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SOCIAL CHOICE SCORING FUNCTIONS*

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Abstract. Let a committee of voters be considering a finite set $A = \{a_1, a_2, \cdots, a_m\}$ of alternatives for election. Each voter is assumed to rank the alternatives according to his preferences in a strict linear order. A social choice function is a rule which, to every finite committee of voters with specified preference orders, assigns a nonempty subset of A, interpreted as the set of "winners". A social choice function is consistent if, whenever two disjoint committees meeting separately choose the same winner(s), then the committees meeting jointly choose precisely these winner(s). The function is symmetric if it does not depend on the names of the various voters and the various alternatives. It is shown that every symmetric, consistent social choice function is obtained (except for ties) in the following way: there is a sequence $s_1, s_2, \cdots, s_m$ of m real numbers such that if every voter gives score s_i to his ith most preferred alternative, then the alternative with highest score (summed over all voters) is the winner.

1. Introduction. A collective decision process may be described in the following terms. A group of individuals—a committee or electorate, for example—is presented with a number of alternatives (motions, or candidates) and the committee members (called *voters*) are to decide collectively which alternatives are best. It is assumed that, by debate, natural predisposition, and so forth, each voter arrives at some ordering of the alternatives in accordance with his preferences. For the present discussion we shall assume that each such *preference order* p is a linear order, i.e., a complete, irreflexive, transitive relation on the set of alternatives $A = \{a_1, a_2, \cdots, a_m\}$. We shall represent p by a column vector

$$p = \begin{pmatrix} a_{i_1} \\ a_{i_2} \\ \cdot \\ \cdot \\ \cdot \\ a_{i_m} \end{pmatrix},$$

where the top alternative is most preferred. A given assignment of preference orders to the voters from a finite set V will be called a preference *profile* for V. A *social choice function* of *order m* is then a function that assignes to every preference profile a nonempty subset of the m-set, A, called a *choice* set. This notion is to be distinguished from a *social preference function*, which associates with each profile a complete (weak) social *ordering* of the alternatives.

When there are only two alternatives to choose from, the method of simple majority rule seems to be the most natural and commonly used social choice function. But for three or more alternatives there is no completely natural extension of simple majority rule, as was pointed out nearly two centuries ago by the Marquis de Condorcet [3]. As a result, a great variety of rules are used in practice for group decision making when three or more alternatives are involved.

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These include the following methods: plurality, Borda, Condorcet, sequential voting (as in the U.S. House of Representatives), exhaustive voting, and double election, to name but a few. For a discussion of some of these methods see Black [2].

What is needed is an axiomatic framework for comparing the merits of these various methods. This type of study was begun by Arrow [1], who identified a set of conditions that permit only dictatorship when three or more alternatives are involved. This result (and later refinements of it) tell us much about what cannot be done, but leaves open the problem of defining what can be done. In the case of social choice functions on two alternatives, some additional results have been obtained, notably by May [5], who characterized simple majority rule by a very pleasing set of axioms, and by Fishburn [4], who characterized the so-called representative systems. In [4], Fishburn also investigates extensively other aspects of social choice functions, but in general he does not consider the effects of varying the size of the electorate, which will be one of our chief interests.

The object of this paper is to study two natural conditions on social choice functions, and to describe precisely the family of all functions satisfying these two conditions.

To agree on notation, let $\mathbb{N}$, the set of nonnegative integers, constitute names for the voters, and let $\mathcal{P}$ denote the set of $m!$ distinct preference orders on the alternative set A. A is assumed to be fixed throughout this paper. For any finite $V \subset \mathbb{N}$, a *profile* is simply a function from V to $\mathcal{P}$, and a social choice function (abbreviated SCF) is a function from the set X of all profiles to the family of nonempty subsets of A. A social choice function is said to be *anonymous* if it depends only on the number of voters associated with each preference order. We can represent the domain of an anonymous SCF by $\mathbb{N}^{m!}$, i.e., the set of all $m!$-tuples with nonnegative integer coordinates, indexed by $\mathcal{P}$, where for any $x \in \mathbb{N}^{m!}$ and $p \in \mathcal{P}$, x_p represents the number of voters having preference order p. The zero vector in $\mathbb{N}^{m!}$ represents the case of no voters—the empty society.

Let S_m be the group of permutations of the index set $\{1, 2, \cdots, m\}$. Each $\sigma \in S_m$ induces permutations (which we also denote by σ) of the alternatives, and hence of the profiles, in the natural way. We say that a SCF f is *neutral* if $f \circ \sigma = \sigma \circ f$ for all $\sigma \in S_m$. If f is both anonymous and neutral, it is said to be *symmetric*. Symmetry simply means that the various voters and the various alternatives are treated equally, i.e., without bias. This seems to be a very natural requirement for most group decision situations.

If f is a symmetric SCF with domain $\mathbb{N}^{m!}$ (indexed by $\mathcal{P}$) then the permutation of coordinates of $\mathbb{N}^{m!}$ induced by σ can be conveniently represented by a permutation matrix M_σ, and we have

$$(1) \qquad f(M_\sigma(x)) = \sigma(f(x)) \quad \text{for all } x \in \mathbb{N}^{m!}.$$

Notice that if f is symmetric and x is a fixed point of M_σ, then by (1), $f(x)$ must be fixed by all powers of σ, and hence may be multiple-valued.

Suppose that each of two individuals, 1 and 2, would choose a_i over all other alternatives if the choice were made by him alone. The Pareto principle then asserts that a_i should be the unique choice of the group consisting of individuals 1 and 2 together. We may extend this notion by requiring that, if A' is the choice of a voter group V', and A'' is the choice set of another voter group

V'' disjoint from V', and if $A' \cap A'' \neq \varnothing$, then the group $V' \cup V''$ should choose precisely the alternatives in $A' \cap A''$. Indeed, in the opinion of both groups, any alternative in $A' \cap A''$ is at least as "good" as any other alternative, while any alternative not in $A' \cap A''$ is not as "good" as any alternative in $A' \cap A''$, in the opinion of at least one of the groups. Any SCF with the above property will be called *consistent.* (This notion was originally introduced in [8].) Independently, a similar notion for social preference functions has been considered by Smith [7].

If f is anonymous, then the consistency condition can be expressed very simply as follows: for all $x', x'' \in \mathbb{N}^{m!}$,

(2) $$f(x' + x'') = f(x') \cap f(x'') \quad \text{whenever } f(x') \cap f(x'') \neq \varnothing.$$

2. Scoring functions. One of the most commonly used social choice functions is the so-called *plurality function*, in which each voter casts one vote for his most preferred alternative, and the alternative(s) with the largest total number of votes constitute the choice set. We may think of this procedure as assigning a score of 1 to each voter's most preferred alternative, a score of 0 to the others, and selecting the alternative(s) with highest total score, summed over all voters.

A second well-known SCF is the *Borda function*, which is defined for m alternatives as follows: let each voter assign score $m - 1$ to his most preferred alternative, score $m - 2$ to his second most preferred alternative, and in general score $m - i$ to his ith most preferred alternative. Then the alternative(s) with highest total score define the choice set for the Borda function.

These two SCF's are examples of the following general class. Given m alternatives and a profile, assign a score of s_i (s_i a real number) to each voter's ith most preferred alternative, and let the choice set consist of the alternative(s) with highest total score. Any SCF obtained in this way will be called a *simple scoring function*, and denoted by f^s, where s is the m-vector $(s_1, s_2, \cdots, s_m)$. s is called a *scoring vector*.

