Games with Perfect Information

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Non-Cooperative Game Theory

- What is it?

  Mathematical study of interactions between rational and self-interested agents.

- Non-Cooperative

  Focus on agents who make their own individual decisions without coalition

Perfect Information

- All players know the game structure.
- Each player, when making any decision, is perfectly informed of all the events that have previously occurred.

Definition of Normal-Form Game

- A finite n-person game, $G =< N, A, u >$
  - $N = \{1, 2, ..., n\}$ is the set of players.
  - $A = \{A_1, A_2, ..., A_n\}$ is a set of available actions. $a = (a_1, a_2, ..., a_n) \in A$ is an action profile (or a pure strategy profile).
  - $u = \{u_1, u_2, ...u_n\}$ is a set of utility functions for n agents.
- A strategy is a complete contingent plan that defines the action an agent will take in all states of the world. Pure strategies (as opposed to mixed strategies) are the same as actions of agents.
- Players move simultaneously.
- Matrix representation for 2-person games

Example: Battle of the Sexes

- A husband and wife want to go to movies. They can select between "Devils wear Parada" and "Iron Man". They prefer to go to the same movie, but while the wife prefers "Devils wear Parada" the husband prefers "Iron Man". They need to make the decision independently.

		Husband	
		DWP	IM
Wife	DWP	10, 5	0, 0
	IM	0, 0	5, 10

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Best-Response Correspondences

- The best-response correspondence $\mathrm{BR}_i(a_{-i}) \in A_i$ are the set of strategies that maximizes agent i's utility given other agents' strategy a_{-i}.
- Compute every agent's best-response correspondences. The fixed points of the best-response correspondences, i.e. $a^* \in \mathrm{BR}(a^*)$, are the NEs of the game.

Example: Continuous Strategy Space

Cournot Competition

- Two suppliers producing a homogeneous good need to choose their production quantity, q_i and q_2. The demand that they are facing is $p(Q) = 1000 - Q$, where $Q = q_1 + q_2$. Unit cost $c > 0$.
- Utility: $u_1(q_1, q_2) = q_1 * [p(q_1 + q_2) - c]$
- Best-response of supplier 1 is $\mathrm{BR}_1(q_2) = \arg\max_{q_1} u_1(q_1, q_2)$

Example: Matching Pennies

- Each of the two players has a penny. They independently choose to display either heads or tails. If the two pennies are the same, player 1 takes both pennies. If they are different, player 2 takes both pennies.

	Heads	Tails
Heads	1, -1	-1, 1
Tails	-1, 1	1, -1

Mixed Strategies

- A mixed strategy of agent i, $\sigma_i \in \mathbf{\Delta}(A_i)$, defines a probability, $\sigma_i(a_i)$ for each pure strategy $a_i \in A_i$.
- Agent i's expected utility of following strategy σ_i is

$$u_i(\sigma) = \sum_{a \in A_i} P_\sigma(a) u_i(a)$$

- The support of σ_i is the set of pure strategies $\{a_i : \sigma(a_i) > 0\}$.
- Pure strategies are special cases of mixed strategies.

Mixed Strategy Nash Equilibrium

- Mixed-strategy prole $\sigma^* = \{\sigma_1^*, \sigma_2^*, ..., \sigma_n^*\}$ is a Nash equilibrium in a game if, for all i,

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*), \quad \forall \sigma_i \in \mathbf{\Delta}(A_i).$$

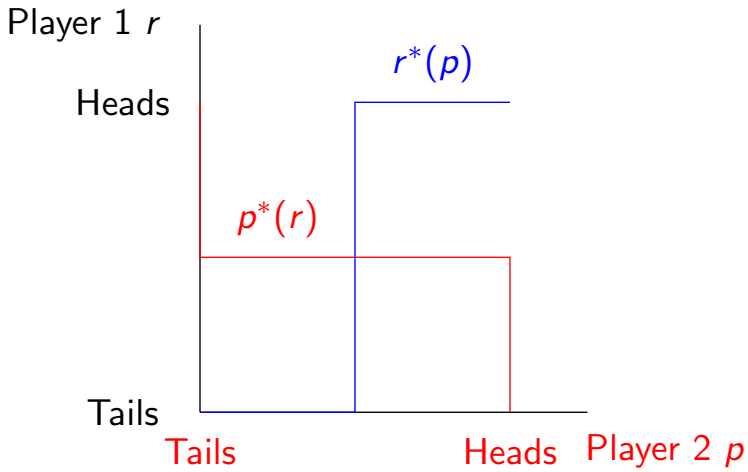
- Theorem (Nash 1951): Every game with a finite number of players and action profiles has at least one Nash equilibrium.
- All pure strategies in the support of agent i at a mixed strategy Nash Equilibrium have the same expected utility.

Finding Mixed Strategy Nash Equilibrium

	Heads	Tails
Heads	1, -1	-1, 1
Tails	-1, 1	1, -1

- Find the fixed point of the Best-Response correspondences.

