Kelly Betting & Prediction Markets
@ CS286r Fall 2012, Harvard

David Pennock
Microsoft Research
A prediction market

- A random variable, e.g.

Will US go into recession in 2012? (Y/N)

- Turned into a financial instrument payoff = realized value of variable

I am entitled to:

$1 if Recession in 2012
$0 if No Recession in 2012
The US Economy will go into Recession during 2012

Last prediction was: $0.59 / share
Today's Change: -
Contract Type: 0-100

Event: US Economy in Recession (*see contract rules for definition*)

View All Un-Matched Predictions

<table>
<thead>
<tr>
<th>Best (highest) price members are buying at</th>
<th>Best (lowest) price members are selling at</th>
</tr>
</thead>
<tbody>
<tr>
<td>Price per share</td>
<td>Quantity</td>
</tr>
<tr>
<td>$0.60</td>
<td>5 shares</td>
</tr>
<tr>
<td>$0.59</td>
<td>56 shares</td>
</tr>
<tr>
<td>$0.37</td>
<td>5 shares</td>
</tr>
</tbody>
</table>
Between 6.0% and 8.0% chance
Kelly betting

• You have $1000
• Market price \( mp \) is 0.05
• You think probability \( p \) is 0.10
• Q: How much should you bet?
Kelly betting

• You have $1000
• Market price mp is 0.05
• You think probability p is 0.10
• Q: How much should you bet?
• Wrong A: $1000
  P(bankruptcy) → 1
Kelly betting

- You have $1000
- Market price $mp$ is 0.05
- You think probability $p$ is 0.10
- Q: How much should you bet?
- Wrong A: fixed $1
  No compounding magic
Kelly betting

• You have $1000
• Market price mp is 0.05
• You think probability p is 0.10
• Q: How much should you bet?
• Optimal A: $52.63
  or f* fraction of your wealth
where $f^* = \frac{(p - mp)}{(1 - mp)}$ “edge/odds”
Why?

- Kelly betting ≡ maximizing log utility
  - Maximizes compounding growth rate
    Maximizes geometric mean of wealth
  - Minimizes expected doubling time
    Minimizes exp time to reach, say, $1M
  - Does not maximize expected wealth
    (“All in” does, but ensures bankruptcy)
Compounding

• You put $10,000 into the bank at 5% rate
  – Compounded annually: $10,000 * 1.05 = 10,500
  – Monthly: $10,000 * (1+.05/12)^{12} = $10,511.62
  – “Secondly”: $10,000 * (1+.05/31.5M)^{31.5M} = $10,512.71
  – Compounded continuously: $10,000 * e^{.05} = $10,512.71

• Good way to compare rates:
  – $W_t = W_0 e^{rt}$
  – $r = 1/t * \ln(W_t/W_0)$
Choosing a bank

<table>
<thead>
<tr>
<th>Institution</th>
<th>APY</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sallie Mae (MMA)</td>
<td>1.05%</td>
</tr>
<tr>
<td>Union Federal Savings Bank</td>
<td>1.05%</td>
</tr>
<tr>
<td>CIT Bank (Member FDIC)</td>
<td>1.05%</td>
</tr>
<tr>
<td>EverBank (MMA)</td>
<td>1.01%</td>
</tr>
<tr>
<td>BARCLAYS (Savings)</td>
<td>1.00%</td>
</tr>
<tr>
<td>ally BANK (Member FDIC)</td>
<td>0.95%</td>
</tr>
</tbody>
</table>
Choosing an investment

- S&P 500, 1988-2011
  \[ r = 18.45\% \]
Choosing an investment

- S&P 500, 1988-2011

<table>
<thead>
<tr>
<th>Year</th>
<th>Annual Return</th>
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<td>16.61%</td>
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Average 18.45%
Kelly = maximize $E[r]$

- Initial wealth $W_0 \rightarrow$ Final wealth $W_1$
  $$W_1 = W_0 e^r$$

- Kelly: Don’t maximize $E[W_1]$
  Instead, maximize $E[r]$
  $$= E[\ln(W_1/W_0)]$$
  $$= E[\ln W_F] - \ln W_0$$
The Long Run

• Ignores variance of r!
• But in the long (enough) run, variance doesn’t matter
• \( W_T = W_0 e^{r_1} e^{r_2} e^{r_3} \ldots \)
  \( \ln W_T = \ln W_0 + r_1 + r_2 + r_3 + \ldots \)
  \( \ln W_T \approx \ln W_0 + t \cdot E[r] \) (law of large #s)
  \( W_T \approx W_0 e^{t \cdot E[r]} \)

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Average 18.45%
The Long Run

- Suppose you could invest for 1000s or millions of years if need be
- Or imagine blackjack – new independent bet every few seconds
- Strategy with higher E[r] will win out with high probability

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Average 18.45%
Decision making under uncertainty, an example

ABC TV’s “Who Wants to be a Millionaire?”
Decision making under uncertainty, an example

\[ v_{15} = \]