Formally, we may define f^s on the domain $\mathbb{N}^{m!}$ (in fact, on $\mathbb{R}^{m!}$) as follows. Given $p \in \mathscr{P}$, let E_p be the $m \times m$ permutation matrix with "1" in the (i, j)th position if and only if a_i is jth most preferred in the preference order p. For every $x \in \mathbb{R}^{m!}$, define $D(x) = \sum_{p \in \mathscr{P}} x_p E_p$ and let $D_i(x)$ denote the ith row of $D(x)$. Then f^s is defined by

$$a_i \in f^s(x) \quad \text{if and only if } D_i(x) \cdot s \geqq D_j(x) \cdot s \quad \text{for all } j,\ 1 \leqq j \leqq m.$$

Except that we make no requirement that a scoring vector satisfy $s_1 > s_2 > \cdots > s_m$, our definition of a simple scoring function is a special case of Fishburn's notion of a summation function [4].

The *trivial* SCF is that f such that $f(x) = A$ for all profiles x. The trivial function can be represented by the simple scoring function $f^{(0,0,\cdots,0)}$.

If f and g are SCF's such that $f(x) \subseteq g(x)$ for all profiles x, we say that f is a *refinement* of g and write $f \lesssim g$. ($f < g$ if $f \lesssim g$ and $f \neq g$.) For any anonymous SCF g and scoring vector $s \in \mathbb{R}^m$, $f^s \circ g$, the *composition* of f^s with g, is defined by

(3) $$a_i \in f^s \circ g(x) \quad \text{if and only if } a_i \in g(x) \quad \text{and} \quad D_i(x) \cdot s \geqq D_j(x) \cdot s, \quad \text{whenever } a_j \in g(x).$$

The meaning of $f^s \circ g$ is that the function f^s is used to resolve ties produced by g. Applying (3) recursively, we define a (*composite*) *scoring function* of order m to be any SCF g of the form $f^{s^\alpha} \circ f^{s^{\alpha-1}} \circ \cdots \circ f^{s^1}$, $\alpha \geqq 1$, where $s^1, s^2, \cdots, s^\alpha \in \mathbb{R}^m$. The expression $f^{s^\alpha} \circ f^{s^{\alpha-1}} \circ \cdots \circ f^{s^1}$ is called a *composition series* for g of *length* α.

The principal aim of this paper will be to establish the following result: a social choice function is symmetric and consistent if and only if it is a scoring function (simple or composite)

3. Extension of domain. Let f be symmetric and consistent. We now show how to extend f in a natural way from the domain $\mathbb{N}^{m!}$ to $\mathbb{Q}^{m!}$, where $\mathbb{Q}$ is the set of all rationals. In other words, we shall extend f to profiles having fractional and negative numbers of voters.

Let $e \in \mathbb{N}^{m!}$ be the vector with "1" in every component. By symmetry, $f(ne) = A$ for all $n \in \mathbb{N}$. Now define $f(x - ne) = f(x)$ for each $n \in \mathbb{N}$. This is well-defined, because if $x' - n'e = x - ne$, then without loss of generality, $n' \geqq n$ and $f(x' - n'e) = f(x') = f(x) \cap f((n' - n)e) = f(x) = f(x - ne)$. This extends f to $\mathbb{Z}^{m!}$ (where $\mathbb{Z}$ is the set of all integers) and it is the unique extension of f to $\mathbb{Z}^{m!}$ that is symmetric and consistent on $\mathbb{Z}^{m!}$. For any positive integer n, and $x \in \mathbb{Z}^{m!}$, consistency implies that $f(nx) = f(x)$, that is, f is *homogeneous*. Now for each positive integer n define $f(x/n) = f(x)$. This is well-defined, because if $x/n = x'/n'$, then $f(x/n) = f(x) = f(n'x) = f(nx') = f(x') = f(x'/n')$. This extends f to $\mathbb{Q}^{m!}$, and it is the unique such extension that is symmetric and consistent on $\mathbb{Q}^{m!}$.

Can a symmetric, consistent f be further extended to $\mathbb{R}^{m!}$ so as to be symmetric and consistent? In § 5 we will show that this can be done, but not necessarily uniquely. For example, the scoring functions $f^{(\sqrt{3},\sqrt{2},1)}$ and $f^{(1,0,0)} \circ f^{(\sqrt{3},\sqrt{2},1)}$ are symmetric and consistent on the domain $\mathbb{R}^{3!}$, and equal on $\mathbb{Q}^{3!}$, but not everywhere equal on $\mathbb{R}^{3!}$. To guarantee uniqueness of the real extension, we need, in addition to symmetry and consistency, the following "continuity" concept. An anonymous f is said to be *continuous* if, whenever $f(x) = \{a_i\}$, then for any profile y there is a sufficiently large integer n such that $f(y + nx) = \{a_i\}$ for all $n' \geqq n$. Thus, continuity is a kind of "domination by large numbers" principle. It means that if a certain committee chooses a unique winner a_i (using f), then given any second committee disjoint from the first, we can replicate the first committee a sufficient number of times so that it will overwhelm the second committee in a combined vote and yield the unique winner $\{a_i\}$

The SCF $f^{(1,0,0)}$ is continuous, while $f^{(0,1,0)} \circ f^{(1,0,0)}$ is not. To see the latter, let x be the profile defined by $x_p = 1$ if

$$p = \begin{pmatrix} a_1 \\ a_3 \\ a_2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a_3 \\ a_2 \\ a_1 \end{pmatrix},$$

and $x_p = 0$ otherwise. Then $f^{(0,1,0)} \circ f^{(1,0,0)}(x) = \{a_3\}$. But if y is the profile representing a single voter with preference

$$\begin{pmatrix} a_1 \\ a_3 \\ a_2 \end{pmatrix},$$

then $f^{(0,1,0)} \circ f^{(1,0,0)}(nx + y) = \{a_1\}$ for all $n \in \mathbb{N}$, hence the function is not continuous.

4. Convexity. Let f be a symmetric and consistent SCF. We may assume that the domain of f is $\mathbb{Q}^{m!}$. For each i, $1 \leqq i \leqq m$, let $R_i = \{x \in \mathbb{Q}^{m!} : a_i \in f(x)\}$. For any $x, y \in R_i$, and rational λ such that $0 \leqq \lambda \leqq 1$, we have $a_i \in f(\lambda x)$ and $a_i \in f((1 - \lambda)x)$, so by consistency $a_i \in f(\lambda x + (1 - \lambda)y)$, that is, $\lambda x + (1 - \lambda)y \in R_i$.

In general, we say that a set $S \subseteq \mathbb{R}^n$ is *$\mathbb{Q}$-convex* if $S \subseteq \mathbb{Q}^n$ and for all $x, y \in S$, and all rational λ, $0 \leqq \lambda \leqq 1$, we have $\lambda x + (1 - \lambda)y \in S$. Thus, each R_i is $\mathbb{Q}$-convex, and in fact is a *$\mathbb{Q}$-convex cone*, since $x \in R_i$ implies $\lambda x \in R_i$ for all rational $\lambda > 0$. To characterize all symmetric and consistent f, we shall need several facts about convex and $\mathbb{Q}$-convex sets.