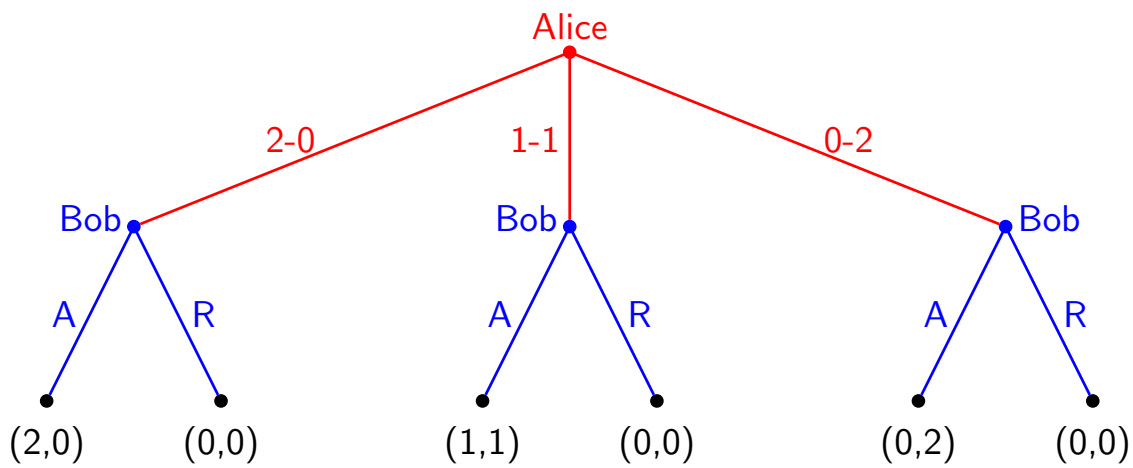
Fixed Point



Extensive-Form Game with Perfect Information

Example: The Sharing Game

- Alice and Bob try to split two indivisible and identical gifts. First, Alice suggests a split: which can be "Alice keeps both", "they each keep one", and "Bob keeps both". Then, Bob chooses whether to Accept or Reject the split. If Bob accepts the split, they each get what the split specifies. If Bob rejects, they each get nothing.



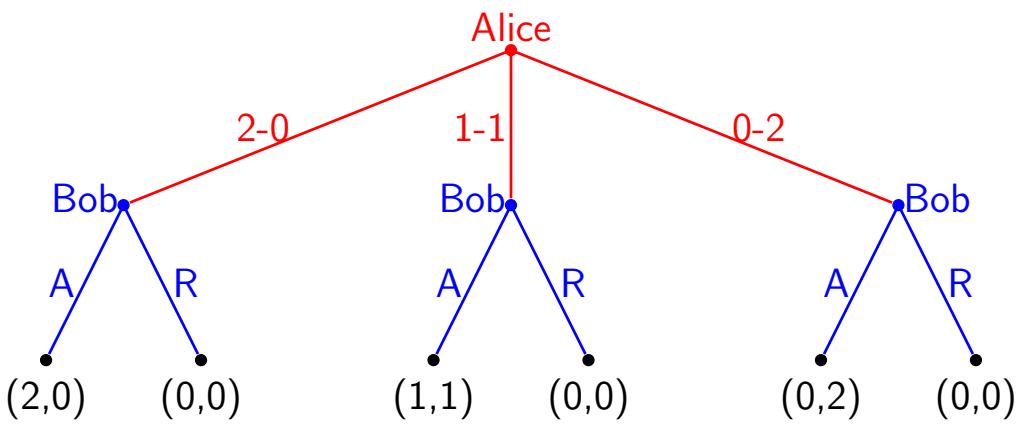
Loosely Speaking...

- Extensive Form
  - A detailed description of the sequential structure of the decision problems encountered by the players in a game.
  - Often represented as a game tree
- Perfect Information
  - All players know the game structure (including the payoff functions at every outcome).
  - Each player, when making any decision, is perfectly informed of all the events that have previously occurred.

Def. of Perfect-Information Extensive-Form Games

- A perfect-information extensive-form game, $G = (N, H, P, u)$
  - $N = \{1, 2, ..., n\}$ is the set of players.

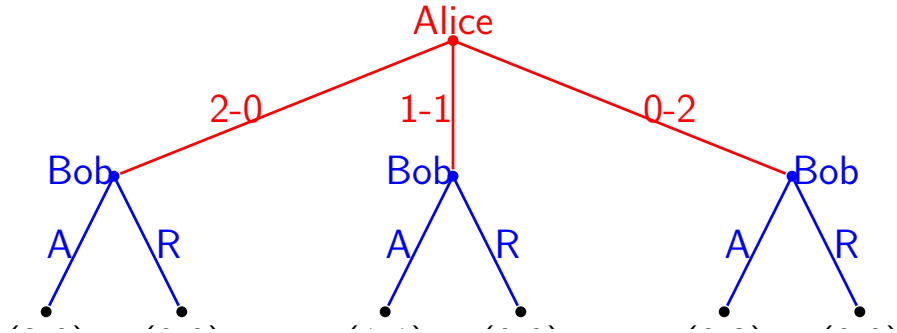
    N={Alice, Bob}



Def. of Perfect-Information Extensive-Form Games

- A perfect-information extensive-form game, $G = (N, H, P, u)$
  - H is a set of sequences (finite or infinite)
    - $\Phi \in H$
    - $h = (a^k)_{k=1,...,K} \in H$ is a history
    - If $(a^k)_{k=1,...,K} \in H$ and $L < K$, then $(a^k)_{k=1,...,L} \in H$
    - $(a^k)_{k=1}^{\infty} \in H$ if $(a^k)_{k=1,...,L} \in H$ for all positive L
    - Z is the set of terminal histories.

