal{P}^{\mathcal{O}}(\text{A-B})} p_{\text{X-Y}} < \sum_{\text{X-Y} \in \mathcal{P}^{\mathcal{O}}(\text{B-A})} p_{\text{X-Y}}$. In this case, by Theorem 3, the efficient exchange mechanism is a threshold Markov mechanism

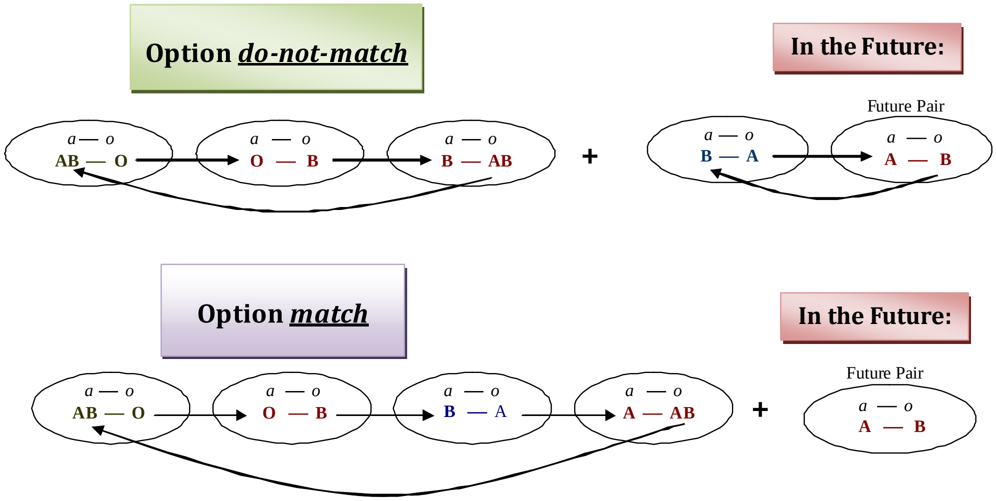


Figure 6: Kidney exchange options when an AB-O$\in \mathcal{P}^{\mathcal{O}}$ (B-A) type pair arrives and there are B-A type pairs in the exchange pool. Note that another feasible exchange for option *do-not-match* is a three-way exchange with AB-O, O-A, and A-AB type pairs.

that takes the B-A type pair number as the relevant state variable, i.e. mechanism $\phi^{0,\underline{s}^*}$ for some threshold $\underline{s}^* \leq 0$. Option *do-not-match* in Figures 5 and 6 is executed whenever there are $|s|$ B-A type pairs with $\underline{s}^* \leq s \leq 0$ and an overdemanded pair in $\mathcal{P}^{\mathcal{O}}$ (B-A) arrives. Under all other states and arrivals the maximal exchanges are conducted. For example, when B-A types are available and an overdemanded pair in $\mathcal{P}^{\mathcal{O}}$ (B-A) arrives, option *match* in Figures 5 and 6 is conducted.

5.2 Computation of the Efficient *Multi-Way* Kidney Exchange Mechanism

In this subsection, we first formulate the underlying arrival process of incompatible pairs to the exchange pool. For any pair type X-Y $\in \mathcal{P}$, let $q_{\text{X-Y}}$ be the probability of a random pair being of type X-Y. We refer to $q_{\text{X-Y}}$ as the **arrival probability** of pair type X-Y $\in \mathcal{T}$. We have $\sum_{\text{X-Y} \in \mathcal{T}} q_{\text{X-Y}} = 1$. A compatible pair does not become available for exchange. Recall that p_c, the probability of tissue-type incompatibility, is the probability of a blood-type compatible pair being available for exchange, and 1 is the probability of a blood-type incompatible pair being available for exchange. We derive the exchange arrival probabilities of each type X-Y, $p_{\text{X-Y}}$, as follows: For each self-demanded and overdemanded type X-Y$\in \mathcal{P}^{\mathcal{O}} \cup \mathcal{P}^{\mathcal{S}}$, we have $p_{\text{X-Y}} = \frac{p_c q_{\text{X-Y}}}{\kappa}$ and for each underdemanded and reciprocally demanded type X-Y$\in \mathcal{P}^{\mathcal{U}} \cup \mathcal{P}^{\mathcal{R}}$, we have $p_{\text{X-Y}} = \frac{q_{\text{X-Y}}}{\kappa}$ where $\kappa = \sum_{\text{X-Y} \in \mathcal{P}^{\mathcal{O}} \cup \mathcal{P}^{\mathcal{S}}} p_c q_{\text{X-Y}} + \sum_{\text{X-Y} \in \mathcal{P}^{\mathcal{U}} \cup \mathcal{P}^{\mathcal{R}}} q_{\text{X-Y}}$. Thus, $\sum_{\text{X-Y} \in \mathcal{P}} p_{\text{X-Y}} = 1$.

5.2.1 Plausibility of the Assumptions for Kidney Exchange

We comment on the efficient kidney exchange mechanism when Assumption 3 is relaxed in Appendix B, i.e., the case when self-demanded types participate in kidney exchange.

Next, we comment on the plausibility of our Assumptions 1, 2, and 4 for kidney exchange:

- Assumption 1 concerns the underlying arrival probabilities of the pairs and the applied matching mechanism. We first show that Assumption 1, i.e. having arbitrarily many underdemanded pairs in the exchange pool in the long run, is a realistic assumption:

Proposition 5 *Suppose that* $p_c\left(q_{AB\text{-}O}+q_{X\text{-}O}\right)+\min\left\{p_c q_{Y\text{-}O}, q_{X\text{-}Y}\right\}<q_{O\text{-}X}$ *for all* $\{X,Y\}=\{A,B\}$, $p_c\left(q_{AB\text{-}O}+q_{AB\text{-}X}\right)+\min\left\{p_c q_{AB\text{-}Y}, q_{Y\text{-}X}\right\}<q_{X\text{-}AB}$ *for all* $\{X,Y\}=\{A,B\}$ *and* $p_c q_{AB\text{-}O}<q_{O\text{-}AB}$. *Then, Assumption 1 holds in the long run regardless of the multi-way dynamic kidney exchange mechanism used.*

Proof of Proposition 5: By Proposition 2, underdemanded type O-A can only be matched in a two-way exchange with its reciprocal type, in an exchange using an AB-O type pair, or in an exchange using a B-O type pair and an A-B type pair such as (B-O, O-A, A-B) (see also Roth, Sönmez, and Ünver 2007). Since

$$p_c\left(q_{\text{AB-O}}+q_{\text{A-O}}\right)+\min\left\{p_c q_{\text{B-O}}, q_{\text{A-B}}\right\}<q_{\text{O-A}},$$

even if all of these types, O-A, AB-O, or B-O and A-B, are used exclusively to match O-A type pairs, there will still be arbitrarily many O-A pairs left in the pool in the long run.

Similarly, an underdemanded type B-AB pair can only be matched in a two-way exchange with its reciprocal type pair, an AB-O type pair, or in an exchange using an AB-A type pair and an A-B type pair such as (AB-A, A-B, B-AB) (see also Roth, Sönmez, and Ünver 2007). Since

$$p_c\left(q_{\text{AB-O}}+q_{\text{AB-B}}\right)+\min\left\{p_c q_{\text{AB-A}}, q_{\text{A-B}}\right\}<q_{\text{B-AB}},$$

even if all of AB-O, AB-B, or AB-A together with A-B type pairs are used exclusively to match B-AB type pairs, arbitrarily many B-AB type pairs will remain in the exchange pool in the long run.

Symmetric versions of these observations hold for O-B and AB-B.

By Proposition 2, an O-AB pair can only be matched using an AB-O pair. Since

$$p_c q_{\text{AB-O}}<q_{\text{O-AB}},$$

even if AB-O type pairs are used exclusively to match AB-O type pairs, arbitrarily many underdemanded type pairs will remain in the long run.

Moreover, all these results are true regardless of the matching mechanism used. $\Diamond$

The hypotheses of the above proposition is very mild, and will hold for sufficiently small tissue-type incompatibility (i.e. crossmatch) probability p_c. Moreover, they hold for real-life blood frequencies. For example, assuming that the recipient and her paired-donor are blood-unrelated, the arrival rates reported at the end of this subsection satisfy these assumptions, when the crossmatch probability is $p_c = 0.11$, as reported by Zenios, Woodle, and Ross (2000).

- Assumption 2 limits the arrival rates of A-B and B-A type pairs as $\frac{1}{2}q_{\text{B-A}} < q_{\text{A-B}} < 2q_{\text{B-A}}$. In reality, Terasaki, Gjertson, and Cecka (1998) report that the arrival rates of A-B and B-A blood types in the US population are $q_{\text{A-B}} = 0.05$ and $q_{\text{B-A}} = 0.03$. Actually, we would expect these numbers to be equal in many cases. Therefore, Assumption 2 is generic enough to cover most realistic arrival rates for kidney exchange.

- Assumption 4 is a limit assumption. However, it is also a somewhat realistic assumption for certain cases. If the recipient is female and has previously borne the child of her paired-donor, then her body is more likely to reject her paired-donor's kidney than other random kidneys due to tissue-type incompatibility (cf. Zenios, Woodle, and Ross 2000).

If we eliminate these assumptions, the structure of the dynamic programming problem will change. Even then, we can create object types and their compatibility relation as a general partial order. Hence, we can classify the pair types as underdemanded, overdemanded, self-demanded, and reciprocally demanded. Using results in dynamic programming, it can be shown that an efficient deterministic Markov mechanism exists.

5.2.2 Computational Approximation for the Efficient Kidney Exchange Mechanism

The threshold values of the efficient multi-way kidney exchange mechanism cannot be solved analytically. In this section, using different parameters for $\frac{\lambda}{\lambda+\varrho}$ and using p_c, $\{q_{\text{X-Y}}\}_{\text{X-Y}\in\mathcal{P}}$ values reported in the medical literature, we numerically compute the efficient matching mechanism's threshold values under Assumptions 1, 2, 3, and 4. Then we use these threshold values to approximate the thresholds when Assumption 3 does not hold and self-demanded types can participate in exchange (see Remark 1 in Appendix B), for subsequent policy simulations.

The algorithm used to compute the efficient threshold values uses Theorem 2 and the value iteration formula in Theorem 4 in Appendix A (see page 259 of Puterman 1994 for a one-sided threshold version of the algorithm). The algorithm makes a 6001 state approximation (3000 negative and 3000 positive states, and state 0) of the infinite countable state space. The crossmatch probability is reported as $p_c = 0.11$ by Zenios, Woodle, and Ross (2001). We also use this number. The medical literature is not precise about the arrival probabilities of the pairs. We have chosen the following way to construct these probabilities: The blood type frequencies of people are widely reported for the US population as follows: For O blood type $q_{\text{O}} = 0.45$, for A blood type $q_{\text{A}} = 0.40$, for B blood type $q_{\text{B}} =$

$\lvert \underline{s}^* \rvert$	$\frac{\lambda}{\lambda+\rho}$								
	0.999995	**0.99999**	**0.99995**	**0.9999**	**0.9995**	**0.999**	**0.995**	**0.99**	**0.95**
$\mathbf{q_{B\text{-}A} = q_{A\text{-}B} = 0.044}$	2	2	2	2	1	1	0	0	0
$\mathbf{q_{B\text{-}A} = 0.039}$ **and** $\mathbf{q_{A\text{-}B} = 0.049}$	292	161	46	28	9	6	2	1	0
$\mathbf{q_{B\text{-}A} = 0.034}$ **and** $\mathbf{q_{A\text{-}B} = 0.054}$	2773	1387	278	147	30	15	3	2	0

Table 1: The threshold number of B-A type pairs for conducting smaller exchanges in the optimal rule.

0.11, and for AB blood type $q_{\text{AB}} = 0.04$.[22] We assume that the pairs are blood type unrelated (such as spouses), and hence the donor and recipient blood types are independently distributed. That is, for any X-Y$\in \mathcal{P}$, we have $q_{\text{X-Y}} = q_{\text{X}} q_{\text{Y}}$.

We also compute the mechanism when $q_{\text{A-B}}$ and $q_{\text{B-A}}$ are not equal to each other. Terasaki, Gjertson, and Cecka (1998) report that in the US, A-B and B-A blood type pairs do not arrive at the same frequency. They report that $q_{\text{A-B}} = 0.05$ and $q_{\text{B-A}} = 0.03$. In our computation, additionally, we use two different probability pairs: (1) $q_{\text{A-B}} = 0.049$ and $q_{\text{B-A}} = 0.039$ and (2) $q_{\text{A-B}} = 0.054$ and $q_{\text{B-A}} = 0.034$.

