CS286r Computational Mechanism Design
Homework 4: Linear Programming and Integer Programming

Spring Term 2002
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Feb 28, 2002

Due: Tuesday 3/5/2002, in the beginning of class. You may use any sources that you want, but you must cite the sources that you use. Teamwork is not allowed.

1. (10 pts) Given primal $\min_x\{c^T x \;:\; Ax \geq b, x \geq 0\}$ and dual $\max_y\{b^T y \;:\; A^T y \leq c, y \geq 0\}$, prove that feasible primal and dual solutions, $\overline{x}$ and $\overline{y}$, are optimal if and only if they satisfy complementary slackness conditions. [**Hint:** use the strong duality property].

2. (5 pts) Formulate the following as linear programs:

   (i) $u = \min\{x_1, x_2\}$, assuming that $0 \leq x_j$ for $j = 1, 2$

   (ii) $v = |x_1 - x_2|$, assuming that $0 \leq x_j$, for $j = 1, 2$

3. (10 pts)

   (i) Show that

   $$\begin{aligned} X &= \{x \in \{0,1\}^4 \;:\; 93x_1 + 49x_2 + 37x_3 + 29x_4 \leq 111\} \\ &= \{x \in \{0,1\}^4 \;:\; 2x_1 + x_2 + x_3 + x_4 \leq 2\} \\ &= \{x \in \{0,1\}^4 \;:\; 2x_1 + x_2 + x_3 + x_4 \leq 2;\; x_1 + x_2 \leq 1;\; x_1 + x_3 \leq 1;\; x_1 + x_4 \leq 1\} \end{aligned}$$

   [**Hint:** first enumerate the feasible solutions defined for the first formulation.]

   Now consider solving the problem $\max\{c^T x \;:\; x \in X\}$ as a linear program, i.e. with $x \in \{0,1\}^4$ replaced by $x \in \mathbb{R}^4_+$.

   (ii) Will the LP relaxation of the integer program provide an upper- or lower-bound?

   (iii) Which formulation of the contraints would you expect to provide the tightest bound? Why?

4. (20 pts) John Harvard is attending a summer school where he must take four courses per day. Each course lasts an hour, but because of the large number of students, each course is repeated several times per day by different teachers. Section i of course k denoted (i,k) meets at the hour t_{ik}, where courses start on the hour between 10am and 7pm. Let T denote the set of start times, and suppose there are m courses, and that each course has n sections. John's preferences for when he takes courses are influenced by the reputation of the teacher, and also the time of day. Let v_{ik} be his value for section (i,k).

   (i) Carefully formulate an integer program to choose a feasible course schedule that maximizes the sum of John's preferences.

   (ii) Modify the formulation in (i), so that John never has more than two consecutive hours of classes without a break.

   (iii) Modify the formulation in (i), so that John chooses the schedule in which he starts his day as late as possible (ignoring the v_{ik} preferences). [**Hint:** you'll need to introduce a new decision variable.]

   (iv) **Extra credit**. How might you formulate the problem to select the latest schedule that also maximizes John's preferences?

5. (10 pts) Consider the 0-1 integer program:

$$\max\left\{\sum_{j=1}^{n} c_j x_j \ : \ \sum_{j=1}^{n} a_{ij} x_j = b_i, \text{ for } i=1,\ldots,m, \ x \in \{0,1\}^n\right\} \qquad (\mathrm{P}_1)$$

   and the 0-1 equality knapsack problem

$$\max\left\{c^T x \ : \ \sum_{j=1}^{n}\left(\sum_{i=1}^{m} u_i a_{ij}\right) x_j = \sum_{i=1}^{m} u_i b_i, x \in \{0,1\}^n\right\} \qquad (\mathrm{P}_2)$$

   where $u \in \mathbb{R}^m$ (a constant vector of real numbers). Show that P_2 is a *relaxation* of P_1, i.e. show that the feasible space in P_1 is (weakly) contained in the feasible space of P_2.

6. (20 pts) In a set covering problem there is a universe U of n elements, a collection of subsets of U, $\mathcal{S} = \{S_1, \ldots, S_k\}$, and a cost function, $c : S \to \mathbb{R}_+$. The problem is to find a minimum cost subcollection of $\mathcal{S}$ that covers all elements in U. Consider the linear program:

$$\begin{aligned} \min_{x_S} \ & \sum_{S \in \mathcal{S}} c(S) x_S && \text{[Cover]} \\ \text{s.t.} \ & \sum_{S \,:\, e \in S} x_S \geq 1, \quad \forall e \in U && (1) \\ & x_S \geq 0, \quad \forall S \in \mathcal{S} \end{aligned}$$

A solution to this LP can provides the minimal cost *fractional* set cover, in which it is possible to accept fractions of a subset $S \in \mathcal{S}$.

(i) Why is the upper-bound, $x_S \leq 1$, $\forall S \in \mathcal{S}$, redundant in this formulation?

(ii) Introduce a variable y_e corresponding to each constraint (1), and construct the dual problem to [Cover].

(iii) Construct the complementary-slackness conditions for the primal and dual formulations.

(iv) Write a short interpretation of each of the CS conditions (max two sentences per condition).

7. Consider a problem with n people to carry out n jobs. Each person is assigned to carry out exactly one job. Individual i has a cost of c_{ij} of an assignment to job j. The problem is to find a minimum cost assignment. The problem can be solved as a linear program (because the optimal solutions are integral):

$$\begin{aligned} \max_{x_{ij}} \quad & \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} && \text{[Assignment]} \\ \text{s.t.} \quad & \sum_{j=1}^{n} x_{ij} = 1, \quad \forall i && (2) \\ & \sum_{i=1}^{n} x_{ij} = 1, \quad \forall j && (3) \\ & x_{ij} \geq 0, \quad \forall i, j \end{aligned}$$

(i) Introduce variable u_i for each constraint (2) and variable v_j for each constraint (3), and formulate the linear programming dual of the assignment problem.

(ii) Construct the complementary-slackness conditions.

(iii) Write a short interpretation of the complementary-slackness conditions (at most two sentences per condition).