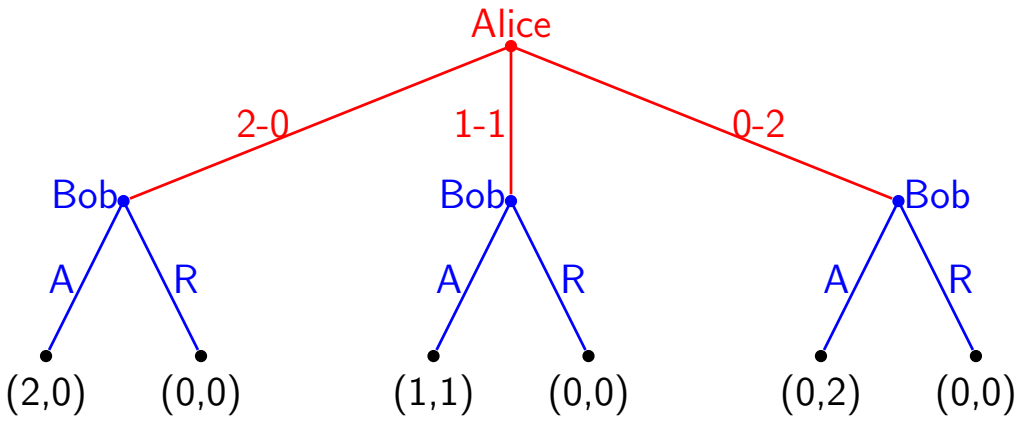
    $H = \{\Phi, 2-0, 1-1, 0-2, (2-0, A), (2-0, R), (1-1, A), (1-1, R), (0-2, A), (0-2, R)\}$



Def. of Perfect-Information Extensive-Form Games

- A perfect-information extensive-form game, $G = (N, H, P, u)$
  - P is the player function, $P : H \backslash Z \rightarrow N$.

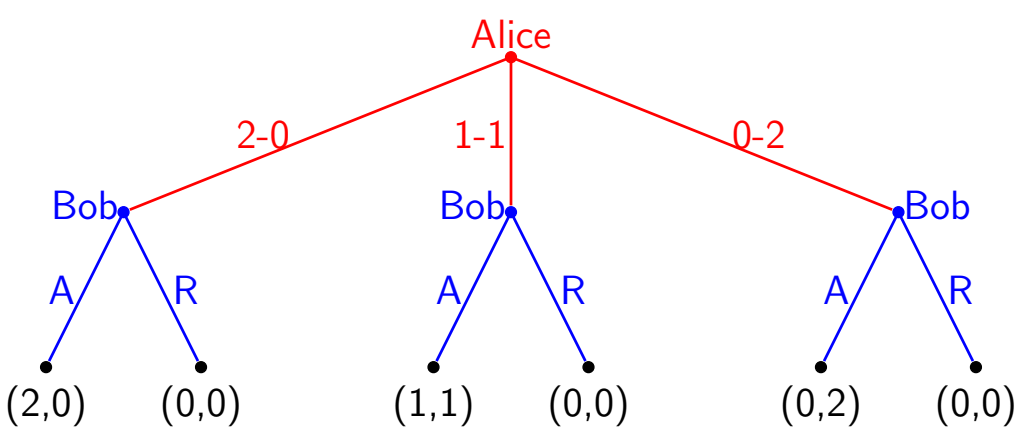
    $P(\Phi)$=Alice
    $P(2-0)$=Bob
    $P(1-1) =$ Bob
    $P(0-2) =$ Bob



Def. of Perfect-Information Extensive-Form Games

- A perfect-information extensive-form game, $G = (N, H, P, u)$
  - $u = \{u_1, u_2, ...u_n\}$ is a set of utility functions, $u_i : Z \rightarrow \mathcal{R}$.