For $S \subseteq \mathbb{R}^n$, let cvx S denote the convex hull of S, aff S the affine hull of S, and $\bar{S}$ the closure of S. If $S \subseteq W \subseteq \mathbb{R}^n$, where W is a flat (affine set), let $\operatorname{int}_W S$ denote the interior of S relative to W, and $\operatorname{ri} S = \operatorname{int}_{\operatorname{aff} S} S$ the *relative interior* of S. The *dimension* of S, dim S, is the dimension of aff S. We shall use the following well-known facts: if C is convex, then $\bar{C}$ and ri C are convex and $\operatorname{ri} \bar{C} = \operatorname{ri} C$, $\overline{\operatorname{ri} C} = \bar{C}$.

LEMMA 1. *$C \subseteq \mathbb{Q}^n$ is $\mathbb{Q}$-convex if and only if $C = \mathbb{Q}^n \cap \operatorname{cvx} C$.*

Proof. If $C = \mathbb{Q}^n \cap \operatorname{cvx} C$ then clearly C is $\mathbb{Q}$-convex.

Conversely, if C is $\mathbb{Q}$-convex, then certainly $C \subseteq \mathbb{Q}^n \cap \operatorname{cvx} C$. Assume for the moment that C is a $\mathbb{Q}$-convex cone containing the origin. For any

$$q \in (\mathbb{Q}^n \cap \operatorname{cvx} C)$$

such that $q \neq 0$,

$$(4) \qquad q = \sum_{i=1}^{k} \lambda_i q^i, \quad \text{where } q^1, q^2, \cdots, q^k \in C \quad \text{and} \quad \lambda_i > 0, \quad \lambda_i \in \mathbb{R}.$$

Assume that among all expressions (4) for q that k is smallest, and we shall show that $\lambda_i \in \mathbb{Q}$, $1 \leqq i \leqq k$. Letting $\lambda_0 = -1$, $q^0 = q$, we can rewrite (4) as

$$(5) \qquad \sum_{i=0}^{k} \lambda_i q^i = 0.$$

If, say, $\lambda_1 \notin \mathbb{Q}$, then considering $\mathbb{R}$ as a vector space over the field $\mathbb{Q}$, let $\{\lambda_0, \lambda_1, \cdots, \lambda_l\}$, $l \geqq 1$, be a basis for $\{\lambda_0, \lambda_1, \cdots, \lambda_k\}$ (renumbering the λ's if necessary), where $\lambda_i = \sum_{j=0}^{l} b_{ij}\lambda_j$, $b_{ij} \in \mathbb{Q}$, $0 \leqq i \leqq k$, and $b_{ii} = 1$, $b_{ij} = 0$, $0 \leqq i \leqq i \neq j \leqq l$. Then (5) implies

$$\sum_{j=0}^{l} \left(\sum_{i=0}^{k} b_{ij} q^i \right) \lambda_j = 0,$$

so by independence,

$$\sum_{i=0}^{k} b_{i1} q^i = 0,$$

and since $b_{01} = 0$,

$$\sum_{i=1}^{k} b_{i1} q^i = 0. \tag{6}$$

Let λ be the greatest real scalar such that $\lambda_i' = \lambda_i - \lambda b_{i1} \geqq 0$, $1 \leqq i \leqq k$. Then $q = \sum_{i=1}^k (\lambda_i - \lambda b_{i1}) q^i$ yields a shorter expression for q, a contradiction. Hence $\lambda_i \in Q$ for $1 \leqq i \leqq k$, and so $q \in C$ by $\mathbb{Q}$-convexity. Thus $C = \mathbb{Q}^n \cap \operatorname{cvx} C$ if C is a $\mathbb{Q}$-convex cone containing the origin. If $C \subseteq \mathbb{Q}^n$ is $\mathbb{Q}$-convex, consider the $\mathbb{Q}$-convex cone $K = \{\lambda(1, x) : \lambda \geqq 0, \lambda \in \mathbb{Q}, x \in C\} \subseteq \mathbb{Q}^{n+1}$. Then

$$\operatorname{cvx} C = \{x \in \mathbb{R}^n : (1, x) \in \operatorname{cvx} K\}.$$

Hence if $x \in \mathbb{Q}^n \cap \operatorname{cvx} C$, then $(1, x) \in \mathbb{Q}^{n+1} \cap \operatorname{cvx} K = K$, so $x \in C$. Thus $C = \mathbb{Q}^n \cap \operatorname{cvx} C$ in any case. □

LEMMA 2. *If $C \subseteq \mathbb{Q}^n$ is $\mathbb{Q}$-convex, then $\bar{C}$ is convex.*

Proof. If $x \in \operatorname{cvx} C$, then $x = \sum_{i=1}^k \lambda_i q^i$ for some finite collection $q^1, q^2, \cdots, q^k \in C$ and $\lambda_i \in \mathbb{R}$, $\lambda_i \geqq 0$, $\sum_{i=1}^k \lambda_i = 1$. For each i, $1 \leqq i \leqq k - 1$, let $\{\lambda_i^n\}$ be a sequence of rationals converging to λ_i, such that $0 \leqq \lambda_i^n \leqq \lambda_i$, and let $\lambda_k^n = 1 - \sum_{i=1}^{k-1} \lambda_i^n$. Then $\lambda_k^n \in \mathbb{Q}$, $\lambda_k^n \geqq 0$ and $\sum_{i=1}^k \lambda_i^n = 1$ for every n, so $x^n = \sum_{i=1}^k \lambda_i^n q^i \in C$, by $\mathbb{Q}$-convexity. Since x^n converges to x, $x \in \bar{C}$. Thus $C \subseteq \operatorname{cvx} C \subseteq \bar{C}$, so $\bar{C} = \overline{\operatorname{cvx} C}$, the latter of which is convex. □

LEMMA 3. *If $C = \bigcup_{i=1}^k S_i$, where $C \subseteq \mathbb{R}^n$ is convex and k is finite, then for some i,* $\dim C = \dim S_i$.

Proof. By induction on k. If $k = 1$ the result is trivial. For $k > 1$, let $H_1 = \operatorname{aff} S_1 \subseteq \operatorname{aff} C = H$. If $\operatorname{ri} C \subseteq H_1$ then $C \subseteq H_1$ so $H_1 = H$ and $\dim C = \dim S_1$. Otherwise, choose $x^0 \in \operatorname{ri} C - H_1$, and $\varepsilon > 0$ so that

$$C' = \{y \in H : \|y - x^0\| < \varepsilon\} \subseteq C - H_1.$$

Then C' is convex, $\dim C' = \dim H = \dim C$ and $C' = \bigcup_{i=2}^k (S_i \cap C')$, so by the induction hypothesis there is an i, $2 \leqq i \leqq k$, such that

$$\dim C = \dim C' = \dim (S_i \cap C') \leqq \dim S_i \leqq \dim C.$$

Hence $\dim S_i = \dim C$. □

5. The main theorem.

THEOREM 1. (i) *A social choice function is symmetric and consistent if and only if it is a scoring function.*

(ii) *A social choice function is symmetric, consistent and continuous if and only if it is a simple scoring function.*

Proof. The "if" parts are left to the reader. To prove the converses, let f be symmetric and consistent; we may take the domain of f to be $\mathbb{Q}^{m!}$. We show first that, where D is the function defined in § 2,

$$f(x) = A \quad \text{for all } x \text{ in } R = \{x \in \mathbb{Q}^{m!} : D(x) = 0\}. \tag{7}$$

If (7) is false, then $m \geqq 2$ and without loss of generality

$$f(x^0) = \{a_1, a_2, \cdots, a_r\} \quad \text{for some } r < m \text{ and } x^0 \in R. \tag{8}$$

