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THE FAIR DIVISION OF A FIXED SUPPLY AMONG A GROWING POPULATION*†

WILLIAM THOMSON

Harvard University

We reconsider the traditional problem of fair division. Division principles should be general enough to accommodate changes in what is to be divided as well as variations in the number of agents among whom the division is to take place. In the usual treatment of the question, this number is assumed to be fixed. Here, we explicitly allow it to vary.

Agents are assumed to have von Neumann–Morgenstern utility functions and division problems are defined as subsets of the utility space. Our approach is axiomatic. Apart from the familiar axioms of Pareto-optimality, scale invariance, continuity and anonymity, we formulate and impose an axiom of *monotonicity with respect to changes in the number of agents*, stating that if sacrifices have to be made to support an additional agent, then everybody should contribute. There is a unique division principle that satisfies them all: it is the natural extension to the n-person case of the two-person solution proposed by Kalai and Smorodinsky.

1. Introduction. We reconsider here the problem of fair division. A *division problem* arises when agents in some group have conflicting preferences over the possible divisions of a list of items to which they are collectively entitled. A *division principle* or *solution* proposes an equitable resolution of all division problems in a given class.

In the usual study of the question, the number of agents n is assumed to be fixed; the present paper allows for variations in n and investigates the existence of division principles which in addition to satisfying a few standard properties, respond appropriately to such variations.

To illustrate what we mean by "appropriately" we take as our starting point the particular outcome reached by applying a given division principle to a specific division problem faced by some group of agents; then, we let an additional agent enter the scene.

We require that the new agent be placed on the same footing as the original agents: all agents have the "same" claims on the aggregate resources. This means that some of these resources will have to be transferred to him, and if the initial distribution was efficient, as we will also demand, some of the original agents will typically lose. It is conceivable, however, that some of them will gain and it is this phenomenon that we will forbid: all of the original agents should share in the new responsibilities of the group. If sacrifices have to be made to support one more person, all should participate.

Although the division of a bundle of goods is important to motivate this paper, we have adopted a somewhat more abstract formulation. We assume agents to have von Neumann–Morgenstern utility functions and we define division problems as subsets of the utility space in a certain class. We investigate the existence of division principles defined on that class and satisfying several axioms: on the one hand, the familiar axioms of *weak Pareto-optimality, anonymity, scale invariance* and *continuity*; and on the other the new axiom of *monotonicity with respect to changes in the number of agents*

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informally described above. We show that there is a unique division principle that satisfies them all: it is the natural generalization of the two-person solution proposed by Kalai and Smorodinsky [2], subsequently referred to as the *KS solution*.

The paper is organized as follows: §2 sets up the framework of analysis. §3 presents the results, the proof of the main theorem being given only for a simple case. §4 is devoted to some final remarks. The appendix contains a general proof of the main theorem.

2. Division problems. Solutions. The Kalai–Smorodinsky solution. An n-person division problem S is a subset of R^n representing in some von Neumann–Morgenstern utility scales the utility vectors available to a group of n agents. Σ^n is the class of n-person division problems satisfying the following assumptions:

(a) S is a *compact convex subset of* R^n_+ *containing at least one strictly positive vector.*

(b) For all $x, y \in R^n_+$, if $x \in S$ and $x \geqslant y$, then $y \in S$. We will say that S is *comprehensive*. (This implies that the projection of S onto a coordinate subspace coincides with the intersection of S with that subspace.)

An *n-person solution* on Σ^n is a function associating to every S in Σ^n a unique point of S, interpreted as the "best" compromise among the agents' conflicting preferences.

The *n-person Kalai–Smorodinsky solution* [2] is defined as follows: for each i, $i = 1, \ldots, n$, let $a_i(S) \equiv \max_{x \in S} x_i$; the maximal (according to the partial ordering of R^n) element of S on the segment connecting the origin to $a(S)$ is the compromise recommended by these authors for S. This n-person solution is denoted K^n.

We offer here an axiomatic characterization of this solution, in a framework that allows the number of agents to vary. This requires a slight generalization of the concepts just introduced.