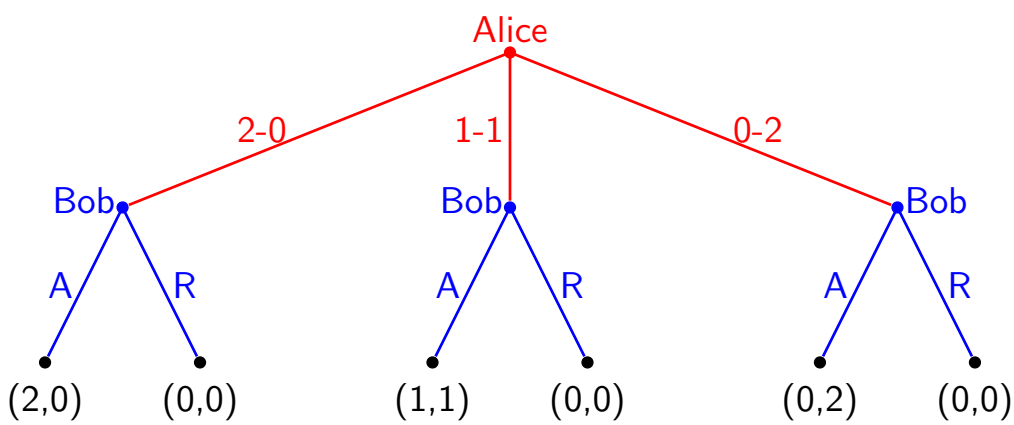
    $u_1((2-0, A)) = 2$
    $u_2((2-0, A)) = 0$
    ...



Pure Strategies in Perfect-Information Extensive-Form Games

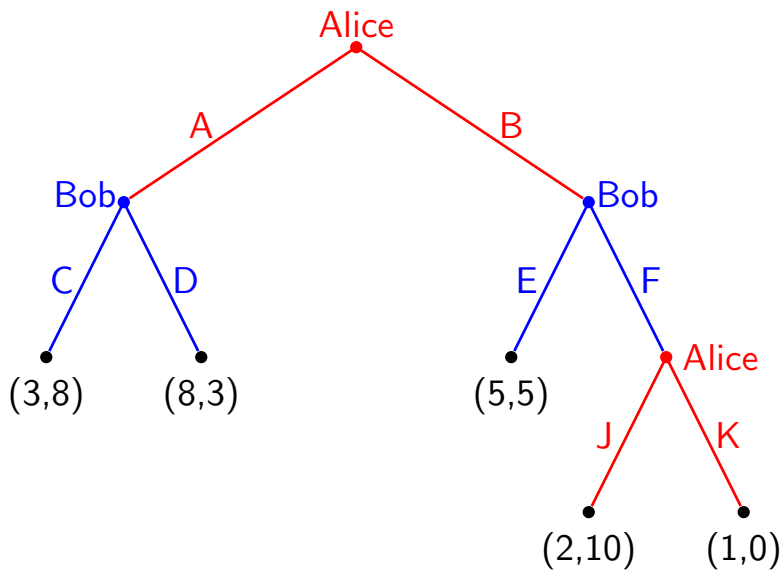
- A pure strategy of player $i \in N$ in an extensive-form game with perfect information, $G = (N, H, P, u)$, is a function that assigns an action in $A(h)$ to each non-terminal history $h \in H \backslash Z$ for which $P(h) = i$.
  - $A(h) = \{a : (h, a) \in H\}$
  - A pure strategy is a contingent plan that specifies the action for player i at every decision node of i.

Pure Strategies for the Sharing Game



$S = \{S_1, S_2\}$
E.g. $s_1 = (2 - 0 \text{ if } h = \Phi)$,
$s_2 = (A \text{ if } h = 2 - 0;\ \text{R if h} = \text{1-1}; R \text{ if } h = 0 - 2)$.

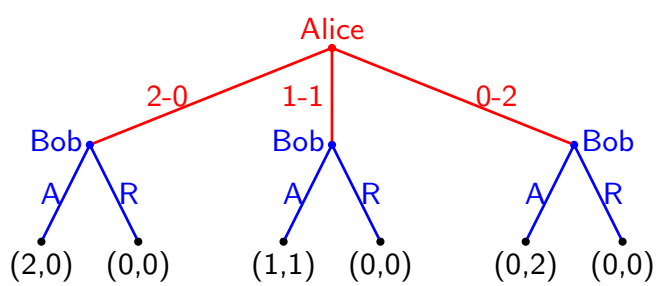
Pure Strategies



$S = \{S_1, S_2\}$
E.g. $s_1 = (A \text{ if } h = \Phi;\ J \text{ if } h = BF)$
$s_2 = (C \text{ if } h = A;\ F \text{ if } h = B)$