Then $y^1 = x^0 + \sum_{j=2}^r M_{(jm)}(x^0) \in R$, and by consistency, $f(y^1) = \{a_1\}$. (If $r = 1$, let $y^1 = x^0$.) For each i, let $R_i = \{x \in R : a_i \in f(x)\}$. Then R_i is a $\mathbb{Q}$-convex cone and $\bar{R}_i$ is convex. Moreover, $\bigcup_{i=1}^m \bar{R}_i = \bar{R}$, and $M_\sigma(\bar{R}_i) = \bar{R}_{\sigma(i)}$, $1 \leqq i \leqq m$, so by Lemma 3 and the symmetry of f, $\operatorname{int}_{\bar{R}} \bar{R}_i \neq \varnothing$ for $1 \leqq i \leqq m$. Suppose that $\operatorname{int}_{\bar{R}} \bar{R}_1 \cap \operatorname{int}_{\bar{R}} \bar{R}_2 \neq \varnothing$. Then choose $x^1 \in \mathbb{Q}^{m!} \cap \operatorname{int}_{\bar{R}} \bar{R}_1 \cap \operatorname{int}_{\bar{R}} \bar{R}_2$ and form $z(\lambda) = \lambda y^1 + (1 - \lambda)x^1$ for each rational λ, $0 < \lambda < 1$. For all sufficiently small such λ,

$$\begin{aligned} z(\lambda) \in \mathbb{Q}^{m!} \cap \operatorname{int}_{\bar{R}} \bar{R}_1 \cap \operatorname{int}_{\bar{R}} \bar{R}_2 &= \mathbb{Q}^{m!} \cap \operatorname{int}_{\bar{R}} \overline{\operatorname{cvx} R_1} \cap \operatorname{int}_{\bar{R}} \overline{\operatorname{cvx} R_2} \\ &= \mathbb{Q}^{m!} \cap \operatorname{int}_{\bar{R}} \operatorname{cvx} R_1 \cap \operatorname{int}_{\bar{R}} \operatorname{cvx} R_2 \subseteq R_1 \cap R_2, \end{aligned}$$

the latter by Lemma 1. Hence $\{a_1, a_2\} \in f(z(\lambda))$. But $f(z(\lambda)) = f(y^1) \cap f(x^1) = \{a_1\}$, by consistency. This contradiction and the symmetry of f imply $\operatorname{int}_{\bar{R}} \bar{R}_i \cap \operatorname{int}_{\bar{R}} \bar{R}_j = \varnothing$ for all $i \neq j$. Now the separation theorem for convex sets implies that for each $i \neq j$ there exists a modulus-one vector $u^{ij} \in \bar{R}$ such that

$$u^{ij} \cdot x \geqq 0 \quad \text{for all } x \in \bar{R}_i, \qquad u^{ij} \cdot x \leqq 0 \quad \text{for all } x \in \bar{R}_j. \tag{9}$$

For $i > j$, choose $u^{ij} = -u^{ji}$, so (9) holds for all $i \neq j$. Let $S_i = \{x \in \bar{R} : u^{ij} \cdot x \geqq 0, \text{ all } j \neq i\}$. Then $\varnothing \neq \operatorname{int}_{\bar{R}} \bar{R}_i \subseteq \operatorname{int}_{\bar{R}} S_i = \{x \in \bar{R} : u^{ij} \cdot x > 0, \text{ all } j \neq i\} \subseteq \bar{R} - \bigcup_{j \neq 1} \bar{R}_j \subseteq \bar{R}_i$, hence $\bar{R}_i = \overline{\operatorname{int}_{\bar{R}} \bar{R}_i} = \overline{\operatorname{int}_{\bar{R}} S_i} = S_i$, that is,

$$\bar{R}_i = \{x \in \bar{R} : u^{ij} \cdot x \geqq 0, \text{all } j \neq i\}. \tag{10}$$

Since $m \geqq 2$, $\bar{R}_i$ has some face of dimension $\dim \bar{R} - 1$. Hence for some $j \neq i$, $\dim(\bar{R}_i \cap \bar{R}_j) = \dim \bar{R} - 1$, and by symmetry this holds for all $i \neq j$. Hence u^{ij} is the *unique* modulus-one vector satisfying (9), so

$$M_\sigma(u^{ij}) = u^{\sigma(i)\sigma(j)} \tag{11}$$

for all permutations σ.

Assume now that $m \geqq 3$, and let $\bar{V} = \operatorname{aff}(\bar{R}_1 \cap \bar{R}_2)$. Choose $x^0 \in \operatorname{int}_{\bar{R}} \bar{R}_3$, and let $y = x^0 + M_{(12)}(x^0)$. Then $y \in \operatorname{int}_{\bar{R}} \bar{R}_3$ and $u^{12} \cdot y = M_{(12)}(u^{12} \cdot y) = u^{21} \cdot y = -u^{12} \cdot y$, so $y \in \bar{V} - (\bar{R}_1 \cap \bar{R}_2)$. Hence $\bar{R}_1 \cap \bar{R}_2$ has a $(\dim \bar{R} - 2)$-dimensional face of form $\bar{R}_1 \cap \bar{R}_2 \cap \bar{R}_k$. By symmetry, we may assume $k = 3$. Then u^{12}, u^{23}, u^{31} are all in $\bar{R}$ and orthogonal to $\bar{R}_1 \cap \bar{R}_2 \cap \bar{R}_3$, so they are dependent:

$$\lambda u^{12} + \lambda' u^{23} + \lambda'' u^{31} = 0, \tag{12}$$

λ, λ', λ'' are not all zero.

u^{12}, u^{23}, u^{31} cannot all be equal because if they were, then (11) would imply that $u^{ij} = \pm u^{12}$ for all $i \neq j$, whence there could be at most two distinct regions $\bar{R}_i$, contrary to the assumption that $m \geqq 3$. Let $\sigma = (1\ 2\ 3)$; then there is an index $p \in \mathscr{P}$ such that the restrictions $\tilde{u}^{12}$, $\tilde{u}^{23}$, $\tilde{u}^{31}$ of u^{12}, u^{23}, u^{31} to the coordinates p, $\sigma(p)$, $\sigma^2(p)$ are not all equal. Since these restrictions are also dependent, they lie in a 2-dimensional plane through the origin, and they are permuted in a 3-cycle by the corresponding restriction of M_σ (which is just a rotation of $\mathbb{R}^3 = \{(x_p, x_{\sigma(p)}, x_{\sigma^2(p)})\}$ having axis $(1, 1, 1)$). Hence $\tilde{u}^{12}, \tilde{u}^{23}, \tilde{u}^{31}$ are distinct and $\tilde{u}^{12} + \tilde{u}^{23} + \tilde{u}^{31} = 0$. Since $\lambda\tilde{u}^{12} + \lambda'\tilde{u}^{23} + \lambda''\tilde{u}^{31} = 0$, the fact $\tilde{u}^{ij}$'s are distinct implies $\lambda = \lambda' = \lambda'' \neq 0$. Hence

$$u^{12} + u^{23} + u^{31} = 0. \tag{13}$$

By (11), the vector u^{12} is invariant under all permutations M_σ, where σ fixes 1 and 2. Hence u_p^{12} has the same value s_{ij} for all preference orders p in which a_1 is ranked ith, a_2 is ranked jth. Let $\mathscr{P}_{ij}$ be the set of all such p. Also, put $s_{ii} = 0$. Then $u^{ij} = -u^{ji}$ and (13) imply

$$s_{ij} = -s_{ji} \quad \text{and} \quad s_{ij} + s_{jk} + s_{ki} = 0 \tag{14}$$

for all distinct i, j, k.