Since we assume in our initial efficient mechanism derivation that self-demanded types are not included, we find the conditional probabilities using the above formula such that arriving pairs are not self-demanded.

Since we do not have a clear prediction of the values of ρ and λ, we use different values in the computation. In particular, we choose different values for $\frac{\lambda}{\lambda+\rho}$ and derive the efficient mechanism. The set of the values we use for $\frac{\lambda}{\lambda+\rho}$ is given as[23]

$$\{0.999995, 0.99999, 0.99995, 0.9999, 0.9995, 0, 999, 0.995, 0.99, 0.95\}.$$

First note that in all cases $\underline{s}^* \leq 0$ and $\overline{s}^* = 0$. We report the threshold value for the stock of B-A type pairs to conduct the smaller exchanges, $|\underline{s}^*|$ in Table 1. For the number of B-A type pairs in the pool larger than $|\underline{s}^*|$, the efficient mechanism requires the largest exchanges.

We observe that, under the most plausible $\frac{\lambda}{\lambda+\rho}$ values,

$$\{0.999995, 0.99999, 0.99999, 0.99995, 0.9999, 0.9995, 0, 999\},$$

the efficient mechanism requires a positive stock of B-A blood type pairs before conducting the largest possible exchanges. Only an unrealistic value such as $\frac{\lambda}{\lambda+\rho}$ =0.995 and lower (requiring only 10 pairs

[22] For example, see the web-page of the Association of American Blood Banks, `http://www.aabb.org` , retrieved on 2/27/2007.

[23] For example, $\frac{\lambda}{\lambda+\rho} = 0.999995$ can be generated by $\lambda = 10000$ and $\rho = 0.05$, which corresponds to 10000 pairs arriving per year and an annual discount rate of 5%. On the other hand, $\frac{\lambda}{\lambda+\rho} = 0.9999$ can be generated by $\lambda = 10000$ and $\rho = 0.10$. We can roughly assume that the discount rate is 5%-10%. Expectations for λ nationwide is around 10000, given that the annual number of conducted live kidney donations is in the 6000-7000 range in the last few years. The lower values for $\frac{\lambda}{\lambda+\rho}$ can also be expected in regional smaller programs. For example, $\frac{\lambda}{\lambda+\rho} = 0.999$ can be generated through $\lambda = 50$ and $\rho = 0.05$.

1 year exchange surplus taking two-way regime numéraire	**$\lambda = 10000$ and $\rho = 0.05$** **$q_{\text{B-A}} = 0.034$ and $q_{\text{A-B}} = 0.054$**
Regime 1: Multi-way	106.59 (0.0004694)
Regime 2: Two-way	100 (0.0004325)

Table 2: Policy simulations for 1 year normalized exchange surplus taking two-way regime numaraire starting from the null state under the two regimes, Regime 1: the one threshold approximation of the optimal rule, Regime 2: The optimal two-way matching rule.

arriving per year at 5% discounting for ρ) may require conducting the largest possible exchanges all the time. We also observe that the threshold value increases with increasing $(q_{\text{A-B}} - q_{\text{B-A}})$ and increasing $\frac{\lambda}{\lambda+\rho}$. For example, when $\frac{\lambda}{\lambda+\rho} = 0.999995$ (with an average of 10000 compatible and incompatible pairs arriving per year and an annual discount rate $\rho = 5\%$), $q_{\text{B-A}} = 0.034$ and $q_{\text{A-B}} = 0.054$, the largest exchanges involving B-A type pairs will be conducted if and only if there are more than 2773 B-A type pairs in the pool.

5.2.3 Simulations on Expected Exchange Surplus

In this subsection, we relax Assumption 3 again, and assume that self-demanded types can participate in exchange. We compute the expected one-year exchange surplus (normalized by the two-way exchange regime) at null state (having no A-B or B-A type pairs, and no self-demanded type pairs) for two different matching mechanisms. We use

- **Regime 1:** The single-state variable, one threshold approximation of the dynamically efficient multi-way matching mechanism, $\hat{\phi}^{\overline{s}^*,\underline{s}^*}$ (cf. Appendix B).
- **Regime 2:** Dynamically efficient two-way matching mechanism ν.

We used the following technique in our simulation. We roughly calibrated our parameters using the US data. We chose λ =10000 (given that 6570 live donor transplants are conducted per year, we assumed that 10000 pairs arrive per year) and $\rho = 0.05$ (that is: the discount rate is 5%) with $\frac{\lambda}{\lambda+\rho}$ =0.999995. We set $q_{\text{B-A}} = 0.034$ and $q_{\text{A-B}} = 0.054$, as reported by Terasaki, Gjertson, and Cecka (1998). We assumed that there were arbitrarily many underdemanded pairs (Assumption 1) and that there was no tissue-type incompatibility between two different pairs (Assumption 4). We simulated the pool for the next arriving 10000 pairs (approximately for 1 year) and calculated the normalized surplus raised taking the two-way exchange as numéraire. We ran this simulation 5000 times. The averages and standard errors of the averages for the 1 year normalized exchange surplus were taken over these 5000 markets (see Table 2). We observed that the efficient multi-way exchange raises about 6.6% more expected surplus than the efficient two-way exchange. This difference is significant at the 1% level using a z-test.

Number of pairs matched in 1 year	$\lambda =$ **10000 and** $\rho =$ **0.05**
	$\mathbf{q_{B\text{-}A}} =$ **0.034 and** $\mathbf{q_{A\text{-}B}} =$ **0.054**
Regime 1: Multi-way	1791.51 (0.79)
Regime 2:Two-way	1680.77 (0.73)
Static: Multi-way	1794.39 (0.80)
Static: Two-Way	1680.77 (0.73)

Table 3: Policy simulations for number of pairs matched in exchanges starting from the null state under the two regimes. With $\lambda = 10000$, on average 4267 pairs enter the exchange. On average, 6733 pairs are compatible and their recipients receive a transplant immediately from their own donors.

We report the number of pairs matched in one year under the two different regimes. We also report the number of pairs that could have been matched if all exchanges were run at the end of the year in a static population. Under the efficient multi-way mechanism, about 6.7% more pairs are matched than the efficient two-way mechanism. This difference is significant at the 1% level using a z-test. Even though the efficient exchange is conducted dynamically, the numbers of pairs matched are close to the maximal possible number (see Table 3). This is an expected result. By Proposition 1, we know that the efficient two-way matching mechanism matches the maximum number of pairs possible, and this table shows that. On the other hand, under the efficient multi-way mechanism, this observation may not be true, since some B-A or A-B type pairs may remain in the pool at the end of the year, and those could have been matched in a static exchange run at the end of the year.[24]

We also report the number of exchanges of different sizes under Regime 1: Efficient Unrestricted Exchange in Table 4 (in approximately 1 year). These are the average numbers found in the above simulation. We observe that the majority of the exchanges are 2-way exchanges, though we observe a substantial number of 3-way and 4-way exchanges. The numbers of larger exchanges are substantially less (less than 2% of all exchanges). We observe that 67% of all exchanges conducted are two-way, 25% are three-way, and only 7% are four-way exchanges. Therefore, the efficient multi-way mechanism does not create a large burden in terms of large exchanges.

6 Conclusions

Having a partial order compatibility structure (which is not a linear order) is the necessary requirement for multi-way dynamically efficient mechanisms having state-dependent features and being different from statically efficient mechanisms. We use a minimal partial order structure to derive dynamically efficient exchange mechanisms in a general exchange model.

We observe three important properties of dynamically efficient mechanisms. They (for both two-

[24]Note that this is only an artifact of our simulation environment, which is terminated after one year. Since the time horizon in reality is infinite, surely these remaining pairs will be matched in the next year.

Number of exchanges in Regime 1: Unrestricted	$\lambda =$ **10000 and** $\rho =$ **0.05** $\mathbf{q_{B\text{-}A}} =$ **0.034 and** $\mathbf{q_{A\text{-}B}} =$ **0.054**
2-way	492.62 (0.32)
3-way	180.51 (0.17)
4-way	49.04 (0.096)
5-way	11.36 (0.047)
6-way	1.83 (0.019)
7-way	0.11 (0.0047)
8-way	0.0016 (0.00056)

Table 4: Average number of exchanges of different sizes in Regime 1.

way and multi-way matching) are not affected by the magnitude of the unit waiting cost c. They conduct at most one exchange at a time. Moreover, whenever an exchange becomes feasible, they conduct it immediately.

In a static setting, Roth, Sönmez, and Ünver (2007) showed that n-way exchanges usually suffice to obtain all benefits from an exchange domain with n object types under a partial order compatibility relation and mild assumptions. In our study, for kidney exchanges, when self-demanded type pairs participate in exchange, the largest possible exchange size is 8 instead of 4 as predicted by the above result, since in a dynamic setting some of the assumptions of the above study do not hold. In the simulations conducted, we showed that exchanges larger than 4-way are extremely rare in a dynamic setting.

The policy simulations show that the threshold values of the efficient kidney exchange mechanism are quite sensitive to the changes in arrival probabilities of A-B and B-A type pairs. Therefore, for our mechanism to have a realistic application, the health authority should measure these arrival rates, precisely.

A final note about incentive properties of dynamically efficient mechanisms will be useful. We can refine the definition of efficient mechanisms as follows: If an X-Y type pair is going to be matched in an exchange and there are multiple X-Y type pairs available in the pool, then the mechanism selects the earliest arriving pair. Suppose that a pair of type X-Y$\in \mathcal{P}$ can manipulate its type and announce it as W-Z$\in \mathcal{P}$ with W$\blacktriangleright$X and Y$\blacktriangleright$Z.[25] It is easy to show that announcing X-Y is the weakly dominant strategy for the pair, i.e. the mechanisms are *strategy-proof*.[26] Entry timing can be another strategic tool. Suppose that each pair, after becoming available, can delay its entry to the pool as a strategic variable. In this case, the above dynamically efficient two-way and multi-way matching mechanisms

[25]Observe that announcing Y$\not\blacktriangleright$Z or W$\not\blacktriangleright$X may result with individually irrational exchanges. Hence, we assume that such manipulations are not possible.

[26]In the context of kidney exchange, since blood types exclusively determine the compatibility between a recipient and a donor of another pair, and since manipulating blood types is extremely difficult, it is almost impossible for a pair to use "compatibility" as a strategic tool to manipulate the dynamic system.

are *delay-proof,* i.e. no pair will benefit by delaying its entry to the pool.

A Appendix: Proofs of Results

Proof of Proposition 2: Suppose that pair i of type X-Y$\in \mathcal{P}^{\mathcal{O}}$ is the only overdemanded pair in the pool, $j \neq i$ is a type Z$_1$-Z$_2 \in \mathcal{P}$ pair in the pool, and $E = (j, j_1, ..., j_k)$ is an exchange that matches pair j. Let A$^\ell$-O$^\ell$ be the type of each pair j_ℓ in E. We have Z$_2$ $\blacktriangleright$A^1, O$^\ell$ $\blacktriangleright$A$^{\ell+1}$ for all $\ell \in \{1, ..., k-1\}$, and A^k $\blacktriangleright$Z$_1$. Two cases are possible:

- Type of $j \in \mathcal{P}^{\mathcal{U}}$: Since Z$_1$ $\blacktriangleright$Z$_2$ and Z$_2$ $\neq$Z$_1$, by acyclicity of $\blacktriangleright$, there exists one pair j_ℓ with O$^\ell$ $\blacktriangleright$A$^\ell$ and O$^\ell$ $\neq$A$^\ell$, i.e., there exists an overdemanded pair $j_\ell \in E$. Since i is the single overdemanded pair in the pool, $j_\ell = i$. By transitivity of $\blacktriangleright$ and by the fact that there are at most two object types at a compatibility level of $\blacktriangleright$, we have (1a) Z$_2$ $\blacktriangleright$X or (1b) Z$_2$ $\not\blacktriangleright$X and X$\not\blacktriangleright$Z$_2$, and yet there exists some $j_m \in E$ with $m < \ell$ such that j_m is of type Z$_2$-X (so that E is individually rational). Similarly, we have (2a) Y$\blacktriangleright$Z$_1$ or (2b) Z$_1$ $\not\blacktriangleright$Y and Y$\not\blacktriangleright$Z$_1$, and yet there exists some $j_m \in E$ with $m > \ell$ such that j_m is of type Y-Z$_1$.[27] This proves necessity.

  - If (1a) and (2a) are satisfied, we can always choose E (in terms of types of pairs) as (Z$_1$-Z$_2$, X-Y) .
  - If (1b) and (2a) are satisfied, we can choose E as (Z$_1$-Z$_2$, Z$_2$-X, X-Y) .
  - If (1a) and (2b) are satisfied, we can choose E as (Z$_1$-Z$_2$, X-Y, Y-Z$_1$) .
  - If (1b) and (2b) are satisfied, we can choose E as (Z$_1$-Z$_2$, Z$_2$-X, X-Y, Y-Z$_1$) .