Normal-Form Representation

A perfect-information extensive-form game $\Rightarrow$ A normal-form game

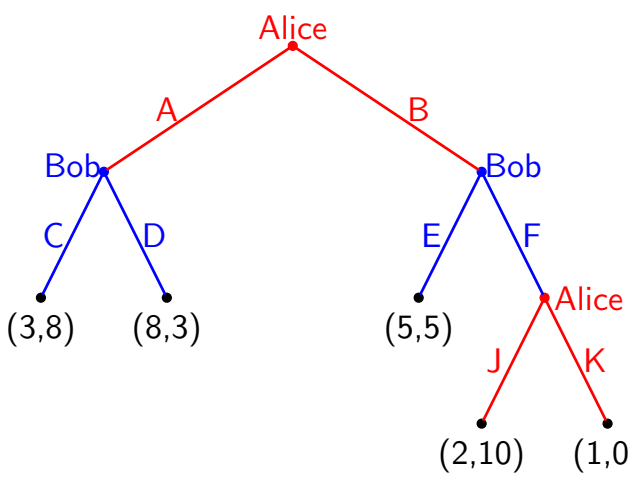


	(A,A,A)	(A,A,R)	(A,R,A)	(A,R,R)	(R,A,A)	(R,A,R)	(R,R,A)	(R,R,R)
2-0	2,0	2,0	2,0	2,0	0,0	0,0	0,0	0,0
1-1	1,1	1,1	0,0	0,0	1,1	1,1	0,0	0,0
0-2	0,2	0,0	0,2	0,0	0,2	0,0	0,2	0,0

A normal-form game $\not\Rightarrow$ A perfect-information extensive-form game

Normal-Form Representation: Example 2

A perfect-information extensive-form game $\Rightarrow$ A normal-form game

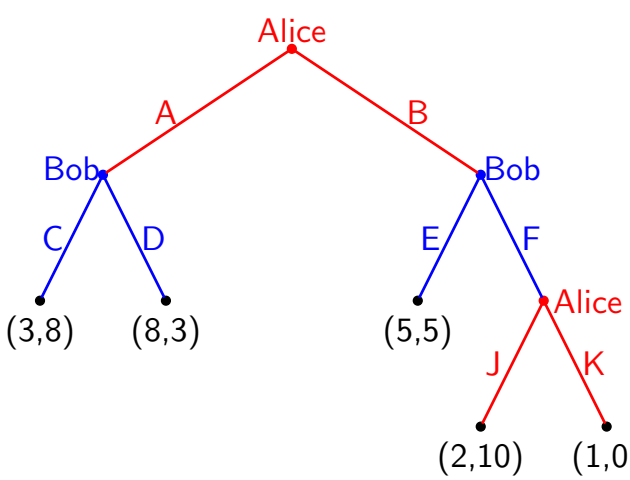


	(C, E)	(C, F)	(D, E)	(D, F)
(A, J)	3, 8	3, 8	8, 3	8, 3
(A, K)	3, 8	3, 8	8, 3	8, 3
(B, J)	5, 5	2,10	5, 5	1, 10
(B, K)	5, 5	1, 0	5, 5	1, 0

A normal-form game $\not\Rightarrow$ A perfect-information extensive-form game

Pure Strategy Nash Equilibrium in Perfect-Information Extensive-Form Games

- A pure strategy profile s is a weak Nash Equilibrium if, for all agents i and for all strategies $s_i' \neq s_i$, $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$. (Same as in normal-form games)



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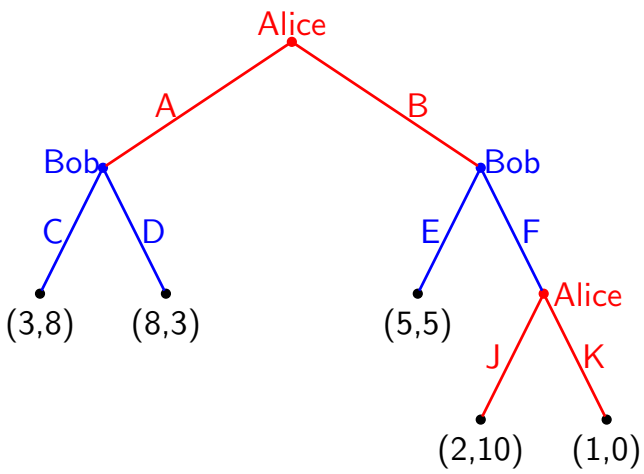

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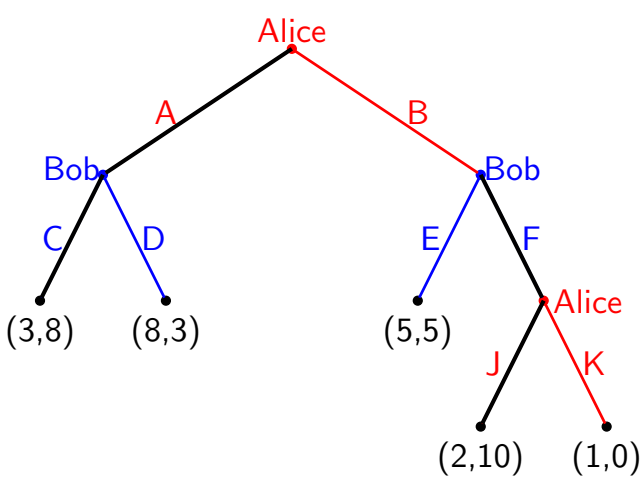