3. The axioms. The characterization result. $I = \{1, 2, \ldots\}$ is the (infinite) universe of "potential" agents. Agent i in I is indexed by the subscript i. $\mathscr{P}$ is the class of finite subsets of I. The elements of $\mathscr{P}$ are denoted $P, Q, \ldots$ and their respective cardinalities $|P|, |Q|, \ldots$. Given P in $\mathscr{P}$, R^P_+ is the Cartesian product of $|P|$ copies of R_+, indexed by the members of P, and Σ^P is the class of subsets of R^P_+ with the properties (a) and (b) previously required of the elements of Σ^n (see §2). Each element of Σ^P represents a division problem that the members of P may conceivably face.

A *solution* F is a list $\{F^P, P \in \mathscr{P}\}$ where each F^P associates to each S in Σ^P a unique point of S, denoted $F^P(S)$, and called the *solution outcome for* S. The *Kalai–Smorodinsky solution* K is the list obtained by setting F^P equal to $K^{|P|}$ (that we will usually write K^P) for all P in $\mathscr{P}$. The axiomatic study of this solution proposed here will involve the following axioms:

Weak Pareto-optimality (WPO). For all $P \in \mathscr{P}$, for all $S \in \Sigma^P$, for all $y \in R^P_+$, if $y > F^P(S)$, then $y \notin S$.

Pareto-optimality (PO). For all $P \in \mathscr{P}$, for all $S \in \Sigma^P$, for all $y \in R^P_+$, if $y \geqslant F^P(S)$, then $y \notin S$.

Anonymity (An). For all $P, P' \in \mathscr{P}$ with $|P| = |P'|$, for all one-to-one functions γ from P to P', for all $S \in \Sigma^P$, for all $S' \in \Sigma^{P'}$, if $S' = \{x' \in \Sigma^{P'} \mid \exists x \in S \text{ s.t. } \forall i \in P, x'_{\gamma(i)} = x_i\}$, then for all $i \in P$, $F^{P'}_{\gamma(i)}(S') = F^P_i(S)$.

Given P in $\mathscr{P}$, let Λ^P be the class of functions λ from R^P_+ into R^P_+ defined as follows: for each i in P, there exists a positive real number a_i such that for each x in R^P_+, $\lambda_i(x) = a_i x_i$. Given S in Σ^P, $\lambda(S)$ is defined to be $\{y \in R^P_+ \mid \exists x \in S \text{ s.t. } y = \lambda(x)\}$. Note that $\lambda(S)$ is also in Σ^P.

Scale Invariance (S. Inv.). For all $P \in \mathscr{P}$, for all $S \in \Sigma^P$, for all $\lambda \in \Lambda^P$, $F^P(\lambda(S)) = \lambda(F^P(S))$.

Continuity (Cont). For all $P \in \mathscr{P}$, for all sequences $\{S^k\}$ of elements of Σ^P converging in the Hausdorff topology to some $S \in \Sigma^P$, $F^P(S^k) \to F^P(S)$.

Monotonicity with respect to changes in the number of agents (Mon). For all $P, Q \in \mathscr{P}$ with $P \subseteq Q$, for all $S \in \Sigma^P$, for all $T \in \Sigma^Q$, if $S = T \cap R^P$, then for all $i \in P$, $F_i^P(S) \geqslant F_i^Q(T)$. (We write $S = T \cap R^P$ with a slight abuse of notation.)

All of these axioms, except Mon, are familiar axioms, appropriately reformulated for the situation under study. Mon relates solution outcomes across cardinalities: T is the division problem faced by the large group Q. S is the division problem that the members of the subgroup P of Q would face if the rights of the agents in $Q \setminus P$ were ignored and each of them were assigned utility 0. The requirement is that none of the members of P should end up better off after the rights of the agents in $Q \setminus P$ have been recognized.

Other Notations. Given P in $\mathscr{P}$, e^P denotes the vector of R^P which has all of its coordinates equal to 1. Given P in $\mathscr{P}$, m subsets $S^1, \ldots, S^m$ of R_+^P, n points $x^1, \ldots, x^n$ of R_+^P, $\operatorname{cch}\{S^1, \ldots, S^m; x^1, \ldots, x^n\}$ denotes the convex and comprehensive hull of $T \equiv S^1 \cup \ldots \cup S^m \cup \{x^1, \ldots, x^n\}$, i.e., it is the smallest convex and comprehensive subset of R_+^P containing T.