  These prove sufficiency.

- Type of $j \in \mathcal{P}^{\mathcal{S}}$: Since Z$_2$ =Z$_1$, when there is another pair h of type Z$_1$-Z$_1$, a two-way exchange consisting of types (Z$_1$-Z$_1$, Z$_1$-Z$_1$) is an individually rational exchange, and E could be chosen using these types. Suppose that there is no other pair of type Z$_1$-Z$_1$ in the pool. Then by acyclicity of $\blacktriangleright$, there exists some pair j_ℓ with O$^\ell$ $\blacktriangleright$A$^\ell$ with O$^\ell$ $\neq$A$^\ell$, i.e., there exists an overdemanded pair $j_\ell \in E$. Since i is the single overdemanded pair in the pool, $j_\ell = i$. By transitivity of $\blacktriangleright$ and by the fact that there are at most two object types at a compatibility level of $\blacktriangleright$, we have (1a) Z$_1$ $\blacktriangleright$X or (1b) Z$_1$ $\not\blacktriangleright$X and X$\not\blacktriangleright$Z$_1$, and yet there exists some $j_m \in E$ with $m < \ell$ such that j_m is of type Z$_1$-X (so that E is individually rational). Similarly, we have (2a) Y$\blacktriangleright$Z$_1$ or (2b) Z$_1$ $\not\blacktriangleright$Y and Y$\not\blacktriangleright$Z$_1$, and yet there exists some $j_m \in E$ with $m > \ell$ such that j_m is of type Y-Z$_1$. These prove necessity.

[27] Equivalently, we could have written Condition (1) as "X$\not\blacktriangleright$Z$_2$ and if Z$_2$ $\not\blacktriangleright$X then there exists some $j_m \in E$ with $m < \ell$ such that j_m is of type Z$_2$-X " as in the hypothesis of the proposition. A similar equivalence is also valid for Condition (2). Thus, these are equivalent to the conditions given in the hypothesis of the proposition.

– If (1a) and (2a) are satisfied, we can always choose E (in terms of types of pairs) as (Z_1-Z_1, X-Y).

– If (1b) and (2a) are satisfied, we can choose E as (Z_1-Z_1, Z_1-X, X-Y).

– If (1a) and (2b) are satisfied, we can choose E as (Z_1-Z_1, X-Y, Y-Z_1).

– If (1b) and (2b) are satisfied, we can choose E as (Z_1-Z_1, Z_1-X, X-Y, Y-Z_1).

These prove sufficiency.

- Type of $j \in \mathcal{P}^{\mathcal{R}}$: We have $Z_2 \not\blacktriangleright Z_1$ and $Z_1 \not\blacktriangleright Z_2$. Suppose there is a pair h of type Z_2-Z_1. Then a two-way exchange consisting of types (Z_1-Z_2, Z_2-Z_1) is an individually rational exchange, and E could be chosen using these types. Suppose that there is no pair of type Z_2-Z_1 in the pool. Then by acyclicity of $\blacktriangleright$, there exists some pair j_ℓ with $O^\ell \blacktriangleright A^\ell$ with $O^\ell \neq A^\ell$, i.e., there exists an overdemanded pair $j_\ell \in E$. Since i is the single overdemanded pair in the pool, $j_\ell = i$. By the fact that there are only two types in the compatibility levels of Z_1 and Z_2, there are no posssible reciprocal types other than Z_1-Z_2 and Z_2-Z_1. Since Z_2-Z_1 type pair does not exist, there is no other pair of the same compatibility level with in Z_1-Z_2 in the exchange E,and by acyclicity we have $Z_2 \blacktriangleright X$ and $Y \blacktriangleright Z_1$ i.e., Z-Y and Z_1-Z_2 are mutually compatible types, proving necessity.

  The types of pairs in E can be chosen as (Z_1-Z_2, X-Y) whenever $Z_2 \blacktriangleright X$ and $Y \blacktriangleright Z_1$, proving sufficiency.

$\Diamond$

Proposition 6 ***(Maximal Exchange Composition Using Overdemanded Types)*** *Under Assumption 1, suppose that X-Y$\in \mathcal{P}^{\mathcal{O}}$ is the type of an overdemanded pair that arrives at the exchange pool. Then, we can conduct an $(n+k+\ell+1)$-way exchange serving*

- *the overdemanded pair of type X-Y;*
- *a maximum of $n = L_X - L_Y$ underdemanded pairs;*
- *one pair from each of the distinct reciprocally demanded types W_1-W_2, W_3-$W_4, \ldots, W_{2k-1}$-$W_{2k} \in \mathcal{P}^{\mathcal{R}}$ such that $W_1, W_2 \blacktriangleright W_3, W_4 \blacktriangleright \ldots \blacktriangleright W_{2k-1}, W_{2k}$ (i.e., these pair types are not reciprocal of another and are ordered according to their compatibility levels), $Y \blacktriangleright W_1$ and $W_{2k} \blacktriangleright X$.*
- *one pair from each of the distinct self-demanded types V_1-$V_1, \ldots, V_\ell$-$V_\ell \in \mathcal{P}^{\mathcal{S}}$ such that (1) $V_1 \not\blacktriangleleft V_2 \not\blacktriangleleft \ldots \not\blacktriangleleft V_\ell$, (2) $Y \blacktriangleright V_1$ or W_1-W_2=Y-V_1, (3) $V_\ell \blacktriangleright X$ or W_{2k-1}-W_{2k} =V_ℓ-X, and (4) if there exists some $d \in \{1, \ldots, \ell-1\}$ such that V_d and V_{d+1} are at the same level then there exists some index $c_d \in \{1, \ldots, k\}$ such that W_{2c_d-1}-W_{2c_d} =V_d-V_{d+1};*

whenever such reciprocally demanded and self-demanded pairs exist in the pool.

Proof of Proposition 6: Suppose the hypothesis of the proposition holds. For notational purposes, let $\mathrm{Z}_{-1}=\mathrm{X}$, $\mathrm{Z}_0=\mathrm{Y}$, $\mathrm{Z}_{2n+1}=\mathrm{X}$, and $\mathrm{Z}_{2n+2}=\mathrm{Y}$. Under Assumption 1, there exists n underdemanded pairs belonging to pair types

$$\mathrm{Z}_1\text{-}\mathrm{Z}_2,\ \mathrm{Z}_3\text{-}\mathrm{Z}_4, ..., \mathrm{Z}_{2n-1}\text{-}\mathrm{Z}_{2n} \in \mathcal{P}^{\mathcal{U}}$$

such that

- for each $m \in \{1, 2..., n\}$, there are m levels between Z_{2m} and Y, that is $L_{\mathrm{Z}_{2m}} = L_{\mathrm{Y}} + m$,
- for each $c \in \{1, 2..., k\}$, there exists an index $m_c \in \{1, 2..., n\}$ such that $\mathrm{Z}_{2m_c}=\mathrm{W}_{2c-1}$ and $\mathrm{Z}_{2m_c+1}=\mathrm{W}_{2c}$,
- for each $d \in \{1, 2..., \ell\}$ with $\mathrm{V}_{d-1} \blacktriangleright \mathrm{V}_d$ (whenever V_{d-1} exists) and $\mathrm{V}_d \blacktriangleright \mathrm{V}_{d+1}$ (whenever V_{d+1} exists), there exists an index $m'_d \in \{1, 2..., k\}$ such that $\mathrm{Z}_{2m'_d}=\mathrm{V}_d$, and
- for each $m \in \{1, 2..., n-1\} \setminus \{m_1, ..., m_k\}$, $\mathrm{Z}_{2m}=\mathrm{Z}_{2m+1}$.

Thus, the exchange E' consisting of pairs belonging to pair types

$$\left(\begin{array}{l}\mathrm{X}\text{-}\mathrm{Y},\ \underbrace{\mathrm{Z}_1}_{=\mathrm{Y}}\text{-}\mathrm{Z}_2,\ \underbrace{\mathrm{Z}_3}_{=\mathrm{Z}_2}\text{-}\mathrm{Z}_4, ..., \mathrm{Z}_{2m_c-1}\text{-}\underbrace{\mathrm{Z}_{2m_c}}_{=\mathrm{W}_{2c-1}},\ \mathrm{W}_{2c-1}\text{-}\mathrm{W}_{2c},\ \underbrace{\mathrm{Z}_{2m_c+1}}_{=\mathrm{W}_{2c}}\text{-}\mathrm{Z}_{2m_c+2}, ...,\ \underbrace{\mathrm{Z}_{2n-1}}_{=\mathrm{Z}_{2n-2}}\text{-}\underbrace{\mathrm{Z}_{2n}}_{=\mathrm{X}}\\ \qquad\qquad \text{for all } c = 1, 2, ..., k.\end{array}\right)$$

is a feasible $(k+n+1)$-way exchange.

We can enlarge exchange E' by inserting the given self-demanded type pairs in order to obtain exchange E'' as follows:

- Recall that for each $d \in \{1, ..., \ell-1\}$ with same level V_{d+1} and V_d, there exists some index $c_d \in \{1, ..., k\}$ such that $\mathrm{W}_{2c_d-1}\text{-}\mathrm{W}_{2c_d}=\mathrm{V}_d\text{-}\mathrm{V}_{d+1}$. In the exchange E' above, we can insert
  - the $\mathrm{V}_d\text{-}\mathrm{V}_d$ type pair between the $\mathrm{Z}_{2m_{c_d}-1}\text{-}\underbrace{\mathrm{Z}_{2m_{c_d}}}_{=\mathrm{W}_{2c_d-1}}$ type pair, and $\mathrm{W}_{2c_d-1}\text{-}\mathrm{W}_{2c_d}$ type reciprocally demanded pair and
  - the $\mathrm{V}_{d+1}\text{-}\mathrm{V}_{d+1}$ type pair between the $\mathrm{W}_{2c_d-1}\text{-}\mathrm{W}_{2c_d}$ type reciprocally demanded pair and the $\underbrace{\mathrm{Z}_{2m_{c_d}+1}}_{=\mathrm{W}_{2c_d}}\text{-}\mathrm{Z}_{2m_{c_d}+2}$ type pair.
- For each $d \in \{1, ..., \ell\}$ with $\mathrm{V}_{d-1} \blacktriangleright \mathrm{V}_d$ (whenever V_{d-1} exists) and $\mathrm{V}_d \blacktriangleright \mathrm{V}_{d+1}$ (whenever V_{d+1} exists), since the object type $\mathrm{Z}_{2m'_d}$ is chosen as $\mathrm{Z}_{2m'_d}=\mathrm{V}_d$, we can insert the $\mathrm{V}_d\text{-}\mathrm{V}_d$ type self-demanded pair, in exchange E', between the pairs $\mathrm{Z}_{2m'_d-1}\text{-}\mathrm{Z}_{2m'_d}$ and $\underbrace{\mathrm{Z}_{2m'_d+1}}_{=\mathrm{Z}_{2m'_d}}\text{-}\mathrm{Z}_{2m'_d+2}$.

Thus, the newly formed exchange E'' serves all of the $n + k + \ell + 1$ pairs including the ones given in the hypothesis of the proposition. $\Diamond$

Proof of Proposition 3: Suppose Assumption 1 holds and for all W$_1$-W$_2$, W$_2$-W$_1$ $\in \mathcal{P}^{\mathcal{R}}$, $2p_{\text{W}_1\text{-W}_2} > p_{\text{W}_2-\text{W}_1}$ and $2p_{\text{W}_2\text{-W}_1} > p_{\text{W}_1-\text{W}_2}$. Let an overdemanded pair i of type X-Y $\in \mathcal{P}^{\mathcal{O}}$ arrive at the exchange pool. We will show that the opportunity cost of holding onto the X-Y type pair and underdemanded pairs, which could be matched immediately, with the expectation of creating a larger exchange in the future is larger than any alternative decision.

First note that pair i will not be used in matching an underdemanded pair that will arrive in the future. Since all underdemanded pairs exist abundantly by Assumption 1, by Corollary 1 we can use it to match $L_{\text{X}} - L_{\text{Y}}$ underdemanded pairs in an exchange E immediately. Moreover suppose that we can match in total n pairs in this exchange (possibly including some reciprocally demanded and self-demanded pairs). In the future, $L_{\text{X}} - L_{\text{Y}}$ is the most underdemanded pairs we can match through pair i, thus we do not hold onto i to match future underdemanded pairs.

