A SET OF INDEPENDENT NECESSARY AND SUFFICIENT CONDITIONS FOR SIMPLE MAJORITY DECISION[1]

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THE PROBLEM of the relation between group choice and individual preferences has been stated by Kenneth J. Arrow in terms of a "social welfare function" that gives group choice as a function of the preferences of the individuals making up the group.[2] One of the conditions that he puts on this function is that group choice concerning a set of alternatives must depend only upon individual preferences concerning the alternatives in that set.[3] In particular, group choice in the presence of just two alternatives depends only upon individual preferences with respect to this pair of alternatives. Since it follows that the pattern of group choice may be built up if we know the group preference for each pair of alternatives, the problem reduces to the case of two alternatives. We shall be concerned with the method of choice by simple majority vote, in order both to elucidate the nature of this familiar institution and to throw further light on Arrow's interesting results.

We assume n individuals and two alternatives x and y. Symbolizing "the ith individual prefers x to y" by xP_iy and "the ith individual is indifferent to x and y" by xI_iy, we assume that for each i one and only one of the following holds: yP_ix, yI_ix, or xP_iy. With each individual we associate a variable D_i that takes the values -1, 0, 1 respectively for each of these situations. Similarly, for the group, we write $D = -1, 0, 1$ according as yPx, yIx, or xPy, i.e., according as the group decision is in favor of y, indifference, or in favor of x.

The function in which we are interested is of the form

$$(1) \qquad D = f(D_1, D_2, \cdots, D_n).$$

It seems appropriate to call it a *group decision function.*[4] It maps the n-fold cartesian product $U \times U \times \cdots \times U$ onto U, where $U = \{-1, 0, 1\}$.[5] We are going to make four very weak assumptions concern-

[1] The author is indebted to K. J. Arrow, L. G. Field, D. Siplfe, and several referees for reading the draft of this paper and making helpful comments.

[2] Kenneth J. Arrow, *Social Choice and Individual Values*, Cowles Commission Monograph No. 12, New York: John Wiley and Sons, 1951, p. 23.

[3] See Arrow, p. 27. The realism of this condition may be questioned, but we introduce it here to provide the link between Arrow's work and the results of this paper.

[4] This terminology seems appropriate since group decision functions, as here defined, and the familiar statistical decision functions are both members of the family of functions whose values are decisions.

[5] The cartesian product $U \times U \times \cdots \times U$ is the set of all different ordered n-tuples $(D_1, D_2, \cdots, D_n)$ whose elements are members of U. Each member of the cartesian product represents a possible set of individual preferences.

ing this function and show that they form a set of independent necessary and sufficient conditions that it be just the familiar method of making group decisions by simple majority. In order to define this method precisely, we let $N(-1)$, $N(0)$, and $N(1)$ stand respectively for the number of -1's, 0's, and 1's in the decision function. Then "simple majority decision" means a decision function that yields $D = -1, 0, 1$ according as $N(1) - N(-1)$ is negative, zero, or positive. An equivalent definition is that $D = -1, 0, 1$ according as $\sum D_i$ is negative, zero, or positive.[6]

The first condition that we put on the group decision function is that each set of individual preferences leads to a defined and unique group choice.

CONDITION I: *The group decision function is defined and single valued for every element of* $U \times U \times \cdots \times U$.

We might describe this condition by saying that the method must be decisive and universally applicable, or more briefly *always decisive*, since it must specify a unique decision (even if this decision is to be indifferent) for any individual preferences.

The second condition is that each individual be treated the same as far as his influence on the outcome is concerned. This means that in $f(D_1, D_2, \cdots, D_n)$ we could interchange any two of the variables without changing the result.

CONDITION II: *The group decision function is a symmetric function of its arguments.*

This condition might well be termed anonymity, since it means that D is determined only by the values of the D_i that appear, regardless of how they are assigned to individuals as indicated by subscripts (names). A more usual label is *equality*.

The third condition is that the method of group decision does not favor either alternative.[7] A precise way of stating this is that if the names of x and y are reversed, the result is not changed. If the names x and y are interchanged, preferences are indicated by different values of the D's. It is a matter of interchanging -1 and 1 wherever they occur as values of D_i or D, since yP_ix, yI_ix, xP_iy become with the new names, xP_iy, xI_iy, and yP_ix. There is no change as far as 0 is concerned since the relation of indifference is assumed to be symmetric, i.e., yI_ix if and only if xI_iy. Thus if we have a statement about alternatives labeled in

[6] Our "simple majority decision" is the same as Arrow's "method of majority decision." See Arrow, p. 46. The function is the same as the so-called "signum function of ΣD_i.

[7] There are many situations where this neutrality is not desirable. It goes without saying that our purpose here is to illuminate the formal characteristics of simple majority decision and not to assert any special value or universality for the stated conditions.

one way, we get an equivalent statement by interchanging x with y and 1 with -1. But x and y do not appear in the decision function (1). Accordingly, if we have a true statement about this function, we must get another true statement from it by replacing each D by its negative. Another way of justifying this is by considering that we might have decided in the first place to assign the values -1 and 1 in the opposite way, and we do not want this to make any difference. Accordingly:

CONDITION III: $f(-D_1, -D_2, \cdots, -D_n) = -f(D_1, D_2, \cdots, D_n)$.

For obvious reasons the mathematical term "odd" does not seem convenient in this context, and we describe this property as *neutrality*.

