CS286r: Assignment 1

Due: Wednesday, September 22 in class

Points will be awarded for clarity, correctness and completeness of the answers. Submissions should be brought to class. You may work in a pair and submit only one solution, but you must understand your solution. *Good luck!*

Total points = 120 points + 10 bonus points.

Problem 1: Kelly Criterion

(1) (Negative Horse Race - 25 points) Consider a horse race with m horses with win probabilities p_1, p_2, ..., p_m. The gambler hopes that a certain horse will lose. He places bets $(b_1, b_2, ..., b_m)$ and $\sum_{i=1}^{m} b_i = 1$. Note that any bet b_i can be negative. After each horse race, he loses his bet b_i if horse i wins, and retains the rest of his bets. Assume that the gambler must invest all of his wealth in every race. Find the growth rate optimal investment strategy b^* and the optimal growth rate W^*.

(2) (St. Petersburg Paradox - 35 points) Many years ago in ancient St. Petersburg the following gambling proposition caused great consternation. You pay a fixed entry fee c to enter the game. A fair coin is flipped repeatedly. The pot starts at 1 dollar and is doubled every time the coin comes up as a head. As soon as the coin comes up as a tail, the game ends and you win all the money in the pot. We use X to denote the payoff of the game, which is a random variable.

   (a) (5 points) Show that the expected payoff for this game is infinite. For this reason, It was argued that a rational trader would be willing to pay an arbitrarily large entry fee to play this game. Discuss why or why not this argument makes sense.

   (b) (5 points) Suppose that a gambler is allowed to pay a fractional entry fee αc to enter the game and accordingly receive αX payoff at the end of the game for some nonnegative α of his choice. Consider a setting in which the St. Petersburg game is repeated for n rounds. The gambler is interested in maximizing his long-term payoff from the game. Suppose that in each round, the gambler invests all of his wealth in the game (i.e. αc equals the gambler's wealth). Let X_i denote the payoff of the game in round i when the gambler pays c. Identify the fair entry fee c^* for this game.

   (c) (15 points) Realistically, the gambler should be allowed to keep a proportion of his money as cash when participating in the repeated St. Petersburg game. Suppose that this is the case. Let $\vec{b} = (b, \bar{b})$, where b is the fraction of the wealth that the gambler invests in the St. Petersburg game and $\bar{b} = 1 - b$ is the fraction that he keeps as cash. Given an expression for the doubling rate $W(\vec{b}, \vec{p})$. Let b^* be the value of b maximizing the doubling rate $W(\vec{b}, \vec{p})$ for a fixed entry fee c. Then answer the following: (1) For what value of the entry fee c does the optimizing value b^* drop below 1? In general, how does b^* vary with c? (2) What is a fair entry fee c?

(d) (Super St. Petersburg - 10 points) Consider a Super St. Petersburg game where $Pr(X = 2^{2^k}) = 2^{-k}, k = 1, 2, ...$ Show that the expected log wealth is infinite for all $b > 0$, for all c.

Now, suppose that we wish to maximize the relative growth rate with respect to some other portfolio, say, $\vec{b} = (\frac{1}{2}, \frac{1}{2})$. Show that there exists a unique b maximizing

$$E \log \frac{\bar{b} + \frac{bX}{c}}{\frac{1}{2} + \frac{1}{2}\frac{X}{c}}.$$

Problem 2: Game Theory

(1) (Strategic Voting - 20 points) Three business partners Alice, Bob, and Carol are voting on whether to give themselves a pay raise. The raise is worth b but each person who votes for the raise incurs a cost of voter resentment equal to $c < b$. The outcome is decided by majority vote.

(a) (5 points) Draw an extensive form game tree for this problem, assuming that the three people vote sequentially and publicly. (Alice votes first. Bob sees Alice's vote and votes next. Carol sees both Alice's and Bob's votes and votes last.)

(b) (5 points) Find a Nash equilibrium for this game using backward induction. Show that it is best to go first.

(c) (10 points) Show that there are two Nash equilibria in which Carol always votes "no" regardless of Alice's and Bob's votes. Clearly state the strategies of these Nash equilibria. Discuss the problems of these equilibria and why they can't be found by backward induction.

(2) (Selten's Horse - 40 points + 10 bonus points) The game in Figure 1 is known as Selten's horse (named after Reinhardt Selten). The game starts with Alice at the top left node. The utilities specified are for Alice, Bob, and Carol in order.

(a) (15 points) Find the two pure strategy Nash equilibria of this game.

(b) (25 points) Find two types of mixed strategy Nash equilibria of this game in which Alice always plays a pure strategy. Prove that one of the two types of Nash equilibria is a sequential equilibrium and the other one is not a sequential equilibrium.

(c) (Optional - 10 points) Are there other types of Nash equilibria of this game? If your answer is yes, characterize these Nash equilibria. If your answer is no, explain why.

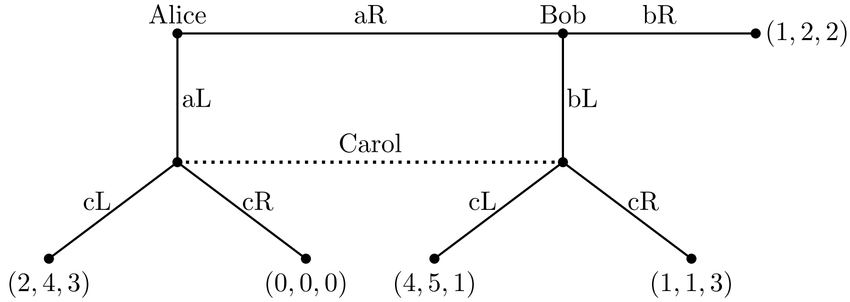

Figure 1: Selten's Horse