For any $x \in \bar{R} = \{x \in \mathbb{R}^{m!} : D(x) = 0\}$, we have, by (14),

$$\begin{aligned} u^{12} \cdot x = \sum_{1 \leqq i,j \leqq m} s_{ij} \sum_{p \in \mathscr{P}_{ij}} x_p &= \sum_{1 \leqq i,j \leqq m} (s_{im} - s_{jm}) \sum_{p \in \mathscr{P}_{ij}} x_p \\ &= \sum_{1 \leqq i \leqq m} s_{im} \left(\sum_{1 \leqq j \leqq m} \sum_{p \in \mathscr{P}_{ij}} x_p \right) \\ &\quad - \sum_{1 \leqq j \leqq m} s_{jm} \left(\sum_{1 \leqq i \leqq m} \sum_{p \in \mathscr{P}_{ij}} x_p \right) \\ &= \sum_{1 \leqq i \leqq m} s_{im} D_{1i}(x) - \sum_{1 \leqq j \leqq m} s_{jm} D_{2j}(x) = 0. \end{aligned}$$

But $u^{12} \in \bar{R}$ and $u^{12} \cdot u^{12} > 0$, so this is a contradiction. Thus (7) is true. (Note that in case $m = 1$ or $m = 2$, (7) is obvious.)

If $D(x) = D(x')$, then $x - x' \in R$, so $f(x) = f(x') \cap f(x - x') = f(x') \cap A = f(x')$. Thus, $f(x)$ depends only on $D(x)$. By definition, the image of D consists of all rational, linear combinations of $m \times m$ permutation matrices. By the Birkhoff–von Neumann theorem, this is precisely the space $\mathscr{D}$ of all rational $m \times m$ matrices with constant row and column sums. Henceforth we shall regard $\mathscr{D}$ as the effective domain of f.

Since the trivial function is a simple scoring function, we may assume that f is nontrivial. Redefine $R = \mathscr{D}$, $R_i = \{D \in \mathscr{D} : a_i \in f(D)\}$, then apply the argument of (8)–(11) to find separating vectors ($m \times m$ matrices) u^{ij} in $\mathscr{D}$ satisfying $u^{ij} = -u^{ji}$ and relations (9), (10), (11). (We now interpret M_σ as acting on matrices $D \in \mathscr{D}$ by interchanging rows in the manner prescribed by σ.) If $s = (s_1, s_2, \cdots, s_m)$ is the ith row of u^{ij}, then since $M_{(ij)}(u^{ij}) = u^{ji} = -u^{ij}$, it must be that the jth row of u^{ij} is $-s$, and all other rows are zero. Therefore $u^{ij} \cdot D \geqq 0$ if and only $D_i \cdot s \geqq D_j \cdot s$, and by symmetry this holds for all $i \neq j$. Hence, by (9), $a_i \in f(D)$ implies $D_i \cdot s \geqq D_j \cdot s$ for all $j \neq i$; in other words, $f \lesssim f^s$.

If $f = f^s$, we are done. Otherwise, suppose inductively that $f < f^{s^\alpha} \circ f^{s^{\alpha-1}} \circ \cdots \circ f^{s^1} = g$ for real m-vectors $s^1, s^2, \cdots, s^\alpha$. Without loss of generality, let $D^0 \in \mathscr{D}$ be such that

$$f(D^0) = \{a_1, a_2, \cdots, a_k\} \subset \{a_1, a_2, \cdots, a_l\} = g(D^0), \tag{15}$$

where $k < l \leqq m$. If $k \geqq 2$, let $D^1 = D^0 + \sum_{j=2}^{k} M_{(jl)}(D^0)$, and if $k = 1$, let $D^1 = D^0$; in either case a_1 is the unique winner under $f(D^1)$, while a_1, a_2 are tied for first under $g(D^1)$. Choose any D^2 such that $g(D^2) = \{a_2\}$; then for all $n > 0$, $g(D^2 + nD^1) = \{a_2\}$. Since $f < g$, $f(D^2 + nD^1) = \{a_2\}$, for all $n > 0$, while $f(D^1) = \{a_1\}$. Hence f could not be continuous (i.e., f continuous implies $f = f^s$, proving (ii)).

To complete the proof of (i), let us concentrate on alternatives a_1 and a_2. Define $R' = \{D \in \mathscr{D} : D_1 \cdot s^1 = D_2 \cdot s^1 > D_j \cdot s^1$ for $j \geqq 3$, and $D_1 \cdot s^i = D_2 \cdot s^i$ for

$i \geqq 2\}$. Further, let $R'_i = \{D \in R' : a_i \in f(D)\}$, where $i = 1, 2$. For all $D \in R'$, $f(D) \subseteq g(D) = \{a_1, a_2\}$, hence $R' = R'_1 \cup R'_2$ and $\bar{R}' = \bar{R}'_1 \cup \bar{R}'_2$. Note that $\bar{R}' = \{D \in \bar{\mathscr{D}} : D_1 \cdot s^1 = D_2 \cdot s^1 \geqq D_j \cdot s^1$ for $j \geqq 3$, and $D_1 \cdot s^i = D_2 \cdot s^i$ for $i \geqq 2\}$ $\supseteq \{D \in \bar{\mathscr{D}} : \{a_1, a_2\} \subseteq g(D)\}$. Then, just as in (8)–(11), (but with $\bar{R}'$ instead of $\bar{R}$) we show that $\bar{R}'_1$, $\bar{R}'_2$ have nonempty, disjoint interiors relative to aff $\bar{R}'$, and $M_{(12)}(\bar{R}'_1) = \bar{R}'_2$. Hence there is an $m \times m$ matrix $u \in \operatorname{aff} \bar{R}'$ of form

$$u = \begin{bmatrix} t_1 & t_2 \cdots & t_m \\ -t_1 & -t_2 \cdots & -t_m \\ & 0 & \end{bmatrix}$$

such that $\bar{R}'_1 = \{D \in \bar{R}' : D_1 \cdot t \geqq D_2 \cdot t\}$.

For any D, $a_1 \in f(D)$ implies $a_1 \in g(D)$. If $\{a_1\} \subsetneqq g(D)$, then without loss of generality $\{a_1, a_2\} \subseteq g(D)$, hence $D \in \bar{R}'_1$ and $D_1 \cdot t \geqq D_2 \cdot t$. Similarly we have $D_1 \cdot t \geqq D_j \cdot t$ for all j such that $a_j \in g(D)$. Hence $a_1 \in f^t \circ g(D)$. If $g(D) = \{a_1\}$, then trivially $a_1 \in f^t \circ g(D)$. By symmetry, it follows from the above that $f \lesssim f^t \circ g = f^t \circ f^{s^\alpha} \circ f^{s^{\alpha-1}} \circ \cdots \circ f^{s^1}$. Moreover, $u \in \bar{R}'$ implies $t \cdot s^\beta = 0$ for $1 \leqq \beta \leqq \alpha$, and we may always choose t so that $|t| = \sum_{i=1}^m t_i = 0$. Therefore the construction must terminate in at most $m - 1$ steps with the conclusion that $f(D) = f^{s^\gamma} \circ f^{s^{\gamma-1}} \circ \cdots \circ f^{s^1}(D)$ for some m-vectors $s^1, \cdots, s^\gamma$ such that $|s^\beta| = 0$ and $s^\alpha \cdot s^\beta = 0$, $1 \leqq \alpha \neq \beta \leqq \gamma$, and for all $D \in \mathscr{D}$. □

Notice that every scoring function actually defines not just a choice set, but in a natural way it weakly orders the whole set of alternatives. Thus, by requiring a social choice function to be symmetric and consistent, we force it in effect to be a social preference function.