The final condition that we place on the decision function is that it respond to changes in individual preferences in a "positive" way. By this we mean that if the group decision is indifference or favorable to x, and if the individual preferences remain the same except that a single individual changes in a way favorable to x, then the group decision becomes favorable to x. More precisely:

CONDITION IV: *If* $D = f(D_1, D_2, \cdots, D_n) = 0$ *or* 1, *and* $D_i' = D_i$ *for all* $i \neq i_0$, *and* $D_{i_0}' > D_{i_0}$, *then* $D' = f(D_1', \cdots, D_n') = 1$.

We call this *positive responsiveness*. It is slightly stronger than Arrow's Condition 2.[8]

We now state our theorem.

THEOREM: *A group decision function is the method of simple majority decision if and only if it is always decisive, egalitarian, neutral, and positively responsive.*

It is easy to see that Conditions I–IV are necessary. Simple majority decision as defined always gives a unique result. Also II holds since the value of the decision function depends only upon $N(-1)$ and $N(1)$ and is therefore independent of the position of the -1's, 0's, and 1's in f. Thirdly, the definition of simple majority decision remains unchanged under an interchange of 1 and -1, so that III is valid. Finally, a change of one vote breaks a tie, so that IV holds.

To show that Conditions I–IV are sufficient, we notice first that II implies that the value of f depends only upon the set of values of the variables and not upon their position in the function. Hence it depends only upon $N(-1)$, $N(0)$, and $N(1)$. It is easy to see that

$$N(-1) = N(1) \qquad \text{implies } D = 0. \tag{2}$$

For suppose that in this case $D = f(\{D_i\}) = 1$. Then from Condition III, $f(\{D_i'\}) = -1$ where $D_i' = -D_i$. But $f(\{D_i'\}) = f(\{D_i\})$ because

[8] See Arrow, p. 25. His condition might well be called positive monotonicity, since it requires merely nonnegative responsiveness.

of the equality of $N(-1)$ and $N(1)$ and the fact that $-0 = 0$. This violates the uniqueness required by Condition I. Similarly, $D \neq -1$, and the only other possibility is $D = 0$. Suppose now that $N(1) = N(-1) + 1$. Then by IV and (2), $D = 1$. By induction, using this result and IV, $D = 1$ for $N(1) = N(-1) + m$ where $0 < m \leqslant n - N(-1)$. Hence

(3) $N(1) > N(-1)$ implies $D = 1$.

From this and Condition III

(4) $N(1) < N(-1)$ implies $D = -1$.

Since (2)–(4) are just the definition of simple majority decision, the sufficiency is proved.

Since we have exhibited a function satisfying Conditions I–IV, they must be consistent. To show independence, it is sufficient to exhibit functions that violate each one while satisfying all the others. We indicate such a function for each condition, leaving it to the reader to verify that each does satisfy all but the specified condition.

(I). $D = 1$ for $N(1) - N(-1) \geqslant 0$, $D = -1$ for $N(1) - N(-1) \leqslant 0$.

(II). $D = -1, 0, 1$ according as $D_1 + N(1) - N(-1)$ is less than, equal to, or greater than zero. (A kind of plural voting.)

(III). $D = -1, 0, 1$ according as $N(1) - 2N(-1)$ is less than, equal to, or greater than zero. (The familiar two-thirds majority rule.)

(IV). $D = -1, 0, 1$ according as $N(1) - N(-1)$ is greater than, equal to, or less than zero. (A more familiar example is the rule for jury decision in which $D = -1, 1$, or 0 according as $N(-1) = n$, $N(1) = n$, or otherwise.)

Arrow's "Possibility Theorem for Two Alternatives" asserts that simple majority decision satisfies his Conditions 2–5 applied to two alternatives [Arrow, pp. 46–48]. It follows that our Conditions I–IV imply his Conditions 2–5. That his are actually weaker conditions may be seen by noting that the example above for which Condition III fails satisfies his Conditions 2–5.[9] In Arrow's terms our theorem may be expressed by saying that any social welfare function (group decision function) that is not based on simple majority decision, i.e., does not decide between any pair of alternatives by majority vote, will either fail to give a definite result in some situation, favor one individual over another, favor one alternative over the other, or fail to respond positively to individual preferences. The fact that Arrow's conditions are still weaker emphasizes the importance of his result that his conditions are incompatible with transitivity [Arrow, p. 59].

[9] See Arrow, pp. 26–30. It is also evident that Arrow's Conditions 2–5 do not imply either Conditions II or IV.

So far we have been concerned only with a single pair of alternatives. Suppose now we have a set of alternatives and that each individual when confronted with any pair either prefers one or the other or is indifferent. For each pair we can construct a decision function. According to the remark made at the beginning of this article, if we accept Arrow's Condition 3, we can build up a function that gives a set of group preferences corresponding to each set of individual preferences. It follows that any social welfare function that satisfies our Conditions I–IV and Arrow's Condition 3 must be constructed in this way. As is well-known, a social welfare function built in this way may lead to group preferences that are nontransitive even if the individual preferences are transitive [Arrow, pp. 2–3, 59]. Nontransitivity of group preferences follows *a fortiori* if the individual preferences are nontransitive. Accordingly any social welfare function satisfying Conditions I–IV for each pair of alternatives will lead to intransitivity unless the possible individual preferences are severely restricted.

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