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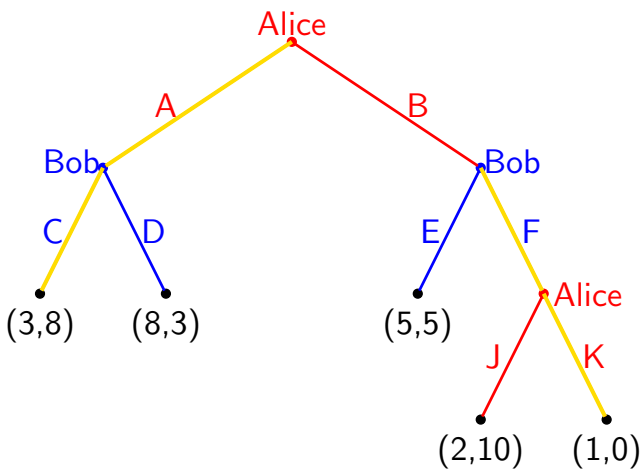

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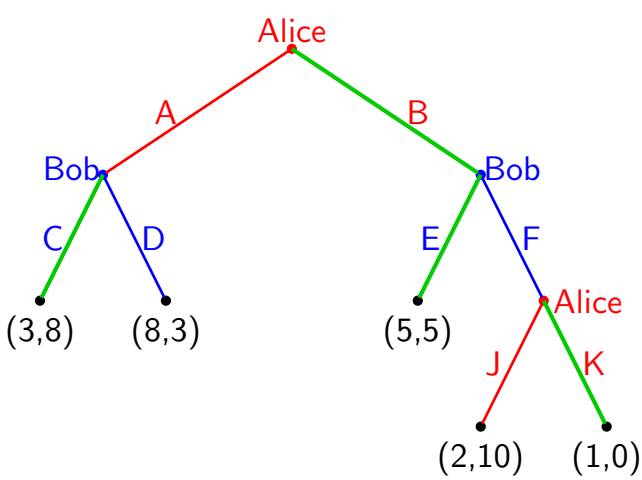

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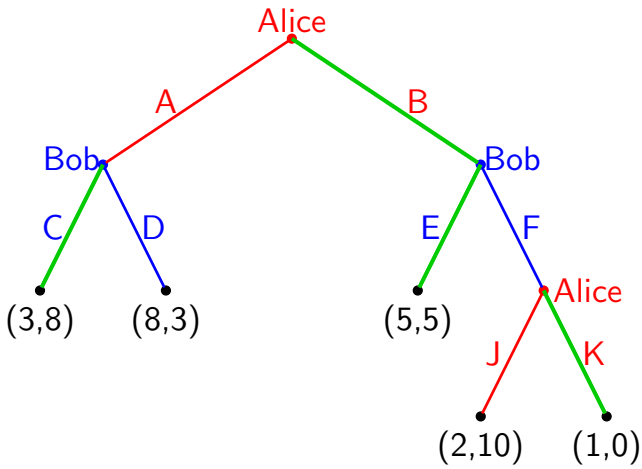
Nash Equilibrium and Non-Credible Threat

- Nash Equilibrium is not a very satisfactory solution concept for perfect-information extensive-form games.



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Let's play a game now

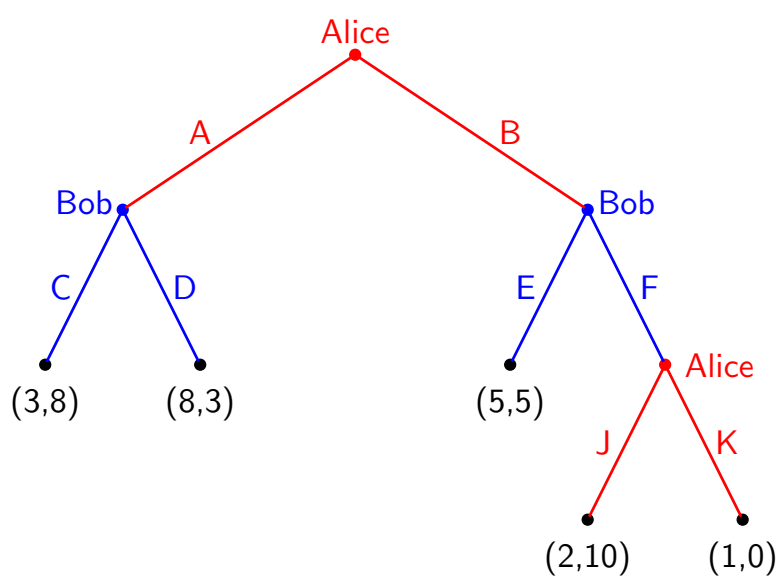
- You are part of a fictitious pride of lions that has a strict hierarchy.
- You are ordered from largest (left) to smallest (right)
- All lions prefer to become larger and can do so by eating the next largest lion if that lion is asleep
- Lions sleep after eating

One day a deer passes by the pride. The largest lion would like to eat it, but is afraid of being eaten by the second largest lion once he falls asleep. Should the largest lion eat the deer?