Since every scoring function of order m is defined, symmetric and consistent on $\mathbb{R}^{m!}$ we have the following.

COROLLARY 1. *Every symmetric, consistent SCF has a symmetric, consistent real extension.*

COROLLARY 2. *Every symmetric, consistent, continuous SCF has a unique consistent, continuous real extension, and this extension is symmetric.*

Proof. If f is symmetric, consistent and continuous, then $f(x) = f^s(x)$ for all $x \in \mathbb{Q}^{m!}$ and some $s \in \mathbb{R}^m$. Let g be a consistent, continuous real extension of f^s, and let $\bar{R}_i = \{x \in \mathbb{R}^{m!} : a_i \in f^s(x)\}$, $\bar{T}_i = \{x \in \mathbb{R}^{m!} : a_i \in g(x)\}$. Now $\bar{R}_i$ is closed, convex, and $\operatorname{int}_{\mathbb{R}^{m!}} \bar{R}_i \neq \varnothing$, so $\bar{R}_i$ is the closure of its rational points. Thus

$$\bar{R}_i = (\overline{\bar{R}_i \cap \mathbb{Q}^{m!}}) = (\overline{\bar{T}_i \cap \mathbb{Q}^{m!}}) \subseteq \bar{T}_i.$$

In particular, $\operatorname{int}_{\mathbb{R}^{m!}} \bar{T}_i \neq \varnothing$; so $\bar{T}_i$ convex implies $(\overline{\bar{T}_i \cap \mathbb{Q}^{m!}}) = \bar{T}_i$, whence $\bar{R}_i = \bar{T}_i$ and $f^s = g$. □

6. Rational equivalence. It is possible for two different composition series $g = f^{s^\alpha} \circ f^{s^{\alpha-1}} \circ \cdots \circ f^{s^1}$ and $h = f^{t^\beta} \circ f^{t^{\beta-1}} \circ f^{t^1}$ to represent the same social choice function in the sense that $g(x) = h(x)$ for all $x \in \mathbb{N}^{m!}$. In this event we in fact must have $g(x) = h(x)$ for all $x \in \mathbb{Q}^{m!}$ (see § 3), and we say that g is *rationally equivalent* to h, written $g \sim h$. If moreover $g(x) = h(x)$ for all $x \in \mathbb{R}^{m!}$, then g is *equivalent* to h, and we write $g \approx h$. A *subcomposition series* of g is a composition series obtained by deleting some nonempty subset of the terms f^{s^β}, $1 \leqq \beta \leqq \alpha$, leaving the others in the given order. (The deletion of all f^{s^β} can be considered to result in the trivial function.) g is *rationally irreducible* if g is not trivial and $g \sim g'$ is false for every subcomposition series g' of g. g is *irreducible* if $g \approx g'$ is false for every such g'. In this section we shall exhibit necessary and sufficient conditions that any two (rationally) irreducible composition series be (rationally) equivalent.

We shall take the domain of all scoring functions to be $\bar{\mathscr{D}}$, the set of all real $m \times m$ matrices with constant row and column sums ($\mathscr{D}$ denotes the set of all rational such matrices).

First we note that, if $\hat{e}$ represents the m-vector with "1" in every component, then for any $s \in \mathbb{R}^m$ and $\lambda, \mu \in \mathbb{R}$, $\lambda > 0$, we have $f^s \approx f^{\lambda s + \mu \hat{e}}$. In particular, we may assume, without loss of generality in the results that follow, that in any composition series the s^β, $1 \leqq \beta \leqq \alpha$, are chosen so that $|s^\beta| = \sum_{i=1}^m s_i^\beta = 0$. We shall also assume that $m \geqq 2$, since for $m = 1$ all SCF's are trivial.

With g and h as above, define $U_0 = V_0 = \mathbb{Q}^m$, and for all γ, $1 \leqq \gamma \leqq \alpha$, let

$$U_\gamma = \{x \in \mathbb{Q}^m : s^\delta \cdot x = 0, 1 \leqq \delta \leqq \gamma\},$$
$$V_\gamma = \{x \in \mathbb{Q}^m : t^\delta \cdot x = 0, 1 \leqq \delta \leqq \gamma\}.$$

U_γ and V_γ are subspaces of $\mathbb{Q}^m$, i.e., they are $\mathbb{Q}$-*subspaces*. For any $\mathbb{Q}$-subspace $U \subseteq \mathbb{Q}^m$, let U^* denote its orthogonal complement in $\mathbb{Q}^m$ and let $\bar{U}$ be the real closure of U, i.e., the real subspace of $\mathbb{R}^m$ spanned by U. $(\bar{U})^* = \overline{(U^*)}$ is its orthogonal complement.

For each $x \in \mathbb{R}^m$, let $[x] = \{\lambda x : \lambda > 0, \lambda \in \mathbb{R}\}$ denote the positive ray through x. With these definitions we can state the following result.

THEOREM 2. *Two rationally irreducible composition series*

$$g = f^{s^\alpha} \circ f^{s^{\alpha-1}} \circ \cdots \circ f^{s^1}$$

and

$$h = f^{t^\beta} \circ f^{t^{\beta-1}} \circ \cdots \circ f^{t^1}$$

are rationally equivalent if and only if $\alpha = \beta$ *and for* $1 \leqq \gamma \leqq \alpha$,

$$U_\gamma = V_\gamma, \tag{16}$$

$$s^\gamma \in [t^\gamma] + \bar{V}^*_{\gamma-1}, \tag{17}$$

$$t^\gamma \in [s^\gamma] + \bar{U}^*_{\gamma-1}. \tag{18}$$

Proof. Let g^γ, $1 \leqq \gamma \leqq \alpha$, denote the subcomposition series $f^{s^\gamma} \circ f^{s^{\gamma-1}} \circ \cdots \circ f^{s^1}$, and similarly define h^γ. Suppose that $\alpha = \beta$ and (18) holds; we shall show by induction on γ that then (16) and (17) hold and $g^\gamma \sim h^\gamma$, hence in particular that $g \sim h$.

If $\gamma = 1$, then $t^1 = \lambda s^1$ for some $\lambda > 0$; hence also $s^1 \in [t^1]$ and $g^1 = f^{s^1} \sim f^{t^1} = h^1$.

Assume then that

$$g^\gamma \sim h^\gamma, U_\gamma = V_\gamma \quad \text{and} \quad s^\gamma \in [t^\gamma] + \bar{V}^*_{\gamma-1} \tag{19}$$

for all $\gamma < \delta$, and we shall prove these statements for $\gamma = \delta$. If $a_i \in g^\delta(D)$, then $D_i \cdot s^\delta \geqq D_j \cdot s^\delta$ for all j such that $\{a_i, a_j\} \subseteq g^{\delta-1}(D)$. Now $\{a_i, a_j\} \subseteq g^{\delta-1}(D)$ implies $(D_i - D_j) \cdot s^\gamma = 0$ for $1 \leqq \gamma \leqq \delta - 1$, hence $D_i - D_j \in U_{\delta-1}$. Since $t^\delta = \lambda s^\delta$

$+ u$ for some $\lambda > 0$ and $u \in \bar{U}^*_{\delta-1}$, $(D_i - D_j) \cdot t^\delta = \lambda(D_i - D_j) \cdot s^\delta + 0 \geqq 0$. Thus $D_i \cdot t^\delta \geqq D_j \; t^\delta$ whenever $\{a_i, a_j\} \subseteq g^{\delta-1}(D) = h^{\delta-1}(D)$ by induction, so $a_i \in h^\delta(D)$.