THEOREM 1. *The Kalai–Smorodinsky solution satisfies WPO, An, S. Inv, Cont and Mon.*

PROOF. That K satisfies WPO, An, S. Inv. and Cont is easily verified. To show that K also satisfies Mon, let P, Q, S, T be as in the statement of that axiom. We have to establish that for all i in P, $K_i^P(S) \geqslant K_i^Q(T)$. To see this, note that by S. Inv, we can assume that $a(T) = e^Q$ and therefore $a(S) = e^P$. This means that $K^Q(T)$ is a point in R^Q of equal coordinates, say a, and similarly, that $K^P(S)$ is a point in R^P of equal coordinates, say b. Then the conclusion that we desire, namely $a \leqslant b$, follows directly from the comprehensiveness of T, and the fact that $K^P(S)$ is weakly Pareto-optimal in S. ∎

THEOREM 2. *A solution $F = \{F^P, P \in \mathscr{P}\}$ satisfies WPO, An, S. Inv and Mon if and only if $F \geqq K$, i.e., for all P in $\mathscr{P}$ and for all S in Σ^P, $F^P(S) \geqq K^P(S)$.*

PROOF. We will limit ourselves here to proving the statement for the case when P has cardinality 2. This case lends itself to a simple geometric illustration (Figure 1). The case of P's of arbitrary cardinality is given in the appendix.

We first observe that by An, F^P depends only on the cardinality of P. To fix the ideas, we then choose $P = \{1, 2\}$ and we observe that by S. Inv it is enough to prove

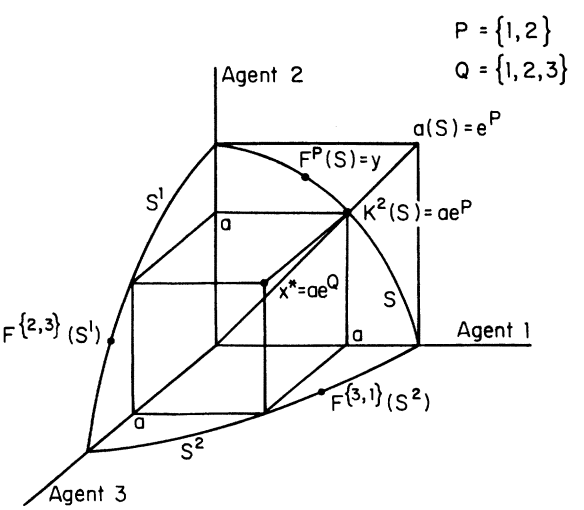

FIGURE 1

that $F^P(S) \geqq K^P(S)$ for all S in Σ^P with $a(S) = e^P$. We argue by contradiction. If this were not the case, there would exist some S in Σ^P with $a(S) = e^P$ but not $F^P(S) \geqq K^P(S)$. Note that the fact that $a(S) = e^P$ implies that $K^P(S)$ has equal coordinates. Let a be their common value. We have either $F_1^P(S) < a$ or $F_2^P(S) < a$. By An we can assume the former. Let $b \equiv F_1^P(S)$.

Now we introduce a third agent whom, by An, we can take to be agent 3, and we construct a division problem T in Σ^Q, where $Q \equiv \{1,2,3\}$, such that each of the three agents is involved in a 2-person subproblem of T analogous to S and in a position analogous to that held by agent 1 in S. This is achieved by first "replicating" S in $R^{\{2,3\}}$ and $R^{\{3,1\}}$ with agents 2 and 3 respectively playing the role played by agent 1 in S. Formally, we introduce

$$S^1 = \left\{(y_2, y_3) \in R^{\{2,3\}} \mid \exists (x_1, x_2) \in S \text{ with } y_2 = x_1 \text{ and } y_3 = x_2\right\},$$

$$S^2 = \left\{(y_3, y_1) \in R^{\{3,1\}} \mid \exists (x_1, x_2) \in S \text{ with } y_3 = x_1 \text{ and } y_1 = x_2\right\}.$$

By An applied twice we have

$$F_2^{\{2,3\}}(S^1) = F_3^{\{1,3\}}(S^2) = F_1^P(S) = b.$$

We next introduce $x^* \equiv ae^Q$ and we finally define $T \subseteq R^Q$ by $T \equiv \text{cch}\{S, S^1, S^2, x^*\}$. It is clear that T is an element of Σ^Q, and that $T \cap R^P = S$, $T \cap R^{\{2,3\}} = S^1$ and $T \cap R^{\{3,1\}} = S^2$.