- $1,000,000$ if correct
- $32,000$ if incorrect
- $500,000$ if walk away
Decision making under uncertainty, an example

if you answer:

$E[v_{15}] = \$1,000,000 \times Pr(\text{correct}) + \$32,000 \times Pr(\text{incorrect})$

if you walk away:

\$500,000
Decision making under uncertainty, an example

if you answer:

\[ E[v_{15}] = 1,000,000 \times 0.5 + 32,000 \times 0.5 = 516,000 \]

if you walk away:

$500,000

you should answer, right?
Decision making under uncertainty, an example

Most people won’t answer: risk averse

\[ U(x) = \log(x) \]

if you answer:

\[ E[u_{15}] = \log(1,000,000) \times 0.5 + \log(32,000) \times 0.5 \]

\[ = \frac{6}{2} + \frac{4.5}{2} = 5.25 \]

if you walk away:

\[ \log(500,000) = 5.7 \]
Decision making under uncertainty, an example

Maximizing $E[u_i]$ for $i < 15$

Q7, $L = \{1, 3\}$

walk

answer

L1

L3

Q7, $L = \{3\}$

log($2k$)

$\sqrt{0.4} \times 0.6$

Q8, $L = \{1, 3\}$

log($1k$)

Q8, $L = \{3\}$

log($2k$)

$\sqrt{0.8} \times 0.2$

log($1k$)
Decision making under uncertainty, in general

\[ \Omega = \text{set of all possible future states of the world} \]
Decision making under uncertainty, in general

\( \omega \) are disjoint exhaustive states of the world

\( \omega_i \): rain tomorrow & Bush elected & Y! stock up & car not stolen & ...

\( \omega_j \): rain tomorrow & Bush elected & Y! stock up & car stolen & ...
Decision making under uncertainty, in general

Equivalent, more natural:

\[ \Omega = 2^n \]

- \( E_i \): rain tomorrow
- \( E_j \): Bush elected
- \( E_k \): Y! stock up
- \( E_l \): car stolen
Decision making under uncertainty, in general

- Preferences, utility
  \[ \omega_i > \omega_j \iff u(\omega_i) > u(\omega_j) \]

- Expected utility
  \[ E[u] = \sum_\omega \Pr(\omega)u(\omega) \]

- Decisions (actions) can affect \( \Pr(\Omega) \)

- What you should do: 
  choose actions to maximize expected utility

- Why?: To avoid being a money pump [de Finetti’ 74], among other reasons...
Preference under uncertainty

- Define a prospect, \( \mu = [p, \omega_1; \omega_2] \)
- Given the following axioms of \( \succeq \):
  - orderability: \( (\omega_1 \succeq \omega_2) \lor (\omega_1 \preceq \omega_2) \lor (\omega_1 \sim \omega_2) \)
  - transitivity: \( (\omega_1 \succeq \omega_2) \land (\omega_2 \succeq \omega_3) \rightarrow (\omega_1 \succeq \omega_3) \)
  - continuity: \( \omega_1 \succeq \omega_2 \succeq \omega_3 \rightarrow \exists p. \omega_2 \sim [p, \omega_1; \omega_3] \)
  - substitution: \( \omega_1 \sim \omega_2 \rightarrow [p, \omega_1; \omega_3] \sim [p, \omega_2; \omega_3] \)
  - monotonicity: \( \omega_1 \succeq \omega_2 \land p > q \rightarrow [p, \omega_1; \omega_2] \succeq [q, \omega_1; \omega_2] \)
  - decomposability:
    \[ [p, \omega_1; [q, \omega_2; \omega_3]] \sim [q, [p, \omega_1; \omega_2]; [p, \omega_1; \omega_3]] \]
- Preference can be represented by a real-valued expected utility function such that:
  \[ u([p, \omega_1; \omega_2]) = p \ u(\omega_1) + (1-p)u(\omega_2) \]
Utility functions

• (\(\theta \in \Theta\) a probability distribution over \(\Omega\))
• \(E[u] : \Theta \rightarrow \mathbb{R}\) represents preferences,
  \(E[u](\theta) \geq E[u](\theta')\) iff \(\theta \geq \theta'\)
• Let \(\phi(\omega) = au(\omega) + b, a > 0\).
  – Then \(E[\phi](\theta) = E[au+b](\theta) = a E[u](\theta) + b\).
  – Since they represent the same preferences, \(\phi\) and \(u\) are strategically equivalent \((\phi \sim u)\).
Utility of money

• Outcomes are dollars

• Risk attitude:
  – risk neutral: \( u(x) \sim x \)
  – risk averse (typical):
    \( u \) concave \( (u''(x) < 0 \text{ for all } x) \)
  – risk prone: \( u \) convex

• Risk aversion function:
  \[ r(x) = -\frac{u''(x)}{u'(x)} \]
Risk aversion & hedging

<table>
<thead>
<tr>
<th>$\omega_1$: car stolen</th>
<th>$\omega_2$: car not stolen</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u(\omega_1) = \log(10,000)$</td>
<td>$u(\omega_2) = \log(20,000)$</td>
</tr>
</tbody>
</table>

- $E[u] = 0.01 \cdot 4 + 0.99 \cdot 4.3 = 4.2980$
- Action: buy $10,000 of insurance for $125
  - $u(\omega_1) = \log(19,875)$
  - $u(\omega_2) = \log(19,875)$
  - $E[u] = 4.2983$
- Even better, buy $5974.68 of insurance for $74.68
  - $u(\omega_1) = \log(15,900)$
  - $u(\omega_2) = \log(19,925)$
  - $E[u] = 4.2984 \iff \text{Optimal}$
What utility function?