We will show that pair i will not be used in matching a self-demanded type pair that will arrive in the future, either. Suppose that V-V is a self-demanded type and a pair of this type can be inserted in exchange E (see Proposition 6 and its proof). Hence, if pair i is used to wait for a V-V type pair to arrive, n pairs in exchange E will wait until the V-V type pair arrives, instead of being matched immediately. A V-V type pair can be matched in several ways: It can be matched with another V-V type pair in a two-way exchange. Or it can be inserted in other exchanges between two pairs such that the object of the first pair is compatible with V and the requirement of the agent of the second pair is also compatible with V. Consider the case in which we match it exclusively with a future V-V type pair j. For the same expected duration that pairs in exchange E wait for pair j to arrive, pair j will wait until the next V-V type pair arrives. Thus, the cost of this exchange is making a future V-V type pair j wait for the same expected duration for a new V-V type pair. This second alternative is less costly than making n (which is larger than 1) pairs of the exchange E wait for the same expected duration; therefore an X-Y type pair i will not be used to match an expected V-V type pair in the future.

Next, we show that pair i will not be used in matching a reciprocally demanded type pair that will arrive in the future.

Suppose that instead of exchange E, we use the type X-Y pair to match one reciprocally demanded pair k of type W$_1$-W$_2$ that will arrive in the future. By Corollary 1, we can match $n+1$ pairs (including k) immediately when k arrives, if we do not conduct exchange E now. Moreover, by Proposition 2, pairs in E (other than pair i) cannot be matched without i. Thus, suppose that we would like to use pair i to serve also this first pair k which will arrive in the future. This causes the exchange E not to be conducted immediately and forces n pairs to wait until pair k arrives. The W$_1$-W$_2$ type pair arrives with a Poisson process with frequency $\lambda p_{\text{W}_1\text{-W}_2}$. If the next W$_1$-W$_2$ type pair arrives at time τ_1, the cost of holding onto the X-Y type pair i and the other $n-1$ pairs in E is

$\int_0^{\tau_1} nce^{-\rho t}dt = \frac{nc}{\rho}\left(1 - e^{-\rho\tau_1}\right)$. Since the exponential $\lambda p_{\text{W}_1\text{-W}_2} e^{-\lambda p_{\text{W}_1\text{-W}_2}\tau_1}$ is the probability density function for the next pair's arrival time τ_1, the expected cost is given by

$$C_1 = E\left[\frac{nc}{\rho}\left(1 - e^{-\rho\tau_1}\right)\right] = \int_0^{\infty} \frac{nc}{\rho}\left(1 - e^{-\rho\tau_1}\right)\lambda p_{\text{W}_1\text{-W}_2} e^{-\lambda p_{\text{W}_1\text{-W}_2}\tau_1} d\tau_1 = \frac{nc}{\lambda p_{\text{W}_1\text{-W}_2} + \rho}.$$

On the other hand, an alternative for the next arriving W_1-W_2 type pair k (other than being matched with pair i in an $(n+1)$-way exchange) is to be matched exclusively with a reciprocal W_2-W_1 type pair that is not currently in the pool, but will arrive in the future The expected discounting until the W_1-W_2 type pair k arrival is

$$E\left[e^{-\rho\tau_1}\right] = \int_0^{\infty} e^{-\rho\tau_1}\lambda p_{\text{W}_1\text{-W}_2} e^{-\lambda p_{\text{W}_1\text{-W}_2}\tau_1} d\tau_1 = \frac{\lambda p_{\text{W}_1\text{-W}_2}}{\lambda p_{\text{W}_1\text{-W}_2} + \rho}.$$

When the W_1-W_2 type pair k arrives, it will wait for a W_2-W_1 type pair and will be matched with it. W_2-W_1 type pairs arrive with a Poisson process with frequency $\lambda p_{\text{W}_2\text{-W}_1}$. Let τ_2 be the arrival time of the next W_2-W_1 type pair. The cost of holding on to the W_1-W_2 type pair k is $\int_0^{\tau_2} ce^{-\rho t}dt = \frac{c}{\rho}\left(1 - e^{-\rho\tau_2}\right)$. Since τ_2 is distributed with the exponential density function $\lambda p_{\text{W}_2\text{-W}_1} e^{-\lambda p_{\text{W}_2\text{-W}_1}\tau_2}$, the current value of expected cost is given by

$$\begin{aligned} C_2 &= \frac{\lambda p_{\text{W}_1\text{-W}_2}}{\lambda p_{\text{W}_1\text{-W}_2} + \rho} E\left[\frac{c}{\rho}\left(1 - e^{-\rho\tau_2}\right)\right] \\ &= \frac{\lambda p_{\text{W}_1\text{-W}_2}}{\lambda p_{\text{W}_1\text{-W}_2} + \rho} . \frac{c}{\lambda p_{\text{W}_2\text{-W}_1} + \rho}, \end{aligned}$$

where the first coefficient $\frac{\lambda p_{\text{W}_1\text{-W}_2}}{\lambda p_{\text{W}_1\text{-W}_2} + \rho}$ refers to the discounting that will occur until the next W_1-W_2 type pair k arrives. Since $2p_{\text{W}_2\text{-W}_1} > p_{\text{W}_1\text{-W}_2}$ and $n \geq 2$, we have $C_1 > C_2$, implying that the X-Y type pair i will never be used to match a future incoming W_1-W_2 type pair k, and will be matched immediately. $\diamondsuit$

Before we start our proof of Proposition 4, we state the following existence and uniqueness theorem (part of Theorems 6.2.3, 6.2.5, and 6.2.10 in Puterman, 1994):

Theorem 4 ***(The Existence and Uniqueness Theorem):*** *Let Z be a countable state set. Let F be a finite action set. Let $\mathcal{V}$ be the set of bounded functions defined from Z to $\mathbb{R}$. Let $0 \leq \delta < 1$. For any $z \in Z$, let*

$$v(z) = \delta \max_{f \in F}\left\{r(z, f) + \textstyle\sum_{\sigma \in Z} p(\sigma | z, f) v(\sigma)\right\},$$

where (i) for all $\sigma \in Z$ and all $f \in F$, $p(\sigma | z, f) \geq 0$, and for all $f \in F$, $\sum_{\sigma \in C} p(\sigma | z, f) = 1$, and (ii) for all $f \in F$, $r(z, f) \in \mathbb{R}$ is bounded. Then:

1. *Function $v \in \mathcal{V}$ exists and is uniquely defined as the limit of the sequence $\{v^m\} \subseteq \mathcal{V}$ (under the sup norm),[28] where v^0 is arbitrary, and for any $m > 0$,*

$$v^m(z) = \delta \max_{f \in F}\left\{r(z, f) + \textstyle\sum_{\sigma \in Z} p(\sigma | z, f) v^{m-1}(\sigma)\right\}.$$

[28] For all $v \in V$, $\|v\| = \sup_{s \in S} |v(s)|$ is the **sup norm** of v.

2. There exists a (deterministic) Markovian mechanism $\phi : Z \to F$ such that for all $z \in Z$,

$$v(z) = \delta \left\{ r(z, \phi(z)) + \textstyle\sum_{\sigma \in Z} p(\sigma | z, \phi(z)) v(\sigma) \right\}.$$

We will use the above theorem in our proof of Proposition 4.

Proof of Proposition 4: Let $\mathrm{W}_1\text{-}\mathrm{W}_2 \in \mathcal{P}^{\mathcal{R}^*}$. Let $F = \{$*do-not-match, match*$\}$ and $f_1 =$*do-not-match*, $f_2 =$*match*. Consider the Bellman equations given in Equations 5, 6, and 7. Let the normalized surplus for choosing the smaller exchanges (action f_1) regarding $\mathrm{W}_1\text{-}\mathrm{W}_2$ be given by

$$r(s_{\mathrm{W}_1\text{-}\mathrm{W}_2}, f_1) = \begin{cases} 2p_{\mathrm{W}_2\text{-}\mathrm{W}_1} & \text{if } s_{\mathrm{W}_1\text{-}\mathrm{W}_2} > 0 \\ 0 & \text{if } s_{\mathrm{W}_1\text{-}\mathrm{W}_2} = 0 \\ 2p_{\mathrm{W}_1\text{-}\mathrm{W}_2} & \text{if } s_{\mathrm{W}_1\text{-}\mathrm{W}_2} < 0 \end{cases}, \tag{8}$$

and the normalized surplus for choosing larger exchanges (action f_2) be given by

$$r(s_{\mathrm{W}_1\text{-}\mathrm{W}_2}, f_2) = \begin{cases} 2p_{\mathrm{W}_2\text{-}\mathrm{W}_1} + \displaystyle\sum_{\mathrm{X}\text{-}\mathrm{Y} \in \mathcal{P}^{\mathcal{O}}(\mathrm{W}_1\text{-}\mathrm{W}_2)} p_{\mathrm{X}\text{-}\mathrm{Y}} & \text{if } s_{\mathrm{W}_1\text{-}\mathrm{W}_2} > 0 \\ 0 & \text{if } s_{\mathrm{W}_1\text{-}\mathrm{W}_2} = 0 \\ 2p_{\mathrm{W}_1\text{-}\mathrm{W}_2} + \displaystyle\sum_{\mathrm{X}\text{-}\mathrm{Y} \in \mathcal{P}^{\mathcal{O}}(\mathrm{W}_2\text{-}\mathrm{W}_1)} p_{\mathrm{X}\text{-}\mathrm{Y}} & \text{if } s_{\mathrm{W}_1\text{-}\mathrm{W}_2} < 0 \end{cases}. \tag{9}$$

When smaller exchanges (action f_1) are chosen, the transition probabilities are given by

$$\begin{aligned} p(s_{\mathrm{W}_1\text{-}\mathrm{W}_2} - 1 \mid s_{\mathrm{W}_1\text{-}\mathrm{W}_2}, f_1) &= p_{\mathrm{W}_2\text{-}\mathrm{W}_1}, \\ p(s_{\mathrm{W}_1\text{-}\mathrm{W}_2} \mid s_{\mathrm{W}_1\text{-}\mathrm{W}_2}, f_1) &= 1 - p_{\mathrm{W}_1\text{-}\mathrm{W}_2} - p_{\mathrm{W}_2\text{-}\mathrm{W}_1}, \\ p(s_{\mathrm{W}_1\text{-}\mathrm{W}_2} + 1 \mid s_{\mathrm{W}_1\text{-}\mathrm{W}_2}, f_1) &= p_{\mathrm{W}_1\text{-}\mathrm{W}_2}. \end{aligned} \tag{10}$$

When larger exchanges (action f_2) are chosen, the transition probabilities are given by

$$\begin{aligned} p(s_{\mathrm{W}_1\text{-}\mathrm{W}_2} - 1 \mid s_{\mathrm{W}_1\text{-}\mathrm{W}_2}, f_2) &= \begin{cases} p_{\mathrm{W}_2\text{-}\mathrm{W}_1} + \displaystyle\sum_{\mathrm{X}\text{-}\mathrm{Y} \in \mathcal{P}^{\mathcal{O}}(\mathrm{W}_1\text{-}\mathrm{W}_2)} p_{\mathrm{X}\text{-}\mathrm{Y}} & \text{if } s_{\mathrm{W}_1\text{-}\mathrm{W}_2} > 0 \\ p_{\mathrm{W}_2\text{-}\mathrm{W}_1} & \text{if } s_{\mathrm{W}_1\text{-}\mathrm{W}_2} \leq 0 \end{cases}, \\ p(s_{\mathrm{W}_1\text{-}\mathrm{W}_2} \mid s_{\mathrm{W}_1\text{-}\mathrm{W}_2}, f_2) &= \begin{cases} 1 - p_{\mathrm{W}_1\text{-}\mathrm{W}_2} - p_{\mathrm{W}_2\text{-}\mathrm{W}_1} - \displaystyle\sum_{\mathrm{X}\text{-}\mathrm{Y} \in \mathcal{P}^{\mathcal{O}}(\mathrm{W}_1\text{-}\mathrm{W}_2)} p_{\mathrm{X}\text{-}\mathrm{Y}} & \text{if } s_{\mathrm{W}_1\text{-}\mathrm{W}_2} > 0 \\ 1 - p_{\mathrm{W}_1\text{-}\mathrm{W}_2} - p_{\mathrm{W}_2\text{-}\mathrm{W}_1} & \text{if } s_{\mathrm{W}_1\text{-}\mathrm{W}_2} = 0 \\ 1 - p_{\mathrm{W}_1\text{-}\mathrm{W}_2} - p_{\mathrm{W}_2\text{-}\mathrm{W}_1} - \displaystyle\sum_{\mathrm{X}\text{-}\mathrm{Y} \in \mathcal{P}^{\mathcal{O}}(\mathrm{W}_2\text{-}\mathrm{W}_1)} p_{\mathrm{X}\text{-}\mathrm{Y}} & \text{if } s_{\mathrm{W}_1\text{-}\mathrm{W}_2} < 0 \end{cases}, \\ p(s_{\mathrm{W}_1\text{-}\mathrm{W}_2} + 1 \mid s_{\mathrm{W}_1\text{-}\mathrm{W}_2}, f_2) &= \begin{cases} p_{\mathrm{W}_1\text{-}\mathrm{W}_2} & \text{if } s_{\mathrm{W}_1\text{-}\mathrm{W}_2} \geq 0 \\ p_{\mathrm{W}_1\text{-}\mathrm{W}_2} + \displaystyle\sum_{\mathrm{X}\text{-}\mathrm{Y} \in \mathcal{P}^{\mathcal{O}}(\mathrm{W}_2\text{-}\mathrm{W}_1)} p_{\mathrm{X}\text{-}\mathrm{Y}} & \text{if } s_{\mathrm{W}_1\text{-}\mathrm{W}_2} < 0 \end{cases}. \end{aligned} \tag{11}$$