Subgame Perfect Equilibrium

- Sequential Rationality: A player's equilibrium strategy should specify optimal actions at every point in the game tree.
- A Subgame Perfect Equilibrium (SPE) of a perfect-information extensive-form game G is a strategy profile s such that for any subgame G' of G, the restriction of s to G' is a NE.
- Every SPE is a NE, but not vice versa.
- **Thm:** Every finite extensive-form game with perfect information has a subgame perfect equilibrium.
  - Finite: The set of sequences H is finite.

Find A SPE: Backward Induction



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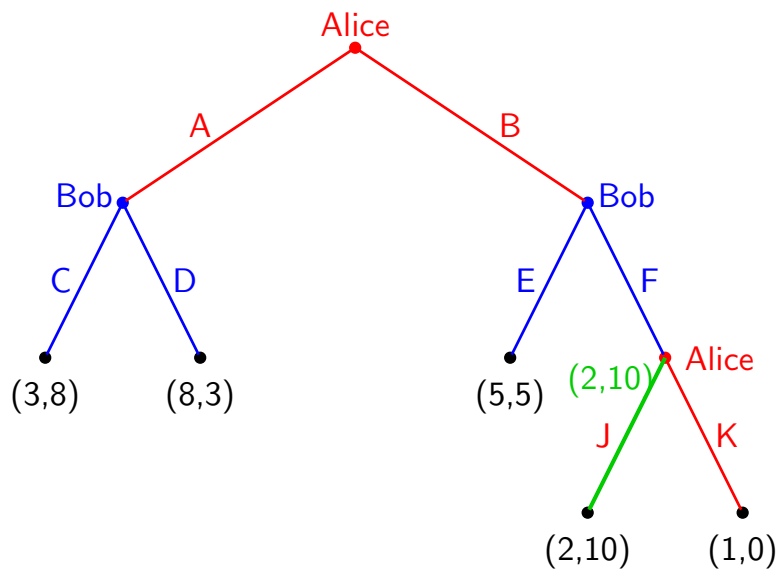

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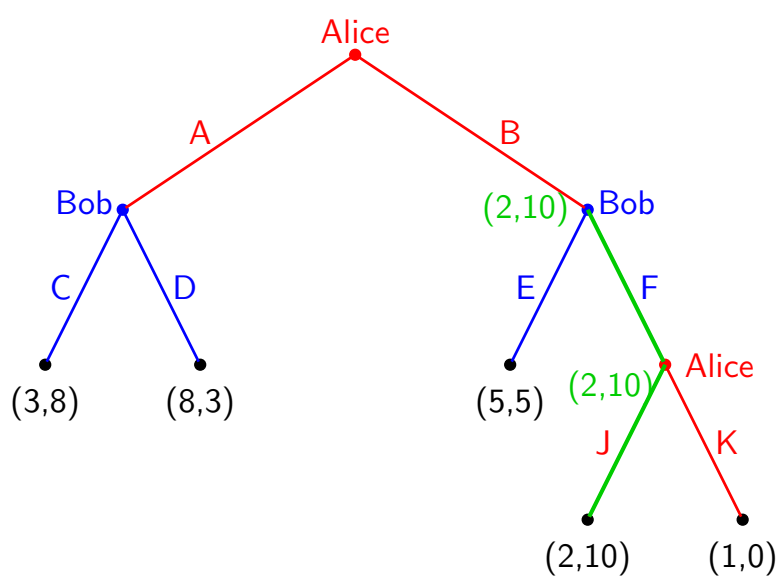

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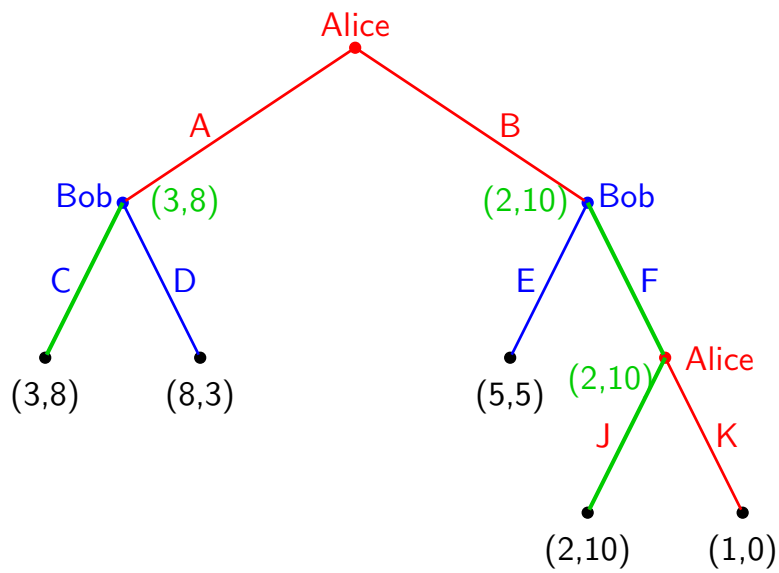
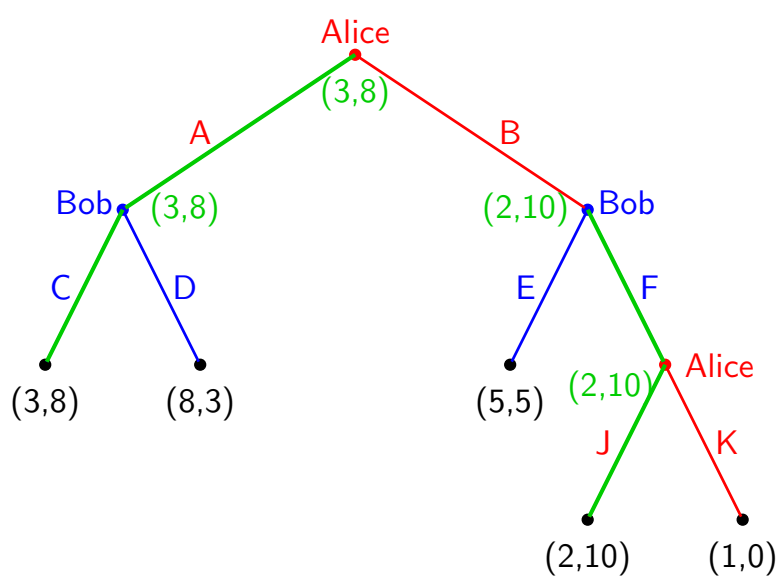