Conversely, if $a_i \in h^\delta(D)$, a similar argument shows that $a_i \in g^\delta(D)$. Thus $g^\delta(D) = h^\delta(D)$. Finally,

$$\begin{aligned} U_\delta &= \{x \in U_{\delta-1} : s^\delta \cdot x = 0\} \\ &= \{x \in V_{\delta-1} : (\lambda t^\delta + u) \cdot x = 0\} \qquad (20) \\ &= \{x \in V_{\delta-1} : t^\delta \cdot x = 0\} = V_\delta, \quad \text{since } \lambda > 0 \text{ and } u \in \bar{U}^*_{\delta-1} = \bar{V}^*_{\delta-1}, \end{aligned}$$

proving the "if" part of the theorem.

To prove the "only if" part, suppose $g^\alpha \sim h^\beta$, where g^α and h^β are rationally irreducible. Without loss of generality, $\alpha \leqq \beta$. First, we shall show by induction on γ, $0 \leqq \gamma \leqq \beta$, that (16)–(18) are valid. For $\gamma = 0$, (16) holds and (17), (18) are vacuous. Suppose then that $U_\gamma = V_\gamma$ for $\gamma < \delta$.

Suppose also that $V_\delta = V_{\delta-1}$. For any $D \in \mathscr{D}$, $\{a_i, a_j\} \subseteq h^{\delta-1}(D)$ implies $D_i \cdot t^\gamma = D_j \cdot t^\gamma$ for $1 \leqq \gamma \leqq \delta - 1$, that is, $D_i - D_j \in V_{\delta-1} = V_\delta$. But then $D_i \cdot t^\delta = D_j \cdot t^\delta$, so $\{a_i, a_j\} \subseteq h^\delta(D)$. It follows that $h^{\delta-1}(D) \subseteq h^\delta(D)$ for all D, and since h^δ is a refinement of $h^{\delta-1}$, $h^\delta(D) = h^{\delta-1}(D)$, so h is reducible, a contradiction. Therefore $V_\delta \subsetneqq V_{\delta-1}$, so $t^\delta \cdot x \neq 0$ for some $x \in V_{\delta-1}$, and $t^\delta \notin \bar{V}^*_{\delta-1}$. Likewise, $s^\delta \notin \bar{U}^*_{\delta-1} = \bar{V}^*_{\delta-1}$.

Hence s^δ, t^δ have nontrivial projections $\bar{s}^\delta$ and $\bar{t}^\delta$ onto $\bar{V}_{\delta-1}$. If $\bar{s}^\delta$ is not a positive multiple of $\bar{t}^\delta$, then $\{x \in \bar{V}_{\delta-1} : \bar{t}^\delta \cdot x > 0 \text{ and } \bar{s}^\delta \cdot x < 0\}$ is nonempty and open, and hence meets $V_{\delta-1}$. Let $x \in V_{\delta-1}$ such that $\bar{t}^\delta \cdot x > 0$ and $\bar{s}^\delta \cdot x < 0$. Form the $m \times m$ matrix D such that $D_1 = D_2 = \cdots = D_{m-1} = x$ and

$$D_m = \left(\sum_{i=1}^{m} x_i \right) \hat{e} - (m-1)x.$$

Then $D \in \mathscr{D}$.

Now $x \in V_{\delta-1} = U_{\delta-1}$ and $s^\gamma \cdot \hat{e} = t^\gamma \cdot \hat{e} = 0$ for all γ implies

$$D \cdot s^\gamma = D \cdot t^\gamma = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \qquad 1 \leqq \gamma < \delta, \tag{21}$$

while

$$D_i \cdot t^\delta = x \cdot t^\delta > 0, \qquad 1 \leqq i \leqq m - 1, \tag{22}$$

and

$$D_m \cdot t^\delta = -(m-1)x \cdot t^\delta < 0 \qquad (\text{since } m \geqq 2). \tag{23}$$

By (21), $g^{\delta-1}(D) = h^{\delta-1}(D) = A$, and so (22), (23) imply

$$h^\delta(D) \subseteq \{a_1, a_2, \cdots, a_{m-1}\}.$$

On the other hand,

$$D_i \cdot s^\delta = x \cdot s^\delta < 0, \qquad 1 \leqq i \leqq m - 1, \tag{24}$$

and

$$D_m \cdot s^\delta = -(m-1)x \cdot s^\delta > 0, \tag{25}$$

so that $g^\delta(D) = \{a_m\}$. Since $h(D) \subseteq h^\delta(D)$ and $g(D) \subseteq g^\delta(D)$, we have $h(D) \neq g(D)$, a contradiction.

Hence $\bar{s}^\delta = \lambda \bar{t}^\delta$ for some $\lambda > 0$ and so $s^\delta \in [t^\delta] + \bar{V}^*_{\delta-1}$. Since $U_{\delta-1} = V_{\delta-1}$, we also have $\bar{U}^*_{\delta-1} = \bar{V}^*_{\delta-1}$, whence $t^\delta \in [s^\delta] + \bar{U}^*_{\delta-1}$. Furthermore, just as in (20) we find that $U_\delta = V_\delta$. Thus (16)–(18) hold for $\gamma = \delta$, hence by induction for all γ, $1 \leqq \gamma \leqq \alpha \leqq \beta$.

If $\alpha < \beta$, then since h is irreducible, $V_\alpha - V_{\alpha+1} \neq \emptyset$ and there is an $x \in V_\alpha - V_{\alpha+1}$ such that $t^{\alpha+1} \cdot x < 0$. Define $D \in \mathcal{D}$ such that $D_i = x$, $1 \leqq i \leq m-1$, and $D_m = \left(\sum_{i=1}^m x_i\right)\hat{e} - (m-1)x$. Since $x \in V_\alpha = U_\alpha$, $g(D) = g^\alpha(D) = A$, while $h(D) \subseteq h^{\alpha+1}(D) = \{a_m\} \neq A$. This contradiction shows that $\alpha = \beta$, and the theorem is proved. $\square$

We say that a scoring function $f^{s^\alpha} \circ f^{s^{\alpha-1}} \circ \cdots \circ f^{s^1}$ is *rational* if $s^1, s^2, \cdots, s^\alpha \in Q^m$.

COROLLARY 1. *Two rationally irreducible, rational scoring functions $f^{s\alpha} \circ \cdots \circ f^{s^1}$ and $f^{t^\beta} \circ f^{t^{\beta-1}} \circ \cdots \circ f^{t^1}$ are rationally equivalent if and only if $\alpha = \beta$ and for all γ, $1 \leqq \gamma \leqq \alpha$,*

$$s^\gamma = b_\gamma t^\gamma + \sum_{\delta=1}^{\gamma-1} b_\delta t^\delta, \quad \textit{where } b_1, b_2, \cdots, b_\gamma \in \mathbb{Q} \textit{ and } b_\gamma > 0, \tag{26}$$

and

$$t^\gamma = c_\gamma s^\gamma + \sum_{\delta=1}^{\gamma-1} c_\delta s^\delta, \quad \textit{where } c_1, c_2, \cdots, c_\gamma \in \mathbb{Q} \textit{ and } c_\gamma > 0. \tag{27}$$

Proof. Under our hypotheses the whole proof of Theorem 2 may be carried out in $\mathbb{Q}^m$ rather than $\mathbb{R}^m$. Hence, in particular, $\bar{V}^*_\gamma$ becomes the $\mathbb{Q}$-subspace generated by $t^1, t^2, \cdots, t^\gamma$ and $\bar{U}^*_\gamma$ the $\mathbb{Q}$-subspace generated by $s^1, s^2, \cdots, s^\gamma$. So (26) and (27) are the same in this case as (17) and (18). $\square$

A similar argument, replacing $\mathbb{Q}^m$ by $\mathbb{R}^m$, yields the following.