Let $\mathcal{V} = \{v : \mathbb{Z} \to \mathbb{R}_+$ such that v is bounded$\}$ be the set of Markov surplus functions for W_1-W_2 types. Let $v^0 \in \mathcal{V}$. For all $m \in \{1, 2, 3, ...\}$ $(\equiv \mathbb{Z}_{++})$, let $v^m \in \mathcal{V}$ be defined through the following recursive system,

$$v^m(s_{W_1\text{-}W_2}) = \max_{f \in \{f_1, f_2\}} w^m(s_{W_1\text{-}W_2}, f) \tag{12}$$

with $w^m : \mathbb{Z} \times \{f_1, f_2\} \to \mathbb{R}_+$ defined for all $f \in \{f_1, f_2\}$ as follows:

$$w^m(s_{W_1\text{-}W_2}, f) = \frac{\lambda}{\lambda+\rho}[r(s_{W_1\text{-}W_2}, f) + \sum_{\sigma = s_{W_1\text{-}W_2}-1}^{s_{W_1\text{-}W_2}+1} \left(p(\sigma \,|s_{W_1\text{-}W_2}, f)\, v^{m-1}(\sigma)\right)]. \tag{13}$$

The state component space for W_1-W_2 and W_2-W_1 types, $\mathbb{Z}$, is countable. Action space $F = \{f_1, f_2\}$ is finite. Since $\lambda > 0$ and $\rho > 0$, we have $0 < \frac{\lambda}{\lambda+\rho} < 1$. Observe that by Equations 10 and 11, for any $s_{W_1\text{-}W_2} \in \mathbb{Z}$ and $f \in F$, $p(\sigma | s_{W_1\text{-}W_2}, f) \geq 0$ for all $\sigma \in \mathbb{Z}$, and, $\sum_{\sigma = s_{W_1\text{-}W_2}-1}^{s_{W_1\text{-}W_2}} p(\sigma | s_{W_1\text{-}W_2}, f) = 1$. By Equations 8 and 9, for any $s_{W_1\text{-}W_2} \in \mathbb{Z}$ and $f \in F$, $r(s_{W_1\text{-}W_2}, f)$ is bounded. Since Equations 8- 13 are directly obtained from the Bellman Equations 5, 6, and 7, by the Existence and Uniqueness Theorem, there is a unique $ES^*_{W_1\text{-}W_2} \in \mathcal{V}$ such that under the sup norm, for all $s \in S$,

$$ES^*_{W_1\text{-}W_2}(s_{W_1\text{-}W_2}) = \lim_{m \to \infty} v^m(s_{W_1\text{-}W_2}).$$

$\Diamond$

The following Lemmata prove Theorem 2:

Lemma 2 *There exist $\overline{s}^* \geq 0$ and $\underline{s}^* \leq 0$ for some $\overline{s}^*, \underline{s}^* \in S$ such that $\phi^{\overline{s}^*, \underline{s}^*}$ is a dynamically efficient multi-way matching mechanism.*

Proof of Lemma 2: Fix W_1-$W_2 \in \mathcal{P}^{\mathcal{R}^*}$. Let $F = \{$*do-not-match, match*$\}$ and $f_1 =$*do-not-match*, $f_2 =$*match*. Let

$$h^* \equiv ES^*_{W_1\text{-}W_2}$$

and

$$z \equiv s_{W_1\text{-}W_2}$$

for notational convenience. The state component space regarding W_1-W_2 and W_2-W_1 types is given by $\mathbb{Z}$. We rewrite the Bellman Equations 5, 6, and 7 as follows: For any $z \in \mathbb{Z}$,

$$h^*(z) = \max_{f \in \{f_1, f_2\}} w(z, f), \tag{14}$$

where

$$w(z, f) = \frac{\lambda}{\lambda+\rho}[r(z, f) + \sum_{\sigma = z-1}^{z+1} p(\sigma \,|z, f)\, h^*(\sigma)], \tag{15}$$

and $r(z, f)$ is defined by Equations 8 and 9, and $p(\sigma \,|z, f)$ is defined by Equations 10 and 11. For all $z \in \mathbb{Z}$,

$$f^z = \arg\max_{f \in \{f_1, f_2\}} w(z, f) \tag{16}$$

such that

$$\text{if } w(z, f_1) = w(z, f_2), \text{ then } f_2 = \arg\max_{f \in \{f_1, f_2\}} w(z, f). \tag{17}$$

For all $z \in \mathbb{Z}$, let

$$\triangle h^*(z) = h^*(z) - h^*(z-1). \tag{18}$$

We prove Lemma 2 using the following four claims:

Claim 1: Suppose that $z > 0$ is such that $f^z = f_2$, and $f^{z+1} = f_1$. Then there is no $k \geq 1$ such that $f^{z+k+1} = f_2$.

Proof of Claim 1: Let $z > 0$ be such that $f^z = f_2$, and $f^{z+1} = f_1$. We prove the claim by contradiction. Suppose there exists some $k \geq 1$ such that

$$f^{z+2} = f_1, ..., f^{z+k} = f_1, \; f^{z+k+1} = f_2.$$

Therefore, by Observation 2 and definitions in Equations 16, 17, and 18,

$$\triangle h^*(z) \leq 1, \triangle h^*(z+1) > 1, \ldots, \triangle h^*(z+k) > 1, \quad \text{and} \quad \triangle h^*(z+k+1) \leq 1. \tag{19}$$

By definitions in Equations 14, 15, 16, and 18; for $r(z, f)$ in Equations 8 and 9; and for $p(\sigma | z, f)$ in Equations 10 and 11, we obtain

$$\begin{aligned}
&\triangle h^*(z+1) \\
&= h^*(z+1) - h^*(z) = w(z+1, f_1) - w(z, f_2) \\
&= \frac{\lambda}{\lambda+\rho} \left[\begin{array}{c} \sum_{\text{X-Y} \in \mathcal{P}^{\mathcal{O}}(\text{W}_1\text{-W}_2)} p_{\text{X-Y}} (\triangle h^*(z) - 1) + p_{\text{W}_2\text{-W}_1} \triangle h^*(z) \\ + (1 - p_{\text{W}_1\text{-W}_2} - p_{\text{W}_2\text{-W}_1}) \triangle h^*(z+1) + p_{\text{W}_1\text{-W}_2} \triangle h^*(z+2) \end{array} \right] \\
&\leq \frac{\lambda}{\lambda+\rho} [p_{\text{W}_2\text{-W}_1} + (1 - p_{\text{W}_1\text{-W}_2} - p_{\text{W}_2\text{-W}_1}) \triangle h^*(z+1) + p_{\text{W}_1\text{-W}_2} \triangle h^*(z+2)] \\
&\qquad \text{since, by } f^z = f_2 \text{, we have } \triangle h^*(z) \leq 1 \text{ (in Inequality System 19)} \\
&< [p_{\text{W}_2\text{-W}_1} + (1 - p_{\text{W}_1\text{-W}_2} - p_{\text{W}_2\text{-W}_1}) \triangle h^*(z+1) + p_{\text{W}_1\text{-W}_2} \triangle h^*(z+2)]. \qquad (20) \\
&\qquad \text{since } \frac{\lambda}{\lambda+\rho} < 1
\end{aligned}$$

We rearrange terms in Inequality 20 to obtain

$$\triangle h^*(z+1) < \frac{p_{\text{W}_2\text{-W}_1}}{p_{\text{W}_1\text{-W}_2} + p_{\text{W}_2\text{-W}_1}} + \frac{p_{\text{W}_1\text{-W}_2} \triangle h^*(z+2)}{p_{\text{W}_1\text{-W}_2} + p_{\text{W}_2\text{-W}_1}}. \tag{21}$$

For all ℓ such that $k \geq \ell > 1$, by definitions in Equations 14, 15, 16, and 18; for $r(z, f)$ in Equations 8 and 9; and for $p(\sigma | z, f)$ in Equations 10 and 11, we obtain

$$
\begin{aligned}
&\triangle h^*(z+\ell) \\
&= h^*(z+\ell) - h^*(z+\ell-1) = w(z+\ell, f_1) - w(z+\ell+1, f_1) \qquad (22)\\
&= \frac{\lambda}{\lambda+\rho}\left[p_{\mathrm{W_2\text{-}W_1}} \triangle h^*(z+\ell-1) + (1 - p_{\mathrm{W_1\text{-}W_2}} - p_{\mathrm{W_2\text{-}W_1}}) \triangle h^*(z+\ell) + p_{\mathrm{W_1\text{-}W_2}} \triangle h^*(z+\ell+1)\right] \\
&< p_{\mathrm{W_2\text{-}W_1}} \triangle h^*(z+\ell-1) + (1 - p_{\mathrm{W_1\text{-}W_2}} - p_{\mathrm{W_2\text{-}W_1}}) \triangle h^*(z+\ell) + p_{\mathrm{W_1\text{-}W_2}} \triangle h^*(z+\ell+1) \qquad (23)\\
&\qquad \text{since } \frac{\lambda}{\lambda+\rho} < 1 \ .
\end{aligned}
$$

We rearrange terms in Inequality 23 to obtain

$$
\triangle h^*(z+\ell) < \frac{p_{\mathrm{W_2\text{-}W_1}} \triangle h^*(z+\ell-1)}{p_{\mathrm{W_1\text{-}W_2}} + p_{\mathrm{W_2\text{-}W_1}}} + \frac{p_{\mathrm{W_1\text{-}W_2}} \triangle h^*(z+\ell+1)}{p_{\mathrm{W_1\text{-}W_2}} + p_{\mathrm{W_2\text{-}W_1}}}. \qquad (24)
$$

Using Inequality 24 for $\ell = k$, and the fact that $\triangle h^*(z+k+1) \leq 1$ (in Equation 19), we have

$$
\begin{aligned}
&(p_{\mathrm{W_1\text{-}W_2}} + p_{\mathrm{W_2\text{-}W_1}}) \triangle h^*(z+k) \\
&\quad < p_{\mathrm{W_2\text{-}W_1}} \triangle h^*(z+k-1) + (1 - p_{\mathrm{W_1\text{-}W_2}} - p_{\mathrm{W_2\text{-}W_1}}) \triangle h^*(z+k) + p_{\mathrm{W_1\text{-}W_2}} \triangle h^*(z+k+1) \\
&\quad \leq p_{\mathrm{W_2\text{-}W_1}} \triangle h^*(z+k-1) + (1 - p_{\mathrm{W_1\text{-}W_2}} - p_{\mathrm{W_2\text{-}W_1}}) \triangle h^*(z+k) + p_{\mathrm{W_1\text{-}W_2}} \ . \qquad (25)
\end{aligned}
$$

We rearrange terms in Inequality 25 to obtain

$$
\triangle h^*(z+k) < \frac{p_{\mathrm{W_2\text{-}W_1}} \triangle h^*(z+k-1)}{p_{\mathrm{W_1\text{-}W_2}} + p_{\mathrm{W_2\text{-}W_1}}} + \frac{p_{\mathrm{W_1\text{-}W_2}}}{p_{\mathrm{W_1\text{-}W_2}} + p_{\mathrm{W_2\text{-}W_1}}}. \qquad (26)
$$

We claim that for any $\ell \in \{1, 2, ..., k-1\}$, we have

$$
\triangle h^*(z+\ell) < \frac{p_{\mathrm{W_1\text{-}W_2}} g(\ell-1) \triangle h^*(z+\ell+1)}{g(\ell)} + \frac{p^{\ell}_{\mathrm{W_2\text{-}W_1}}}{g(\ell)}, \qquad (27)
$$

where

$$
g(\ell) = \sum_{i=0}^{\ell} p^{i}_{\mathrm{W_1\text{-}W_2}} p^{\ell-i}_{\mathrm{W_2\text{-}W_1}}. \qquad (28)
$$

We will prove that Inequality 27 holds using Inequalities 21 and 24 by induction.