By Mon, we conclude that $F_1^Q(T) \leqslant b$, i.e., what agent 1 gets in T should be no greater than what he gets in $T \cap R^P = S$, $F_1^P(S)$. Since, by construction, T is invariant under rotations of the agents, we also conclude that $F_2^Q(T) \leqslant b$ and $F_3^Q(T) \leqslant b$. These three inequalities together yield $F^Q(T) \leqq be^Q$. However, $x^* = ae^Q$ belongs to T and since $b < a$, we obtain $F^Q(T) < x^*$ in contradiction with WPO. ∎

Given P in $\mathscr{P}$, let now $\tilde{\Sigma}^P$ be the subset of Σ^P of division problems S that satisfy

(c) For all $x, y \in S$, if $y \geqslant x$, there exists $z \in S$ with $z > x$.

For such a division problem, any weakly Pareto-optimal point x is in fact Pareto-optimal. Then, we have

COROLLARY 1. *If a solution $F = \{F^P, P \in \mathscr{P}\}$ satisfies WPO, An, S. Inv and Mon, then for all P in $\mathscr{P}$ and for all S in $\tilde{\Sigma}^P$, $F^P(S) = K^P(S)$.*

This is a straightforward consequence of Theorem 2. We also have:

COROLLARY 2. *There is no solution satisfying PO, An, S. Inv and Mon.*

PROOF. It relies on the following example, in which $P \equiv \{1,2\}$, $Q \equiv \{1,2,3\}$, $T \equiv \text{cch}\{(2,1,1), (0,2,0), (0,0,2)\}$. One easily checks that $K^Q(T) = e^Q$. If $F = \{F^P, P \in \mathscr{P}\}$ is a solution satisfying PO, An, S. Inv and Mon, then by Theorem 2 and PO, $F^Q(T) = (2,1,1)$. Let $S \equiv T \cap R^P = \text{cch}\{(2,1), (0,2)\}$. We have $K^P(S) = (4/3)e^P$. Since this point is Pareto-optimal for S, Theorem 2 yields that $F^P(S) = (4/3)e^P$. But then we have $F_1^Q(T) = 2 > 4/3 = F_1^P(T \cap R^P)$, in contradiction with Mon. ∎

This result shows that we have to be satisfied with WPO if we want the other axioms. Theorem 2 suggests that there may be solutions other than the KS solution that satisfy WPO, An, S. Inv and Mon. This is formally established in the next lemma.

LEMMA 1. *The Kalai–Smorodinsky solution is not the only solution to satisfy WPO, An, S. Inv and Mon.*

PROOF. It relies on the following example: let $Q \equiv \{1,2,3\}$ and T in Σ^Q be defined as $T \equiv \text{cch}\{(1,0,1), (0,1,1)\}$. Set $F^Q(T) \equiv (1/2, 1/2, 1)$, For any T' in Σ^Q which can be obtained from T by a positive linear transformation, set $F^Q(T')$ equal to the image

of $(1/2, 1/2, 1)$ under this transformation. For any other T' in Σ^Q, set $F^Q(T') = K^Q(T')$. For any other Q' in $\mathscr{P}$ of cardinality 3, define $F^{Q'}$ from F^Q by applying An. For any other P in $\mathscr{P}$, set $F^P = K^P$. We omit the verification that the solution so constructed satisfies the four axioms. ∎

The solution, however, is not continuous. At this point we will, therefore, impose continuity. Then we have:

THEOREM 3. *If a solution satisfies WPO, An, S. Inv, Cont and Mon, then it is the Kalai–Smorodinsky solution.*

PROOF. It is a straightforward consequence of Theorem 2 and the fact that for all P in $\mathscr{P}$, any element of Σ^P can be approximated by a sequence of elements of $\tilde{\Sigma}^P$. ∎

Theorems 1 and 3 together constitute the announced characterization of the Kalai–Smorodinsky solution.

4. Concluding comments.

4.1. *Restriction on the domain of admissible division problems.* If we had worked directly on the domain $\tilde{\Sigma}^P$ of division problems satisfying (a) (b) *and* (c), we would have:

THEOREM 4. *A solution $F = \{F^P, P \in \mathscr{P}\}$, where each F^P is now defined on $\tilde{\Sigma}^P$, satisfies PO, An, S. Inv and Mon if and only if it is the Kalai–Smorodinsky solution.*

PROOF. Sufficiency is a straightforward consequence of Theorem 1 since on the smaller domain used here WPO is equivalent to PO.

