CS286r Computational Mechanism Design
Homework 2: Mechanism Design

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Due: Tuesday 2/26/2002, in the beginning of class. You may use any sources that you want, but you must cite the sources that you use. Teamwork is not allowed.

1. Consider a second-price sealed-bid (Vickrey) auction of one item, with bidders, i, with values, $v_i \in [0, \overline{v}]$, and quasilinear preferences, i.e. with $u_i(v_i, p) = v_i - p$, given price p.

   (a) (10 pts) Show that bid $b_i(\theta_i) = v_i$ for all values, $v_i \in [0, \overline{v}]$, is a weakly dominant strategy for each bidder i. [Prove this from first principles, do not use the fact that the Vickrey auction is a special case of the Groves mechanism].

   (b) (5 pts) Let $b_{(k)}$ denote the kth highest bid. Suppose that the seller introduces a reservation price, $r \in [0, 1]$, such that the item is only sold if $b_{(1)} \geq r$, for price $p = \max[r, b_{(2)}]$. Show that truthful bidding remains a weakly dominant strategy for bidders.

   (c) (5 pts) Consider the special case of an auction with a single bidder, with a Uniformly distributed value $v_1 \sim U(0, \overline{v})$. In addition, suppose that the seller has value, v_0, for the item. Verify that strategies, $r^*(v_0) = (v_0 + \overline{v})/2$, $b_1^*(v_1) = v_1$ form a Bayesian-Nash eq. of this reserve-price Vickrey auction.

   (d) (5 pts) In fact, $((v_0 + \overline{v})/2, v_1, \ldots, v_N)$, is also the Bayes-Nash eq. of the auction with N bidders, each with value v_i. Assuming, $\overline{v} = 1$ and $v_0 = 0$, determine the seller's expected revenue for the special case of **two** bidders. [**Hint:** construct an expression, by case analysis of the bids received, for the expected revenue to the seller. The fact, $E[v_{(2)} | v_{(2)} \geq 1/2] = 2/3$, where $v_{(2)}$ is the second-highest value across two bidders, will be useful.]

   (e) (5 pts) For this two-bidder case, compare the expected revenue in the reserve price Vickrey auction to that in the Vickrey auction with no reserve price, and provide an intuitive argument about the effect on allocative-efficiency. [**Hint:** The following fact is very helpful: the expected k^{th}

highest value among n values independently drawn from the uniform distribution on $[\underline{v}, \overline{v}]$ is $\underline{v} + \left(\frac{n+1-k}{n+1}\right)(\overline{v} - \underline{v})$.]

2. Consider a problem in which the outcome space, $\mathcal{O} \subset \mathbb{R}$, and each agent i, with type θ_i, has *single-peaked* preferences, $u_i(o, \theta_i)$ over outcomes. In particular, each agent, i, with type θ_i, has a *peak*, $p_i(\theta_i) \in \mathcal{O}$, such that $p(\theta_i) \geq d > d'$ or $d' > d \geq p(\theta_i)$ imply that $u_i(d, \theta_i) > u_i(d', \theta_i)$ (p.10–11, M.Jackson "Mechanism Theory" handout).

   (a) (10 pts) Show that the "median selection" mechanism, in which each agent declares its peak and the mechanism selects the median (with a tie break in the case of an even number of agents) is *strategyproof*, and implements a *Pareto Optimal* outcome.

   (b) (5 pts) Let N denote the number of agents. Suppose, in addition, that the mechanism can position its own $N-1$ "phantom peaks", before the peaks from the agents are received. Show that the median selection mechanism applied to the combined, $2N-1$, peaks remains strategyproof.

   (c) (5 pts) In combination with the phantom peaks, the median selection mechanism can implement a rich variety of outcomes. Describe a method to position the peaks to implement the kth order statistic *of the peaks announced by agents*, for some $1 \leq k \leq N$. (i.e. implement the outcome at the kth largest peak)

3. Consider the design of a mechanism for a simple bilateral trading problem, in which there is a single seller (agent 1), with a single item, and a single buyer (agent 2). The outcome of the mechanism defines an *allocation*, (x_1, x_2), where $x_i \in \{0, 1\}$ and $x_i = 1$ if agent i receives the item in the allocation, and defines *payments* (p_1, p_2) by the agents to the mechanism. Let v_i denote the value of agent i for the item, and suppose quasilinear preferences, such that $u_i(x_i, p_i) = x_i v_i - p_i$ is the utility of agent i for outcome (x_1, x_2, p_1, p_2).

   (a) (10 pts) Specify the Vickrey-Clarke-Groves mechanism for the problem; i.e. define the strategy space, the rule to select the allocation based on agent strategies, and the rule to select the payments based on agent strategies.

   (b) (5 pts) Provide a simple example to show that the VCG mechanism for the exchange is not (*ex post*) weak budget-balanced.

   (c) (5 pts) Is it possible to build an exchange mechanism that leads to an efficient allocation in a *dominant strategy* equilibrium, and is also *ex post* weak budget-balanced and *interim* individual-rational? What about in *Bayes-Nash* equilibrium? [Either refer to the appropriate impossibility theorem, or describe in brief terms the appropriate mechanism.]

4. (10 pts) Show that if $f : \Theta \rightarrow \mathcal{O}$ is truthfully implementable in dominant strategies when the set of possible types is Θ_i for $i = 1, \ldots, N$ [i.e. the

direct revelation mechanism, $\mathcal{M} = (\Theta, f)$, is strategyproof], then when each agent i's set of possible types is $\hat{\Theta}_i \subset \Theta_i$ (for $i = 1, \ldots, N$) the social choice function $\hat{f} : \hat{\Theta} \rightarrow \mathcal{O}$ satisfying $\hat{f}(\theta) = f(\theta)$ for all $\theta \in \hat{\Theta}$ is truthfully implementable in dominant strategies.