- Let $\ell = 1$. Observe that $g(0) = 1$ and $g(1) = p_{\mathrm{W_1\text{-}W_2}} + p_{\mathrm{W_2\text{-}W_1}}$ using the definition of g (in Equation 28). Therefore, by Inequality 21,

$$
\triangle h^*(z+1) < \frac{p_{\mathrm{W_1\text{-}W_2}} g(0) \triangle h^*(z+2)}{g(1)} + \frac{p_{\mathrm{W_2\text{-}W_1}}}{g(1)}.
$$

- Let $\ell \in \{2, ..., k-1\}$. In the inductive step, assume that $\triangle h^*(z+\ell-1) < \frac{p_{\mathrm{W}_1\text{-}\mathrm{W}_2} g(\ell-2) \triangle h^*(z+\ell)}{g(\ell-1)} + \frac{p_{\mathrm{W}_2\text{-}\mathrm{W}_1}^{\ell-1}}{g(\ell-1)}$. We substitute the right-hand side of this inequality for $\triangle h^*(z+\ell-1)$ in Inequality 24 to obtain

$$\triangle h^*(z+\ell) < \frac{p_{\mathrm{W}_2\text{-}\mathrm{W}_1}}{p_{\mathrm{W}_1\text{-}\mathrm{W}_2} + p_{\mathrm{W}_2\text{-}\mathrm{W}_1}} \left(\frac{p_{\mathrm{W}_1\text{-}\mathrm{W}_2} g(\ell-2) \triangle h^*(z+\ell)}{g(\ell-1)} + \frac{p_{\mathrm{W}_2\text{-}\mathrm{W}_1}^{\ell-1}}{g(\ell-1)} \right) + \frac{p_{\mathrm{W}_1\text{-}\mathrm{W}_2} \triangle h^*(z+\ell+1)}{p_{\mathrm{W}_1\text{-}\mathrm{W}_2} + p_{\mathrm{W}_2\text{-}\mathrm{W}_1}}. \tag{29}$$

We rearrange terms in Inequality 29 to obtain

$$\triangle h^*(z+\ell) < \frac{p_{\mathrm{W}_1\text{-}\mathrm{W}_2} g(\ell-1) \triangle h^*(z+\ell+1) + p_{\mathrm{W}_2\text{-}\mathrm{W}_1}^{\ell}}{\left[(p_{\mathrm{W}_1\text{-}\mathrm{W}_2} + p_{\mathrm{W}_2\text{-}\mathrm{W}_1}) g(\ell-1) - p_{\mathrm{W}_2\text{-}\mathrm{W}_1} p_{\mathrm{W}_1\text{-}\mathrm{W}_2} g(\ell-2)\right]}. \tag{30}$$

Using the definition of g in Equation 28, we observe that

$$(p_{\mathrm{W}_1\text{-}\mathrm{W}_2} + p_{\mathrm{W}_2\text{-}\mathrm{W}_1}) g(\ell-1) - p_{\mathrm{W}_2\text{-}\mathrm{W}_1} p_{\mathrm{W}_1\text{-}\mathrm{W}_2} g(\ell-2) = g(\ell). \tag{31}$$

Substituting $g(\ell)$ for the left-hand side of Equation 31 in Inequality 30, we obtain

$$\triangle h^*(z+\ell) < \frac{p_{\mathrm{W}_1\text{-}\mathrm{W}_2} g(\ell-1) \triangle h^*(z+\ell+1)}{g(\ell)} + \frac{p_{\mathrm{W}_2\text{-}\mathrm{W}_1}^{\ell}}{g(\ell)},$$

completing the induction.

We have $\triangle h^*(z+k-1) < \frac{p_{\mathrm{W}_1\text{-}\mathrm{W}_2} g(k-2) \triangle h^*(z+k)}{g(k-1)} + \frac{p_{\mathrm{W}_2\text{-}\mathrm{W}_1}^{k-1}}{g(k-1)}$ by Inequality 27. We substitute the right-hand side of this inequality for $\triangle h^*(z+k-1)$ in Inequality 26 to obtain the following inequality:

$$\triangle h^*(z+k) < \frac{p_{\mathrm{W}_2\text{-}\mathrm{W}_1}}{p_{\mathrm{W}_1\text{-}\mathrm{W}_2} + p_{\mathrm{W}_2\text{-}\mathrm{W}_1}} \left(\frac{p_{\mathrm{W}_1\text{-}\mathrm{W}_2} g(k-2) \triangle h^*(z+k)}{g(k-1)} + \frac{p_{\mathrm{W}_2\text{-}\mathrm{W}_1}^{k-1}}{g(k-1)} \right) + \frac{p_{\mathrm{W}_1\text{-}\mathrm{W}_2}}{p_{\mathrm{W}_1\text{-}\mathrm{W}_2} + p_{\mathrm{W}_2\text{-}\mathrm{W}_1}}. \tag{32}$$

Rearranging terms in Inequality 32, we obtain

$$\triangle h^*(z+k) < \frac{p_{\mathrm{W}_2\text{-}\mathrm{W}_1}^{k} + p_{\mathrm{W}_1\text{-}\mathrm{W}_2} g(k-1)}{(p_{\mathrm{W}_1\text{-}\mathrm{W}_2} + p_{\mathrm{W}_2\text{-}\mathrm{W}_1}) g(k-1) - p_{\mathrm{W}_2\text{-}\mathrm{W}_1} p_{\mathrm{W}_1\text{-}\mathrm{W}_2} g(k-2)}. \tag{33}$$

Using the definition of g (in Equation 28), we observe that

$$p_{\mathrm{W}_2\text{-}\mathrm{W}_1}^{k} + p_{\mathrm{W}_1\text{-}\mathrm{W}_2} g(k-1) = g(k) \text{ and} \tag{34}$$

$$(p_{\mathrm{W}_1\text{-}\mathrm{W}_2} + p_{\mathrm{W}_2\text{-}\mathrm{W}_1}) g(k-1) - p_{\mathrm{W}_2\text{-}\mathrm{W}_1} p_{\mathrm{W}_1\text{-}\mathrm{W}_2} g(k-2) = g(k). \tag{35}$$

Substituting $g(k)$ for the left-hand side terms of Equations 34 and 35, Inequality 33 can be rewritten as

$$\triangle h^*(z+k) < 1. \tag{36}$$

However, Inequality 36 through Observation 2 contradict the claim that $f^{z+k} = f_1$ and $\triangle h^* (z+k) > 1$ (stated in Inequality System 19). We showed that for any $z > 0$ whenever $f^z = f_2$ and $f^{z+1} = f_1$, there is no $k \geq 1$ such that $f^{z+k+1} = f_2$, completing the proof of Claim 1. $\square$

Claim 2: There exists $z' \geq 0$ such that for all $z > z'$ we have $f^z = f_2$.

Proof of Claim 2: Consider a scenario in which there are infinitely many W_1-W_2 type pairs available at the exchange pool. That is, the state component is $z = \infty$. Every incoming overdemanded pair of one of the types in $\mathcal{P}^{\mathcal{O}} (\mathrm{W}_1\text{-}\mathrm{W}_2)$ can be used in an exchange that matches a W_1-W_2 type pair. After such an exchange, there will still be infinitely many W_1-W_2 type pairs, implying that incoming W_2-W_1 type pairs are not affected by the previous decision of choosing largest possible exchanges. Therefore, at state component $z = \infty$, the efficient action is f_2 (option *match*, conducting largest possible exchanges). Therefore, every incoming W_2-W_1 type pair will be matched in a two-way exchange with a W_1-W_2 type pair, and every incoming pair of one of the types in $\mathcal{P}^{\mathcal{O}} (\mathrm{W}_1\text{-}\mathrm{W}_2)$ will be matched efficiently serving a W_1-W_2 type pair. Since we are discussing the next incoming pairs, this surplus should be discounted with $E\left[e^{-\rho \tau_1}\right] = \frac{\lambda}{\lambda+\rho}$ The exchange surplus for the first matched W_1-W_2 or W_2-W_1 type pair in this scenario is

$$\overline{h}^1 = \frac{\lambda}{\lambda+\rho}\left[\sum_{\text{X-Y}\in\mathcal{P}^{\mathcal{O}}(\mathrm{W}_1\text{-}\mathrm{W}_2)} p_{\text{X-Y}}\left(\frac{c}{\rho}\right) + p_{\mathrm{W}_2\text{-}\mathrm{W}_1}\left(2\frac{c}{\rho}\right)\right].$$

Similarly, the current value of the exchange surplus for the second matched W_1-W_2 or W_2-W_1 type pair is $\overline{h}^2 = \frac{\lambda}{\lambda+\rho}\overline{h}^1$, ..., and the current value of the exchange surplus for the k^{th} matched W_1-W_2 or W_2-W_1 type pair is $\overline{h}^k = \left(\frac{\lambda}{\lambda+\rho}\right)^{k-1}\overline{h}^1$. Therefore, the total exchange surplus of state component ∞ is

$$\begin{aligned} h(\infty) &= \sum_{k=1}^{\infty}\overline{h}^k = \sum_{k=1}^{\infty}\left(\frac{\lambda}{\lambda+\rho}\right)^{k-1}\overline{h}^1 = \frac{1}{1-\left(\frac{\lambda}{\lambda+\rho}\right)}\overline{h}^1 \\ &= \frac{\lambda c}{\rho^2}\left(\sum_{\text{X-Y}\in\mathcal{P}^{\mathcal{O}}(\mathrm{W}_1\text{-}\mathrm{W}_2)} p_{\text{X-Y}} + 2p_{\mathrm{W}_2\text{-}\mathrm{W}_1}\right). \end{aligned}$$

By normalizing $h(\infty)$ by $\frac{c}{\rho}$, we obtain

$$h^*(\infty) = \frac{\lambda}{\rho}\left(\sum_{\text{X-Y}\in\mathcal{P}^{\mathcal{O}}(\mathrm{W}_1\text{-}\mathrm{W}_2)} p_{\text{X-Y}} + 2p_{\mathrm{W}_2\text{-}\mathrm{W}_1}\right).$$

Clearly, the normalized exchange surplus at state component ∞ is an upper-bound for the normalized efficient exchange surplus for $z \to \infty$. Suppose that there is no $z' > 0$ such that for all $z > z'$, $f^z = f_2$. By Claim 1, there exists some $z' > 0$ such that for all $z > z'$, $f^z = f_1$ and $h^*(z) \geq h^*(z-1) + 1$ (by Observation 2). Therefore, for any $z > z'$,

$$h^*(z) \geq (z - z') + h^*(z').$$

Then as $z \to \infty$, $h^*(z) \to \infty$, contradicting the fact that $h^*(\infty)$ is bounded. This and Claim 1 imply that there exists some $z' > 0$ such that for all $z > z'$, $f^z = f_2$. $\square$

We state the following two claims, whose proofs are symmetric versions of the proofs of Claims 1 and 2:

Claim 3: Suppose that $z < 0$ is such that $f^z = f_2$, and $f^{z-1} = f_1$. Then there is no $k \geq 1$ such that $f^{z-k-1} = f_2$.

Claim 4: There exists $z'' \leq 0$ such that for all $z < z''$ we have $f^z = f_2$.