COROLLARY 2. *Two irreducible scoring functions $f^{s^\alpha} \circ f^{s^{\alpha-1}} \circ \cdots \circ f^{s^1}$ and $f^{t^\beta} \circ f^{t^{\beta-1}} \circ \cdots \circ f^{t^1}$ are equivalent if and only if $\alpha = \beta$ and for all γ, $1 \leqq \gamma \leqq \alpha$,*

$$s^\gamma = b_\gamma t^\gamma + \sum_{\delta=1}^{\gamma-1} b_\delta t^\delta, \quad \textit{where } b_1, b_2, \cdots, b_\gamma \in \mathbb{R} \textit{ and } b_\gamma > 0. \tag{28}$$

$$t^\gamma = c_\gamma s^\gamma + \sum_{\delta=1}^{\gamma-1} c_\delta s^\delta, \quad \textit{where } c_1, c_2, \cdots, c_\gamma \in \mathbb{R} \textit{ and } c_\gamma > 0. \tag{29}$$

Example 1. Consider the scoring function

$$g = f^{(5,-4,-1,0)} \circ f^{(3,\sqrt{2},-2,-1-\sqrt{2})}.$$

Then

$$U_1 = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{Q}^4 : 3x_1 - 2x_3 - x_4 = 0 \text{ and } x_2 - x_4 = 0\},$$

so that

$$\bar{U}_1^* = \{b(3,0,-2,-1) + c(0,1,0,-1) : b, c \in \mathbb{R}\}.$$

Hence the family of rationally irreducible scoring functions which are rationally equivalent to g are precisely those of form $f^{t^2} \circ f^{t^1}$, where

$$t^2 \in \{a(5,-4,-1,0) + b(3,0,-2,-1) + c(0,1,0,-1) : a, b, c \in \mathbb{R}, a > 0\},$$
$$t^1 \in \{d(3,\sqrt{2},-2,-1-\sqrt{2}) : d \in \mathbb{R}, d > 0\}.$$

The family of irreducible scoring functions equivalent to the above g are those of form

$$f^{a(5,-4,-1,0)+b(3,\sqrt{2},-2,-1-\sqrt{2})} \circ f^{c(3,\sqrt{2},-2,-1-\sqrt{2})},$$

where $a, b, c \in \mathbb{R}$ and $a, c > 0$.

7. The Borda function. We say that a SCF f has the *cancellation property* if, whenever x is a profile such that the number of voters preferring a_i to a_j equals the number preferring a_j to a_i for all pairs $a_i \neq a_j$, then $f(x) = A$. Any profile of this type will be called *balanced*. We say that a SCF is *faithful* if the choice set for a single individual is the singleton set consisting of that individual's most preferred alternative. In other words, a SCF is faithful if "socially most preferred" and "individually most preferred" have the same meaning when society consists of a single individual.

The *Borda function* of order m is the simple scoring function f^s defined by $s = (m, m-1, \cdots, 2, 1)$. Clearly, any m-vector s for which $s_1 > s_2$ and $s_i - s_{i+1} = s_{i+1} - s_{i+2}$ for $1 \leqq i \leqq m-2$ defines the same function.

THEOREM 3. *For any fixed number m of alternatives, there is one and only one social choice function that is neutral, consistent, faithful, and has the cancellation property—namely, Borda's function.*

Proof. The Borda function clearly has the given properties. Conversely, let f have these properties. As shown in [8, Lemma 5], any SCF that is consistent and has the cancellation property is anonymous. Hence f is symmetric and consistent, so it is representable by a scoring function on the domain $\mathbb{Q}^{m!}$, say $f(x) = f^{s^\alpha} \circ f^{s^{\alpha-1}} \circ \cdots \circ f^{s^1}(x)$ for all $x \in \mathbb{Q}^{m!}$. We shall show that for every β, $1 \leqq \beta \leqq \alpha$,

$$s_i^\beta - s_{i+1}^\beta = s_{i+1}^\beta - s_{i+2}^\beta, \qquad 1 \leqq i \leqq m-2. \tag{30}$$

Fix i, $1 \leqq i \leqq m-2$. Define $x \in \mathbb{Q}^{m!}$ such that $x_p = 1$ for

$$p = \begin{pmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_m \end{pmatrix}, \begin{pmatrix} a_m \\ a_{m-1} \\ \cdot \\ \cdot \\ \cdot \\ a_1 \end{pmatrix},$$

and $x_p = 0$ otherwise. Define $y \in \mathbb{Q}^{m!}$ such that $y_p = 1$ for

$$p = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ a_{m-1} \\ a_m \end{pmatrix}, \begin{pmatrix} a_m \\ a_{m-1} \\ \vdots \\ a_{i+2} \\ a_i \\ a_{i+1} \\ a_{i-1} \\ \vdots \\ a_2 \\ a_1 \end{pmatrix}, \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{i-1} \\ a_{i+2} \\ a_{i+1} \\ a_i \\ a_{i+3} \\ \vdots \\ a_{m-1} \\ a_m \end{pmatrix}, \begin{pmatrix} a_m \\ a_{m-1} \\ \vdots \\ a_{i+3} \\ a_{i+1} \\ a_i \\ a_{i+2} \\ a_{i-1} \\ \vdots \\ a_2 \\ a_1 \end{pmatrix},$$

and $y_p = 0$ otherwise.

Then x and y are balanced, so $f(x) = f(y) = A$. Hence for $1 \leqq \beta \leqq \alpha$, we must have

$$D_i(x) \cdot s^\beta = D_{i+1}(x) \cdot s^\beta = D_{i+2}(x) \cdot s^\beta$$

or

$$s_i^\beta + s_{m-i+1}^\beta = s_{i+1}^\beta + s_{m-i}^\beta = s_{i+2}^\beta + s_{m-i-1}^\beta. \tag{31}$$

Also,

$$D_i(y) \cdot s^\beta = D_{i+1}(y) \cdot s^\beta, \qquad 1 \leqq \beta \leqq \alpha,$$

that is,

$$s_i^\beta + s_{i+2}^\beta + 2s_{m-i}^\beta = 2s_{i+1}^\beta + s_{m-i-1}^\beta + s_{m-i+1}^\beta$$

or

$$(s_i^\beta - s_{i+1}^\beta) + (s_{m-i}^\beta - s_{m-i+1}^\beta) = (s_{i+1}^\beta - s_{i+2}^\beta) + (s_{m-i-1}^\beta - s_{m-i}^\beta),$$

and the latter combined with (31) gives (30).

Since f is faithful, we must have $s_1^1 > s_2^1$. Combined with (30) this implies that f^{s^1} is the Borda function. But (30) also implies that $s^\beta = \lambda^\beta s^1 + \mu^\beta$ for appropriate λ^β and $\mu^\beta \in \mathbb{R}$, hence $f^{s^\alpha} \circ f^{s^{\alpha-1}} \circ \cdots \circ f^{s^1} \approx f^{s^1}$. $\square$

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