- Economist: Only you can choose
- Engineer: Let’s find out
Performance of Prediction Markets with Kelly Bettors

David Pennock, Microsoft*
John Langford, Microsoft*
Alina Beygelzimer (IBM)

*work conducted at Yahoo!
Wisdom of crowds

When to guess: if you’re in the 99.7th percentile

More:
http://www.overcomingbias.com/2007/02/how_and_when_to.html
Can a prediction market learn?

Typical
- Mkt >> avg expert
- Static

This paper
- Mkt >? best expert
- Dynamic/Learning (experts algos)
- Assumption: Agents optimize growth rate (Kelly)
Market Model

subjective probability

$p_1, u_1, \ldots, p_n, u_n$

utility for money

$1 \text{ if } E_1$

$1 \text{ if } E_2$

$\vdots$

$1 \text{ if } E_s$

Competitive equilibrium prices

$mp^{<E_1>}, mp^{<E_2>}, \ldots$

$\equiv \sum q_i(mp) = 0$
Market Model

subjective probability
$p_1, \ln$

utility for money
$p_i, \ln$

$p_n, \ln$

$\text{Competitive equilibrium prices } mp^{<E_1>}, mp^{<E_2>}, \ldots$

$\equiv \Sigma q_i(mp)=0$

$q_i(mp)=w_i/mp^*(p_i-mp)/(1-mp)$
Wealth-weighted average

• Assume all agents optimize via Kelly and are “price takers”
• Then $mp = \sum w_i p_i$
Fractional Kelly betting

- Bet $\lambda f^*$ fraction of your wealth, $\lambda \in [0,1]$.
- Why? Ad hoc reasons
  - Full Kelly is too risky (finite horizon)
  - I’m not confident of $p$ (2nd-order belief)
- Our (new) reason
  - $\lambda$ fraction Kelly behaviorally equiv to full Kelly with revised belief $\lambda p + (1-\lambda) mp$
  - Has Bayesian justification
Wealth-weighted average

• Assume all agents optimize via Kelly and are “price takers”
• Then \( mp = \sum w_i p_i \)
• Fractional Kelly: \( mp = \sum \lambda_i w_i p_i \)
Wealth dynamics

• Agents trade in a binary pred market
  Event outcome is revealed
  Wealth is redistrib. (losers pay winners)

**Repeat**

• After each round
  If event happens: \( w_i \leftarrow w_i \frac{p_i}{pm} \)
  If event doesn’t: \( w_i \leftarrow w_i \frac{(1-p_i)}{(1-pm)} \)

• Wealth is redistributed like Bayes rule!
Wealth dynamics

- Single security
- Multiperiod market
- Agents with log util
- Fixed beliefs
Fig. 1. a) Price (black line) versus the observed frequency (gray line) of the event over fifty time periods. The market consists of one hundred full-Kelly agents with initial wealth $w_i = 1$. b) Wealth after fifteen time periods versus belief for fifty Kelly agents. The event has occurred in three of the fifteen trials. The solid line is the posterior Beta distribution consistent with the observation of three successes in fifteen independent Bernoulli trials.
Wealth dynamics: Regret

• Theorem:
Mkt log loss < \min_i \text{agent } i \text{ log loss} - \ln w_i
• Applies for all agents, all outcomes, even adversarial
• Same bound as experts algorithms
⇒ No “price of anarchy”
Learning the Kelly fraction

- Competitive equilibrium: $\lambda = 1$
- Rational expectations equil: $\lambda = 0$
- Practical (wisdom of crowds): $\lambda = \varepsilon$
- A proposal: Fractional Kelly as an experts algorithm btw yourself and the market
  - Start with $\lambda = 0.5$
  - If right, increase $\lambda$; If wrong, decrease
  - Won’t do much worse than market (0)
  - Won’t do much worse than original prior p
Summary and Questions

• Self-interested Kelly bettors implement Bayes’ rule, minimize market’s log loss
• If mkt & agents care about log loss: win-win
• Questions
  – Can non-Kelly bettors be induced to minimize log loss?
  – Can Kelly bettors be induced to minimize squared loss, 0/1 loss, etc.?
  – Can/should we encourage Kelly betting?
Recommended Read

• Great science writing (Pulitzer prize winner)
• Fun read
• Accessible yet accurate
• Well researched
• Mob meets politics meets Time Warner meets Wall Street meets Vegas meets Bell Labs scientists