By Claims 1 and 2, there exists $\overline{s}^*_{\text{W}_1\text{-W}_2} \geq 0$ such that $f^z = f_2$ for all $z > \overline{s}^*_{\text{W}_1\text{-W}_2}$ and $f^z = f_1$ for all $0 \leq z < \overline{s}^*_{\text{W}_1\text{-W}_2}$. By Claims 3 and 4 there exists $\underline{s}^*_{\text{W}_1\text{-W}_2} \leq 0$ such that $f^z = f_2$ for all $z < \underline{s}^*_{\text{W}_1\text{-W}_2}$ and $f^z = f_1$ for all $0 \geq z \geq \underline{s}^*_{\text{W}_1\text{-W}_2}$. Since W$_1$-W$_2 \in \mathcal{P}^{\mathcal{R}^*}$ is arbitrary, the threshold mechanism $\phi^{\overline{s}^*,\underline{s}^*}$ is an efficient matching mechanism. $\lozenge$

Lemma 3 *For each W_1-$W_2 \in \mathcal{P}^{\mathcal{R}^*}$,*

$$ES^*_{W_1\text{-}W_2}(0) < \frac{\lambda}{\rho}\left(p_{W_1\text{-}W_2} + p_{W_2\text{-}W_1}\right)$$

Proof of Lemma 3: Fix W$_1$-W$_2 \in \mathcal{P}^{\mathcal{R}^*}$. Consider the state component $z = 0$. If W$_1$-W$_2$ and W$_2$-W$_1$ type pairs could be matched as soon as they arrived at the exchange pool, the decision problem of the health authority would be trivial and it would match the overdemanded type pairs in the largest possible exchanges. That is, since no W$_1$-W$_2$ or W$_2$-W$_1$ type pairs remain in the pool unmatched, whenever an X-Y $\in \mathcal{P}^{\mathcal{O}}(\text{W}_1\text{-W}_2) \cup \mathcal{P}^{\mathcal{O}}(\text{W}_2\text{-W}_1)$ type overdemanded pair arrives at the exchange pool, it will be matched in an exchange without a W$_1$-W$_2$ or W$_2$-W$_1$ type pair that matches the maximum number of underdemanded pairs possible and possibly some other reciprocally demanded pairs. Let the associated exchange surplus with this process be $\overline{ES_{\text{W}_1\text{-W}_2}}(0)$. Since in reality W$_1$-W$_2$ and W$_2$-W$_1$ type pairs are not matched as soon as they arrive, $\overline{ES_{\text{W}_1\text{-W}_2}}(0) > ES_{\text{W}_1\text{-W}_2}(0)$. The exchange surplus related to a pair is $\frac{c}{\rho}$. Since we are discussing the next incoming pair, this surplus should be discounted with $E\left[e^{-\rho\tau_1}\right] = \frac{\lambda}{\lambda+\rho}$, implying that the associated exchange surplus is

$$\overline{ES_{\text{W}_1\text{-W}_2}}^1 = \frac{\lambda}{\lambda+\rho}\left[\left(p_{\text{W}_1\text{-W}_2} + p_{\text{W}_2\text{-W}_1}\right)\frac{c}{\rho}\right].$$

Similarly, the exchange surplus associated with the second incoming pair is $\overline{ES_{\text{W}_1\text{-W}_2}}^2 = \frac{\lambda}{\lambda+\rho}\overline{ES_{\text{W}_1\text{-W}_2}}^1$, ..., and the exchange surplus associated with the k^{th} incoming pair is $\overline{ES_{\text{W}_1\text{-W}_2}}^k = \left(\frac{\lambda}{\lambda+\rho}\right)^{k-1}\overline{ES_{\text{W}_1\text{-W}_2}}^1$. Therefore,

$$\begin{aligned}\overline{ES_{\text{W}_1\text{-W}_2}}(0) &= \sum_{k=1}^{\infty}\overline{ES_{\text{W}_1\text{-W}_2}}^k = \sum_{k=1}^{\infty}\left(\frac{\lambda}{\lambda+\rho}\right)^{k-1}\overline{ES_{\text{W}_1\text{-W}_2}}^1 = \frac{1}{1-\left(\frac{\lambda}{\lambda+\rho}\right)}\overline{ES_{\text{W}_1\text{-W}_2}}^1 \\ &= \frac{\lambda c}{\rho^2}\left(p_{\text{W}_1\text{-W}_2} + p_{\text{W}_2\text{-W}_1}\right).\end{aligned}$$

Recall that $ES^*_{\mathrm{W_1\text{-}W_2}}(0) = \frac{\rho}{c} ES_{\mathrm{W_1\text{-}W_2}}(0)$. Hence,

$$ES^*_{\mathrm{W_1\text{-}W_2}}(0) = \frac{\rho}{c} ES_{\mathrm{W_1\text{-}W_2}}(0) < \frac{\rho}{c}\overline{ES_{\mathrm{W_1\text{-}W_2}}}(0) = \frac{\lambda}{\rho}\left(p_{\mathrm{W_1\text{-}W_2}} + p_{\mathrm{W_2\text{-}W_1}}\right).$$

$\Diamond$

Lemma 4 *For each W_1-$W_2 \in \mathcal{P}^{\mathcal{R}^*}$, we have $\overline{s}^*_{W_1\text{-}W_2} \geq 0$ and $\underline{s}^*_{W_1\text{-}W_2} = 0$, or $\overline{s}^*_{W_1\text{-}W_2} = 0$ and $\underline{s}^*_{W_1\text{-}W_2} \leq 0$.*

Proof of Lemma 4: Fix $\mathrm{W_1\text{-}W_2} \in \mathcal{P}^{\mathcal{R}^*}$. We prove the lemma by contradiction. Suppose that there exist some $\overline{s}^*_{\mathrm{W_1\text{-}W_2}} > 0$ and $\underline{s}^*_{\mathrm{W_1\text{-}W_2}} < 0$ such that $\phi^{\overline{s}^*,\underline{s}^*}$ is efficient. Since $\overline{s}^*_{\mathrm{W_1\text{-}W_2}} > 0$, action f_1 (*do-not-match* $\mathrm{W_1}$-$\mathrm{W_2}$ type pair and choose the smaller exchange option) is chosen at state component 1, whenever an action needs to be taken. By the Bellman Equation 5, the *normalized* exchange surplus related to action f_1 is $ES^*_{\mathrm{W_1\text{-}W_2}}(1)$, the *normalized* exchange surplus related to action f_2 (*match* $\mathrm{W_1}$-$\mathrm{W_2}$ type pair and choose the larger exchange option) is $ES^*_{\mathrm{W_1\text{-}W_2}}(0)+1$, and we have $ES^*_{\mathrm{W_1\text{-}W_2}}(1) \geq ES^*_{\mathrm{W_1\text{-}W_2}}(0)+1$ (by Observation 2 and since in case of equality f_2 is chosen). Similarly, since $\underline{s}^*_{\mathrm{W_1\text{-}W_2}} < 0$, action f_1, that is: the smaller exchange, is chosen at state component -1, whenever an action needs to be taken. By the Bellman Equation 6, the *normalized* exchange surplus related to action f_1 is $ES^*_{\mathrm{W_1\text{-}W_2}}(-1)$, the *normalized* exchange surplus related to action f_2 (the larger exchange) is $ES^*_{\mathrm{W_1\text{-}W_2}}(0)+1$, and we have $ES^*_{\mathrm{W_1\text{-}W_2}}(-1) \geq ES^*_{\mathrm{W_1\text{-}W_2}}(0)+1$ (by Observation 3). We recall the Bellman Equation for state component 0 as follows (Equation 7):

$$ES^*_{\mathrm{W_1\text{-}W_2}}(0) = \frac{\lambda}{\lambda+\rho}\left[\begin{array}{c}\left(\sum\limits_{\mathrm{X\text{-}Y}\in\mathcal{P}^{\mathcal{O}}(\mathrm{W_1\text{-}W_2})\cup\mathcal{P}^{\mathcal{U}}\cup\mathcal{P}^{\mathcal{R}}\setminus\{\mathrm{W_1\text{-}W_2},\mathrm{W_2\text{-}W_1}\}} p_{\mathrm{X\text{-}Y}}\right) ES^*_{\mathrm{W_1\text{-}W_2}}(0) \\ +p_{\mathrm{W_1\text{-}W_2}} ES^*_{\mathrm{W_1\text{-}W_2}}(1) + p_{\mathrm{W_2\text{-}W_1}}\left(ES^*_{\mathrm{W_1\text{-}W_2}}(-1)\right)\end{array}\right].$$

We replace $ES^*(1)$ by the smaller number $ES^*(0)+1$ and $ES^*(-1)$ by the smaller number $ES^*(0)+1$ in the above expression to obtain the following inequality:

$$ES^*_{\mathrm{W_1\text{-}W_2}}(0) \geq \frac{\lambda}{\lambda+\rho}\left[\begin{array}{c}\left(\sum\limits_{\mathrm{X\text{-}Y}\in\mathcal{P}^{\mathcal{O}}(\mathrm{W_1\text{-}W_2})\cup\mathcal{P}^{\mathcal{U}}\cup\mathcal{P}^{\mathcal{R}}\setminus\{\mathrm{W_1\text{-}W_2},\mathrm{W_2\text{-}W_1}\}} p_{\mathrm{X\text{-}Y}}\right) ES^*_{\mathrm{W_1\text{-}W_2}}(0) \\ +p_{\mathrm{W_1\text{-}W_2}}\left(ES^*(0)+1\right) + p_{\mathrm{W_2\text{-}W_1}}\left(ES^*(0)+1\right)\end{array}\right].$$

Arranging the terms in the above inequality, we obtain

$$ES^*_{\mathrm{W_1\text{-}W_2}}(0) \geq \frac{\lambda}{\rho}\left(p_{\mathrm{W_1\text{-}W_2}} + p_{\mathrm{W_2\text{-}W_1}}\right),$$

contradicting Lemma 3. Therefore, we have $\overline{s}^*_{\mathrm{W_1\text{-}W_2}} \geq 0$ and $\underline{s}^*_{\mathrm{W_1\text{-}W_2}} = 0$, or $\overline{s}^*_{\mathrm{W_1\text{-}W_2}} = 0$ and $\underline{s}^*_{\mathrm{W_1\text{-}W_2}} \leq 0$. $\Diamond$

Proof of Theorem 2: The proof follows directly from Lemmata 2, 3, and 4. $\Diamond$

B Appendix: On the Efficient Kidney Exchange Mechanism When Self-Demanded Types Participate in Exchange

In this appendix, we retain Assumptions 1, 2, and 4, and relax Assumption 3, that is: we assume that self-demanded type pairs also participate in exchange. When there are self-demanded types in the exchange pool, under Assumptions 1, 2, and 4, the full state of the matching mechanism should be denoted not only by the difference between the number of A-B and B-A type pairs but also by four other variables that denote the number of O-O, A-A, B-B, and AB-AB type pairs. Next, we outline the intuition behind the derivation of the structure of the efficient mechanism under Assumptions 1, 2, and 4. A formal derivation using Bellman Equations is complicated because of the high dimensionality of the state space. However, we can make use of the underlying structure of the problem and our results in the previous subsection in explaining the intuition:

Let $\phi^{\overline{s}^*,\underline{s}^*}$ be the efficient matching mechanism under Assumptions 1, 2, and 4. Without loss of generality, let $\overline{s}^* \geq 0$ and $\underline{s}^* = 0$. Suppose Assumptions 1, 2, and 4 still apply, while self-demanded types can participate in exchange. Two cases can arise in the pool:

- When a self-demanded type pair arrives: Suppose this pair is of type X-X. If there is another type X-X pair available in the exchange pool, then we obtain a two-way exchange by matching these two pairs together immediately. Observe that this exchange is efficient, since self-demanded pairs cannot save any underdemanded pairs by Proposition 2. Therefore, under the efficient mechanism, there will be 0 or 1 self-demanded type pairs in the pool.

- When a non-self-demanded type pair arrives: Let $E = (i_1, .., i_k)$ be a feasible exchange without any self-demanded types (let $i_{k+1} \equiv i_1$). If there exists a self-demanded X-X type pair i available in the exchange pool such that there are pairs i_ℓ and $i_{\ell+1}$ with X blood-type donor and X blood-type recipient, respectively, then we can insert pair i between pairs i_ℓ and $i_{\ell+1}$ and obtain a feasible exchange E'. This exchange is better than E, since (1) self-demanded types cannot save any underdemanded types, (2) overdemanded types are most efficiently used in saving underdemanded types, and finally, (3) self-demanded types can otherwise be matched with only same-type pairs if they are not inserted in larger exchanges. So, we need to enlarge the exchanges as much as possible by inserting all possible existing self-demanded type pairs.

  By the above argument, and given the fact that efficient mechanism under Assumptions 1, 2, 3, and 4 is a threshold mechanism, the efficient mechanism under Assumptions 1, 2, and 3 is a *generalized threshold mechanism*, with a threshold number of A-B (or B-A type pairs) to conduct smaller exchanges (the largest exchanges without the A-B or B-A type pairs), where the threshold number depends on the existence or absence of self-demanded type pairs at each state.

  Let E_S be the possible smaller exchange and E_L be the possible larger exchange without any self-demanded type pairs. We have the following possibilities for the pair types in E_S and E_L:

state s	**pair types in $\mathbf{E_S}$**	**pair types in $\mathbf{E_L}$**
s < 0	(A-O, O-A)	(A-O, O-B, B-A)
When |s| B-A type	(AB-B, B-AB)	(AB-B, B-A, A-AB)
pairs exist	(AB-O, O-A, A-AB), (AB-O, O-B, B-AB)	(AB-O, O-B, B-A, A-AB)
s > 0	(B-O, O-B)	(B-O, O-A, A-B)
When s A-B type	(AB-A, A-AB)	(AB-A, A-B, B-AB)
pairs exist	(AB-O, O-A, A-AB), (AB-O, O-B, B-AB)	(AB-O, O-A, A-B, B-AB)

Three cases are possible:

– X-X $\in$ {O-O, AB-AB}: Observe that type X-X pair can be inserted in E_S if and only if it can be inserted in E_L in each case. Therefore, the existence of 1 X-X type pair or the absence of X-X type pairs has no effect on the thresholds, since in either case the marginal gain of the larger exchange is only 1 pair. Therefore, whenever such an X-X type pair exists, inserting the X-X type pair in E_S or E_L, whichever is chosen under the thresholds $\overline{s}^*$ and $\underline{s}_*$, is the efficient action.