Find A SPE: Backward Induction



Note on Computational Complexity

- Finding NE for general normal-form games requires time exponential in the size of the normal form.
- The induced normal form of an extensive-form game is exponentially larger than the original representation.
- Algorithm of backward induction requires time linear in the size of the extensive-form game. (Depth-first transverse)
- For zero-sum extensive-form games, we can slightly improve the running time.

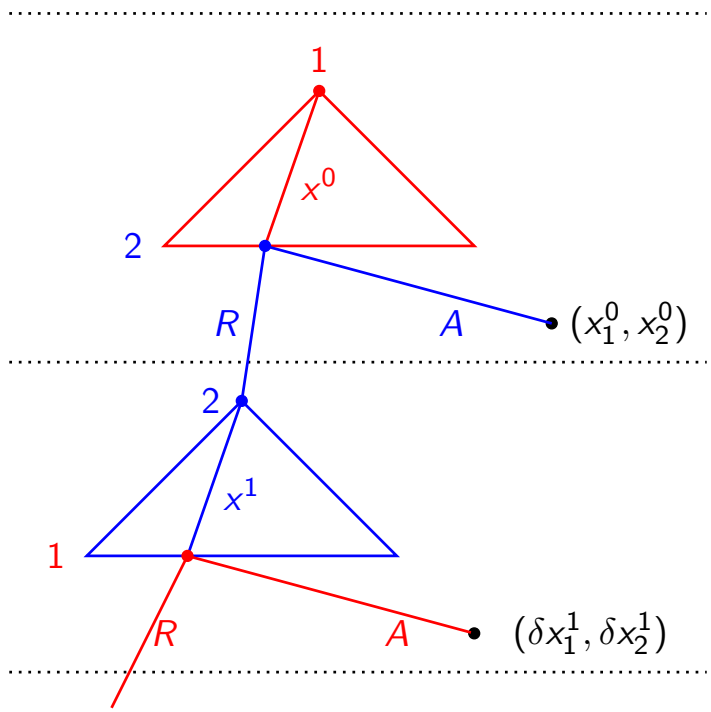
A Bargaining Game: Split-the-Pie

- Two players trying to split a desirable pie. The set of all possible agreements X is the set of all divisions of the pie,

$$X = \{(x_1, x_2) : x_i \geq 0 \text{ for } i = 1, 2 \text{ and } x_1 + x_2 = 1\}.$$

- The first move of the game occurs in period 0, when player 1 makes a proposal $x^0 \in X$, then player 2 either accepts or rejects. Acceptance ends the game while rejection leads to period 1, in which player 2 makes a proposal $x^1 \in X$, which player 1 has to accept or reject. Again, acceptance ends the game; rejection leads to period 2, in which it is once again player 1's turn to make a proposal. The game continues in this fashion so long as no offer has been accepted.
- $u_i(x, t) = \delta^t x_i$ if proposal x has been accepted in period t, $\delta \in (0, 1)$.
- $u_i = 0$ if no agreement has reached.

Split-the-Pie as A Perfect-Information Extensive-Form Game



Nash Equilibria of the Split-the-Pie Game

- The set of NEs is very large. For example, for any $x \in X$ there is a NE in which the players immediately agree on x.

  E.g. Player 1 always propose (0.99, 0.01) and only accepts a proposal (0.99, 0.01).

SPE of the Split-the-Pie Game

The unique SPE of the game is

- Player 1 always proposes $(\frac{1}{1+\delta}, \frac{\delta}{1+\delta})$ and accepts proposals that has $x_1 \geq \frac{\delta}{1+\delta}$.
- Player 2 always proposes $(\frac{\delta}{1+\delta}, \frac{1}{1+\delta})$ and accepts proposals that has $x_2 \geq \frac{\delta}{1+\delta}$.