Necessity requires an adaptation of Theorem 2. In the proof given above for the case $|P| = 2$, set x^* equal to ce^Q with $c = (a + b)/2$, instead of $x^* = ae^Q$. The proof presented in the appendix, for arbitrary $|P|$, would require a slightly more complicated adaptation. ∎

It is important to note that no use is made of Cont in this theorem.

4.2. *The case of a finite number of potential agents.* Instead of assuming that the number of potential agents is infinite, we suppose here that there is some positive integer n representing the maximal number of agents among whom a division is ever to be considered. $I = \{1, \ldots, n\}$ is now the universe of potential agents. We correspondingly introduce the class $\mathscr{P}^n$ of subsets of I and we defined a *solution on* $\mathscr{P}^n$ as a finite list $\{F^P, P \in \mathscr{P}^n\}$, where each F^P is as before. The axioms introduced in §2, formulated for solutions, are rewritten to apply to solutions on $\mathscr{P}^n$ by simply replacing $\mathscr{P}$ by $\mathscr{P}^n$ in their statements. The question now arises whether the KS solution can be characterized as the only solution on $\mathscr{P}^n$ to satisfy WPO, An, S. Inv, Cont and Mon.

The sufficiency (Theorem 1) would not be affected, of course, by this domain restriction. But the necessity would. The general proof of Theorem 2 given in the appendix consists in constructing for each P in $\mathscr{P}$ and each division problem S in Σ^P another division problem involving $3|P| - 2$ agents; therefore if *F is a solution on $\mathscr{P}^n$ satisfying WPO, An, S. Inv and Mon, then for all P in $\mathscr{P}^n$ such that $3|P| - 2 \leqslant n$, $F^P \geqq K^P$.* Other methods of proof can strengthen this result: for instance, one can show that if $|P| = 2$, then $F^P \geqq K^P$ be introducing only *one* additional agent, as is done in §3; also that if $|P| = 3$, then $F^P \geqq K^P$ by introducing only *two* additional agents. Since no attempt was made in the proof of Theorem 2 to minimize the number of additional agents (a question that seems to be worth investigating), the underlined result can indeed be strengthened.

It is clear, however, that the solution on $\mathscr{P}^n$ obtained by setting $F^P = K^P$ for all P in $\mathscr{P}^n$ will *not* be the only one to satisfy the five axioms. Another example is obtained by setting $F^P = K^P$ for all P in $\mathscr{P}^n$ different from I, and F^I equal to the n-person

solution which associates with every S in Σ^I the maximizer of the "Nash product" $\prod_{i=1}^n x_i$ over the points of S satisfying the inequalities required by Mon and the selection just made of the other components of F. (The set over which this maximization takes place is not empty since $K^I(S)$ is in it. It is also easy to verify that the solution on $\mathscr{P}^n$ so defined satisfies the five axioms.)

4.3. *Independence of irrelevant alternatives.* If the axiom of scale invariance is replaced by the axiom of *Independence of Irrelevant Alternatives* stating that the solution outcome of any choice problem remains the solution outcome of any subproblem that still contains it (see Nash [4]), but the other axioms are left unchanged, a characterization of the *egalitarian solution* is obtained. This is the solution associating to every S the unique maximal element of S with equal coordinates. This result is proved in Thomson [7].

4.4. *Other remarks on domains.* Kalai and Smorodinsky's paper is written for the class of *two-person bargaining problems* as introduced by Nash [4]. (See also Luce and Raiffa [3].) These are pairs (S,d) where S is a convex and compact subset of R^2 representing the utility vectors available to the two agents, and d, the disagreement point, is a point of S strictly dominated by some other point of S. Our formulation amounts to assuming that d is the origin. We could as well have left d arbitrary and strengthened the axiom of scale invariance to include invariance under translations of the origins of the utility functions. We chose our formulation to simplify the notation. Requiring that all points of S dominate d, as we have, guarantees that individual rationality will be satisfied. (This is the condition that no agent ends up worse off than he would be at the disagreement point.) Our condition of comprehensiveness will hold in particular if free disposability of utilities is assumed. It is an important condition, since without it the n-person KS solution would select outcomes that would not necessarily be weakly Pareto-optimal, as soon as $n \geqslant 3$, as discussed by Roth [5].