– X-X = A-A: Consider the case when there is 1 A-A type pair. If $\underline{s}^* < 0$ let the pair types of E_S be (AB-B, B-AB) and the pair types of E_L be (AB-B, B-A, A-AB), and if $\overline{s}^* > 0$, let the pair types of E_S be (B-O, O-B) and the pair types of E_L be (B-O, O-A, A-B). In each case, the A-A pair cannot be inserted in the smaller exchange, but it can be inserted in the larger exchange. Therefore, the marginal gain of the larger exchange is 2 pairs (with A-A type pair). Recall that when no A-A pair exists, the marginal gain of the larger exchange is only 1 pair. Therefore, the threshold for smaller exchanges with an A-A type pair cannot exceed the threshold without an A-A type pair in absolute value, $|\underline{s}^*|$ or $\overline{s}^*$, whichever applies. For all other possibilities for E_S and E_L pair types, the A-A type pair can be inserted in E_S if and only if it can be inserted in E_L; hence the threshold $|\underline{s}^*|$ or $\overline{s}^*$ is still valid for smaller exchanges.

– X-X = B-B: the symmetric argument for the case X-X = A-A applies by interchanging the roles of A and B blood types.

Based on this intuition, we state the following remark:

Remark 1 *Suppose Assumptions 1, 2, and 4 hold, i.e., self-demanded type pairs can also participate in exchange. Let $\phi^{\overline{s}^*,\underline{s}^*}$ be the dynamically efficient kidney exchange mechanism under Assumptions 1, 2, 3, and 4. Consider the case $\overline{s}^* > 0$ and $\underline{s}^* = 0$. Then, there exist thresholds $0 \leq \overline{s}^*_{A\text{-}A} \leq \overline{s}^*$ and $0 \leq \overline{s}^*_{B\text{-}B} \leq \overline{s}^*$ such that under an efficient mechanism whenever a decision is required between two exchanges – the largest exchange with an A-B type pair (option* match*) or the largest exchange without an A-B type pair (option* do-not-match*) – the smaller exchange is chosen if and only if the number of A-B type pairs, s, satisfies*

- $\overline{s}^*_{A\text{-}A} \geq s \geq 0$, *if an A-A type pair exists and a B-O type pair arrives,*

- $\overline{s}^*_{B\text{-}B} \geq s \geq 0$, *if a B-B type pair exists and an AB-A type pair arrives,*

- $\overline{s}^* \geq s \geq 0$, *otherwise.*

If these conditions are not satisfied, the largest exchanges are conducted as soon as they become feasible. The efficient mechanism is symmetrically defined for the case $\overline{s}^* = 0$ *and* $\underline{s}^* < 0$ *with thresholds* $\underline{s}^*$, $0 \geq \underline{s}^*_{A\text{-}A} \geq \underline{s}^*$, *and* $0 \geq \underline{s}^*_{B\text{-}B} \geq \underline{s}^*$.[29]

We can state a single state variable approximation of the efficient mechanism as follows under Assumptions 1, 2, and 4 with $\underline{s}^*_{\text{A-A}} = \underline{s}^*_{\text{B-B}} = \underline{s}^*$ and $\overline{s}^*_{\text{A-A}} = \overline{s}^*_{\text{B-B}} = \overline{s}^*$:

- When a self-demanded type pair arrives: Suppose this pair is of type X-X. If there is another type X-X pair available in the exchange pool, then we obtain a two-way exchange by matching these two pairs together immediately.

- When a non-self-demanded type pair arrives: Let $E = (i_1, .., i_k)$ be the efficient exchange according to the efficient mechanism $\phi^{\overline{s}^*, \underline{s}^*}$ without taking the existence of self-demanded types into consideration (let $i_{k+1} \equiv i_1$). If there exists a self-demanded X-X type pair i available in the exchange pool such that there are pairs i_ℓ and $i_{\ell+1}$ with an X blood-type donor and an X blood-type recipient, respectively, then we can insert pair i between pairs i_ℓ and $i_{\ell+1}$ and obtain a feasible exchange E'. We repeat the process with E' until no feasible self-demanded type pair remains to be inserted. We conduct the final exchange obtained.

Let this mechanism be called $\hat{\phi}^{\overline{s}^*, \underline{s}^*}$. We conduct policy simulations using this mechanism.

[29] There could be only one ambiguity in the definition: Suppose that an AB-O type pair becomes available for exchange and a smaller exchange is chosen. Moreover, suppose the number of A-B type pairs is s (such that $0 < s < s^*$). In this case, smaller exchanges are chosen, but we can form two types of exchanges with the AB-O pair.

- We can have a three-way exchange with AB-O, O-A, and A-AB type pairs, or

- We can have a three-way exchange with AB-O, O-B, and B-AB type pairs.

As explained before, by Assumption 1, choosing either exchange is fine when there are no self-demanded types. Consider the case in which there is one A-A and one B-B type pair in the exchange pool. Depending on which exchange is chosen either an A-A or a B-B type pair can be inserted in the exchange, but not both (cf. Proposition 2). When there are self-demanded types, the optimal rule always chooses either the first type of exchange or the second type of exchange depending on the arrival probabilities.

Since in reality incompatible AB-O type pairs are rare, the impact of the tie-breaker is minimal. Moreover, we can make the following approximations: $\overline{s}^*_{\text{A-A}} = \overline{s}^*_{\text{B-B}} = \overline{s}^*$ and $\underline{s}^*_{\text{A-A}} = \underline{s}^*_{\text{B-B}} = \underline{s}^*$.

References

[1] Abdulkadiroğlu, A. and S. Loertscher (2006). Dynamic house allocation. Working paper, Duke University and Columbia University.

[2] Abdulkadiroğlu, A. and T. Sönmez (1999). House allocation with existing tenants. *Journal of Economic Theory 88*, 233-260.

[3] Abecassis, M., M. Adams, P. Adams, R. M. Arnold, C. R. Atkins, M. L. Barr, W. M. Bennett, M. Bia, D. M. Briscoe, J. Burdick, R. J. Corry, J. Davis, F. L. Delmonico, R. S. Gaston, W. Harmon, C. L. Jacobs, J. Kahn, A. Leichtman, C. Miller, D. Moss, J. M. Newmann, L. S. Rosen, L. Siminoff, A Spital, V. A. Starnes, C. Thomas, L. S. Tyler, L. Williams, F. H. Wright, and S. Youngner (2000). Consensus statement on the live organ donor. *Journal of the American Medical Association 284*, 2919-2926.

[4] Athey, S. and I. Segal (2007). An optimal dynamic mechanism. Working paper, Harvard University and Stanford University.

[5] Bergemann, D. and J. Välimaki (2006). Optimal dynamic auctions. Working paper, Yale University and Helsinki School of Economics.

[6] Bloch, F. and D. Cantala (2009). Markovian Assignment Rules. Working paper, Ecole Polytechnique and El Colegio de México.

[7] Delmonico, F. L. (2004). Exchanging kidneys – Advances in living-donor transplantation. *New England Journal of Medicine 350*, 1812-1814.

[8] Ehlers, L. (2002). Coalitional strategy-proof house allocation. *Journal of Economic Theory 105*, 298-317.

[9] Ehlers, L., B. Klaus, and S. Papai (2002). Strategy-proofness and population-monotonicity in house allocation problems. *Journal of Mathematical Economics 38*, 329-339.

[10] Ergin, H. (2000). Consistency in house allocation problems. *Journal of Mathematical Economics 34*, 77-97.

[11] Gershkov, A. and B. Moldovanu (2007). Dynamic revenue maximization with heterogeneous objects: A mechanism design approach. *American Economic Journal: Microeconomics*, forthcoming.

[12] Gershkov, A. and B. Moldovanu (2008). Learning about the future and dynamic efficiency. *American Economic Review*, forthcoming.

[13] Gjertson, D. W. and J. M. Cecka (2000). Living unrelated donor kidney transplantation. *Kidney International 58*, 491-499.

[14] Jackson, M. O. and T. R. Palfrey (1998). Efficiency and voluntary implementation in markets with repeated pairwise bargaining. *Econometrica 66*, 1353–1388.

[15] Kesten, O. (2004). Coalitional strategy-proofness and resource monotonicity for house allocation problems. Working paper, Carnegie Mellon University.

[16] de Klerk, M., K. M. Keizer, F. H. J. Claas, M. Witvliet, B. J. J. M. Haase-Kromwijk, and W. Weimar (2005). The Dutch national living donor kidney exchange program. *American Journal of Transplantation 5*, 2302-2305.

[17] Kurino, M. (2008). House allocation with overlapping agents: A dynamic mechanism design approach. Working paper, University of Pittsburgh.

[18] Overbeck, I., M. Bartels, O. Decker, J. Harms, J. Hauss, and J. Fangmann (2005). Changes in quality of life after renal transplantation. *Transplantation Proceedings 37*, 1618–1621.

[19] Papai, S. (2000). Strategyproof assignment by hierarchical exchange. *Econometrica 68*, 1403-1433.

[20] Park, K., J. H. Lee, K. H. Huh, S. I. Kim, and Y. S. Kim (2004). Exchange living donor kidney transplantation: Diminution of donor organ shortage. *Transplantation Proceedings 36*, 2949–2951.

[21] Puterman, M. L. (1994). *Markov Decision Processes Discrete Stochastic Dynamic Programming.* John Wiley and Sons: Hoboken, New Jersey.

[22] Rapaport, F. T. (1986). The case for a living emotionally related international kidney donor exchange registry. *Transplantation Proceedings 18*, 5-9.

[23] Roth, A. E. (1982). Incentive compatibility in a market with indivisibilities. *Economics Letters 9*, 127-132.

[24] Roth, A. E. and A. Postlewaite (1977). Weak versus strong domination in a market with indivisible goods. *Journal of Mathematical Economics 4*, 131-137.

[25] Roth, A. E., T. Sönmez, and M. U. Ünver (2004). Kidney exchange, *Quarterly Journal of Economics 119*, 457-488.

[26] Roth, A. E., T. Sönmez, and M. U. Ünver (2005a). Pairwise kidney exchange. *Journal of Economic Theory 125*, 151-188.

[27] Roth, A. E., T. Sönmez, and M. U. Ünver (2005b). A kidney exchange clearinghouse in New England. *American Economic Review Papers and Proceedings 95(2),* 376-380.

[28] Roth, A. E., T. Sönmez, and M. U. Ünver (2007). Efficient kidney exchange: coincidence of wants in markets with compatibility-based preferences. *American Economic Review 97*, 828-851.

[29] Said, M. (2009). Auctions with dynamic populations: Efficiency and revenue maximization. Working paper, Yale University.

[30] Saidman, S. L., A. E. Roth, T. Sönmez, M. U. Ünver, and F. L. Delmonico (2006). Increasing the opportunity of live kidney donation by matching for two and three way exchanges. *Transplantation 81*, 773-782.

[31] Shapley, L. S. and H. Scarf (1974). On cores and indivisibility. *Journal of Mathematical Economics 1*, 23-37.

[32] Skreta, V. (2006). Sequentially optimal mechanisms. *Review of Economic Studies 73*, 1085-1111.

[33] Schweitzer, E. J., A. Wiland, D. Evans, M. Novak, I. Connerny, L. Norris, J. O. Colonna, B. Philosophe, A. C. Farney, B. E. Jarrell, and S. T. Bartlett (1998). The shrinking renal replacement therapy break-even point. *Transplantation 107*, 1702–1708.

[34] Sönmez, T. and M. U. Ünver (2005). House allocation with existing tenants: an equivalence," *Games and Economic Behavior 52*, 153-185.

[35] Sönmez, T. and M. U. Ünver (2006). Kidney exchange with good samaritan donors: A characterization. Working paper, Boston College and University of Pittsburgh.

[36] Terasaki, P. I., D. W. Gjertson, and J. M. Cecka (1998). Paired kidney exchange is not a solution to ABO incompatibility. *Transplantation 65*, 291.

[37] Zenios, S. A. (2002). Optimal control of a paired-didney exchange program. *Management Science 48*, 328-342.

[38] Zenios, S. A., E. S. Woodle, and L. F. Ross (2001). Primum non nocere: avoiding harm to vulnerable wait list candidates in an indirect kidney exchange. *Transplantation 72*, 648-54.