The family Σ^n of division problems used in this paper is of particular interest to economists since the image in the utility space of the set of possible distributions of a given bundle of infinitely divisible and freely disposable commodities, among a group of n agents with continuous, concave and nondecreasing utility functions, normalized by assigning zero utility to the zero bundle, is an element of Σ^n. (No randomization is needed then to obtain convexity.) An element of $\tilde{\Sigma}^n$ is obtained by requiring that utility functions be in fact strictly increasing.

Finally, we note that the axiomatic characterization offered by Kalai and Smorodinsky for their solution in the 2-person case, can be extended in various ways to the n-person case, n being considered a fixed number (see Roth [5], Segal [6], and Imai [1], who is concerned with a lexicographic version of the solution).

Appendix. This appendix contains the general proof of Theorem 2.

THEOREM 2. *A solution $F = \{F^P, P \in \mathscr{P}\}$ satisfies WPO, An, S. Inv and Mon if and only if $F \geqq K$.*

PROOF. Suppose by way of contradiction that for some P in $\mathscr{P}$ and some S in Σ^P, it is not the case that $y \geqq x$, where $y \equiv F^P(S)$ and $x \equiv K^P(S)$. Let i_o in P be such that $y_{i_o} = \min_{i \in P} y_i \equiv b$. By S. Inv, we can assume that S is normalized so that $a(S) = e^P$, and by An, that P contains agent 1 and that in fact $i_o = 1$. x is the only weakly Pareto-optimal point of S with equal coordinates, say a. We have $a > b$.

The proof consists in constructing a division problem T involving the original members of P as well as some new agents, and such that each of them faces one subproblem identical to the one faced by agent 1 in S.

Let $P^1 \equiv P \backslash \{1\}$, P^2 and P^3 be two elements of $\mathscr{P}$ disjoint from each other as well as disjoint from P^1 and such that $|P^2| = |P^3| = |P^1|$ and let $Q \equiv P \cup P^2 \cup P^3$. We construct T in Σ^Q by first specifying its intersections with various subspaces of R^Q of dimension equal to $|P|$. First, let γ^1 be a one-to-one function from P^1 to P^2. For each i in P^1, let $S^i \subseteq R_+^{P^2+\cup\{i\}}$ be defined by

$$S^i \equiv \left\{ x \in R_+^{P^2 \cup \{i\}} \mid \exists y \in S \text{ s.t. } x_i = y_1 \text{ and } \forall j \in P^1, x_{\gamma^1(j)} = y_j \right\}.$$

S^i is a replica of S in which agent i plays the role played by agent 1 in S, and the agents in P^2 play the role played by the agents in P^1.

The operation that was just performed for each of the members of P^1 in relation to P^2 is then successively carried out for each of the members of P^2 in relation to P^3, and for each of the members of P^3 in relation to P^1. More precisely, let γ^2 be a one-to-one function from P^2 to P^3, and for each i in P^2 let $S^i \subseteq R_+^{P^3 \cup \{i\}}$ be defined by

$$S^i \equiv \left\{ x \in R^{P^3 \cup \{i\}} \mid \exists y \in S \text{ s.t. } x_i = y_1 \text{ and } \forall j \in P^1, x_{\gamma^2 o \gamma^1(1)} = y_j \right\}$$

and finally, for each i in P^3, let $S^i \subseteq R^{P^1 \cup \{i\}}$ be defined by

$$S^i \equiv \left\{ x \in R_+^{P^1 \cup \{i\}} \mid \exists y \in S \text{ s.t. } x_i = y_1 \text{ and } \forall j \in P^1, x_j = y_j \right\}.$$

Let P_i denote $P^2 \cup \{i\}$ for i in P^1, $P^3 \cup \{i\}$ for i in P^2 and $P^1 \cup \{i\}$ for i in P^3.

Obviously, for each i in $\bigcup_{j=1}^3 P^j$, S^i belongs to Σ^{P_i} and An permits to conclude that $F_i^{P_i}(S^i) = F_1^P(S) = b$. Finally let $x^* \equiv ae^Q$ and $T \subseteq R^Q$ be defined by $T \equiv \text{cch}\{S, S^i, i \in \bigcup_{j=1}^3 P^j; x^*\}$. It is clear that T is in Σ^Q, that $T \cap R^P = S$ and for each i in $\bigcup_{j=1}^3 P^j$, $T \cap \mathrm{R}^{P_i} = S^i$.

Applying Mon to agent 1 by comparing what he gets in T to what he gets in $T \cap R^P = S$, we obtain $F_1^Q(T) \leqslant F_1^P(S) = b$.

Similarly, applying Mon to each agent i in $\bigcup_{j=1}^3 P^j$ by comparing what he gets in T to what he gets in $T \cap R^{P_i} = S^i$, we obtain $F_i^Q(T) \leqslant F_i^{P_i}(S^i) = b$.

These inequalities yield $F^Q(T) \leqq be^Q$. However, $x^* = ae^Q$ belongs to T and since $a > b$, we obtain a violation of WPO. ∎

The argument is illustrated in Figures 2 and 3. In Figure 2, it is schematically indicated with which group each agent is associated so as to face a division problem identical to that faced by agent 1 in S.

In Figure 3, the case $|P| = 2$ is illustrated somewhat more precisely, with $P = \{1,2,3\}$, $P^1 = \{2,3\}$, $P^2 = \{4,5\}$, $P^3 = \{6,7\}$ and $Q = \{1, \ldots, 7\}$. The division problem which is to be replicated involves agent $i_o = 1$, and agents 2 and 3. It is indicated

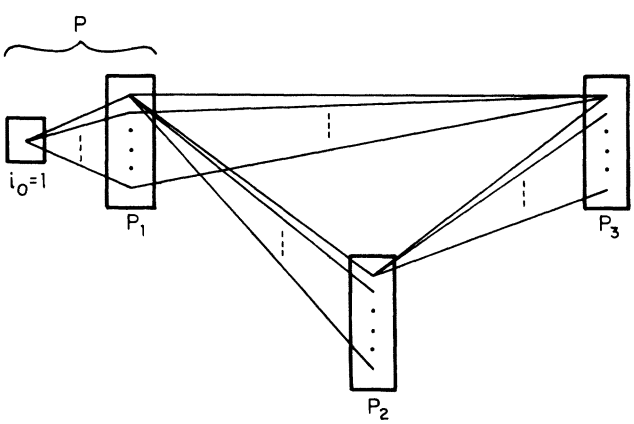

FIGURE 2

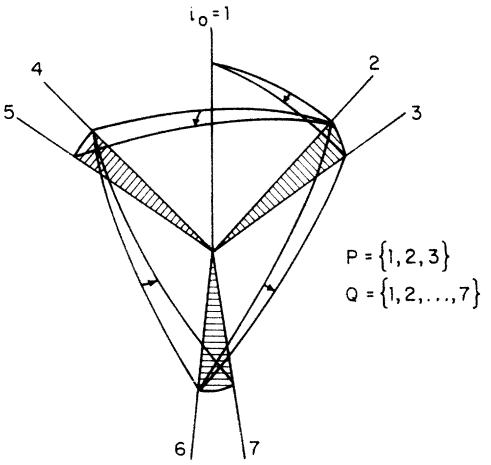

FIGURE 3

by its three intersections with the three coordinate planes and the completion in the third dimension, represented by the arrow. Agents 2 and 3 are each to the group $\{4,5\}$ as agents 4, 5 are each to the group $\{6,7\}$ and agents 6, 7 are each to the group $\{2,3\}$ and as agent 1 is to the group $\{2,3\}$. This means that agent 2 is to agent 3 as agent 4 is to agent 5 and as agent 6 is to agent 7 (yielding the shaded division problems). The proof first specifies the intersections S^i of T with the following subspaces of R^Q: $R^{\{123\}}$, $R^{\{245\}}$, $R^{\{345\}}$, $R^{\{467\}}$, $R^{\{567\}}$, $R^{\{623\}}$, $R^{\{723\}}$. The construction of T is completed by adding the point x^* of coordinates all equal to a, and taking the convex and comprehensive hull of S, the S^i and x^*.

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DEPARTMENT OF ECONOMICS, HARVARD UNIVERSITY, CAMBRIDGE, MASSACHUSETTS 02138