Game Theory

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Basics

Agents I, strategy $s_i \in S_i$.

Def. A *strategy*, s_i, is a complete contingent plan; defines the action an agent will take in all states of the world.

Essentially, the *joint strategies* of agents, $s = (s_1, \ldots, s_I)$ define the outcome of the game, and the utility of agents.

Def. The payoff function, $u_i(s)$, defines agent i's utility for strategy profile $s = (s_1, \ldots, s_I)$.

Wrapped up in the term "utility" is an implicit assumption [vNM47] that *rational* agents behave as *expected utility maximizers*.

Def. A strategic-form game $G = (S_1, \ldots, S_I; u_1, \ldots, u_I)$.

Example: Ascending-price Auction

State of the world, (p, x), defines the ask price $p \geq 0$ and whether agent is holding the item $x \in \{0, 1\}$.

Strategy defines the bid $b_i(p, x)$ that agent i will take for every state (p, x):

$$b_{\mathrm{BR}}(p, x) = \begin{cases} p & \text{, if } x = 0 \text{ and } p < v_i \\ \text{no bid} & \text{, otherwise.} \end{cases}$$

Example 1

[Prisoner's Dilemma]

Two people are arrested for a crime. If a suspect testifies, and the other does not testify (DC), then released and receives a reward. If neither testifies (CC), both released. If both testify (DD), go to prison and collect reward.

	C	D
C	1,1	-1,2
D	2,-1	0,0

Def. Strategy profile s^* is a (weak) dominant-strategy equilibrium of a game if, for all i,

$$u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i}), \qquad \forall s_i \in S_i, \forall s_{-i} \in S_{-i}$$

Example: SPSB

Agents I, values v_i. Bidding strategy $b_i \in [0, \infty)$.

$$u_i(b_i, b_{-i}, v_i) = \begin{cases} v_i - \max_{j \neq i} b_j & \text{, if } b_i > \max_{j \neq i} b_j \\ 0 & \text{, otherwise} \end{cases}$$

Given value v_i, strategy $b_i^*(v_i, b_{-i}) = v_i$ is a (weakly) dominant strategy, for all b_{-i}.

Let $b' = \max_{j \neq i} b_j$. If $b' < v_i$ then any bid $b' < b_i(v_i)$ is optimal. If $b' \geq v_i$, then any bid $b' \geq b_i(v_i)$ is optimal. Bid $b_i(v_i) = v_i$ solves both cases.

Example 2

	L	M	R
U	4,3	5,1	6,2
M	2,1	8,4	3,6
D	3,0	9,6	2,8

Example 3

	L	M	R
U	0,4	4,0	5,3
M	4,0	0,4	5,3
B	3,5	3,5	6,6

Nash Equilibrium

Def. A pure-strategy profile s^* is a *Nash equilibrium* if

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*), \qquad \forall s_i \in S_i, \forall i$$

Iterated Elimination of Dominated Strategies

Maintain a removed set of strategies, $R_i \subseteq S_i$, for each agent i. Initially, $R_i = \emptyset$.

- Choose an agent, i, and a strictly dominated strategy, $s_i \in (S_i \setminus R_i)$; i.e., such that some $s_i' \in (S_i \setminus R_i)$ satisfies:

$$u_i(s_i', s_{-i}) > u_i(s_i, s_{-i}), \quad \forall s_{-i} \in S_{-i} \setminus R_{-i}$$

- Add s_i to "remove set", R_i.
- Continue.

Thm. If a unique profile, s^*, survives, then it is the unique Nash equilibrium of the game.

Thm. If a profile, s^*, is a Nash eq., then it *must* survive iterated elimination.

Best-response Correspondences

[Brute force Nash eq. computation!]

First, apply iterated elimination of strictly dominated strategies, to get a candidate set $C_i \subseteq S_i$ of strategies for each agent.

Def. The best-response correspondence, $\mathrm{br}_i(s_{-i}) \subseteq S_i$, for agent i, computes the set of utility-maximizing strategies, given strategies s_{-i} from other agents.

Then, *compute* the BR correspondence for every agent, and *search* for a strategy profile s, with $s_i \in \mathrm{br}_i(s_{-i})$, for all i.

Example: Simple Competition Model

Consider a "Cournot" competition model. Two suppliers, produce quantity q_1, q_2 of a homogeneous good. Unit price is $p(Q) = [a - Q]^+$, for $Q = q_1 + q_2$.

Utility:

$$u_1(q_1, q_2) = q_1(a - (q_1 + q_2)) - q_1 c$$

Best-response:

$$q_1^*(q_2) = \max_{0 \leq q_1 \leq \infty} q_1(a - (q_1 + q_2) - c)$$

Compute first order conditions, and solve.

Example 4

[Matching Pennies]

Each player has a penny and must choose whether to display it with heads facing up or down. If the pennies match, agent 1 wins, if the pennies do not match, agent 2 wins.

	H	T
H	1,-1	-1,1
T	-1,1	1,-1

Nash Equilibrium: Mixed

Def. A mixed strategy, $\sigma_i \in \Sigma_i$, defines a prob., $\sigma(s_i)$, for each strategy $s_i \in S_i$ in agent i's strategy space; write $u_i(\sigma)$ for expected payoff.

Def. A mixed-strategy profile σ^* is a *Nash equilibrium* if

$$u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i, \sigma_{-i}^*), \qquad \forall \sigma_i \in \Sigma_i, \forall i$$

Thm. (Nash 1950) Every finite strategic-form game has a mixed-strategy equilibrium.

Note. All strategies in the *support*, $\{s_i \ : \ \sigma(s_i) > 0\}$, of mixed-strategy σ_i must have the same expected utility $u_i(s_i, \sigma_{-i}^*)$.

Example: Matching Pennies

	H	T
H	1,-1	-1,1
T	-1,1	1,-1

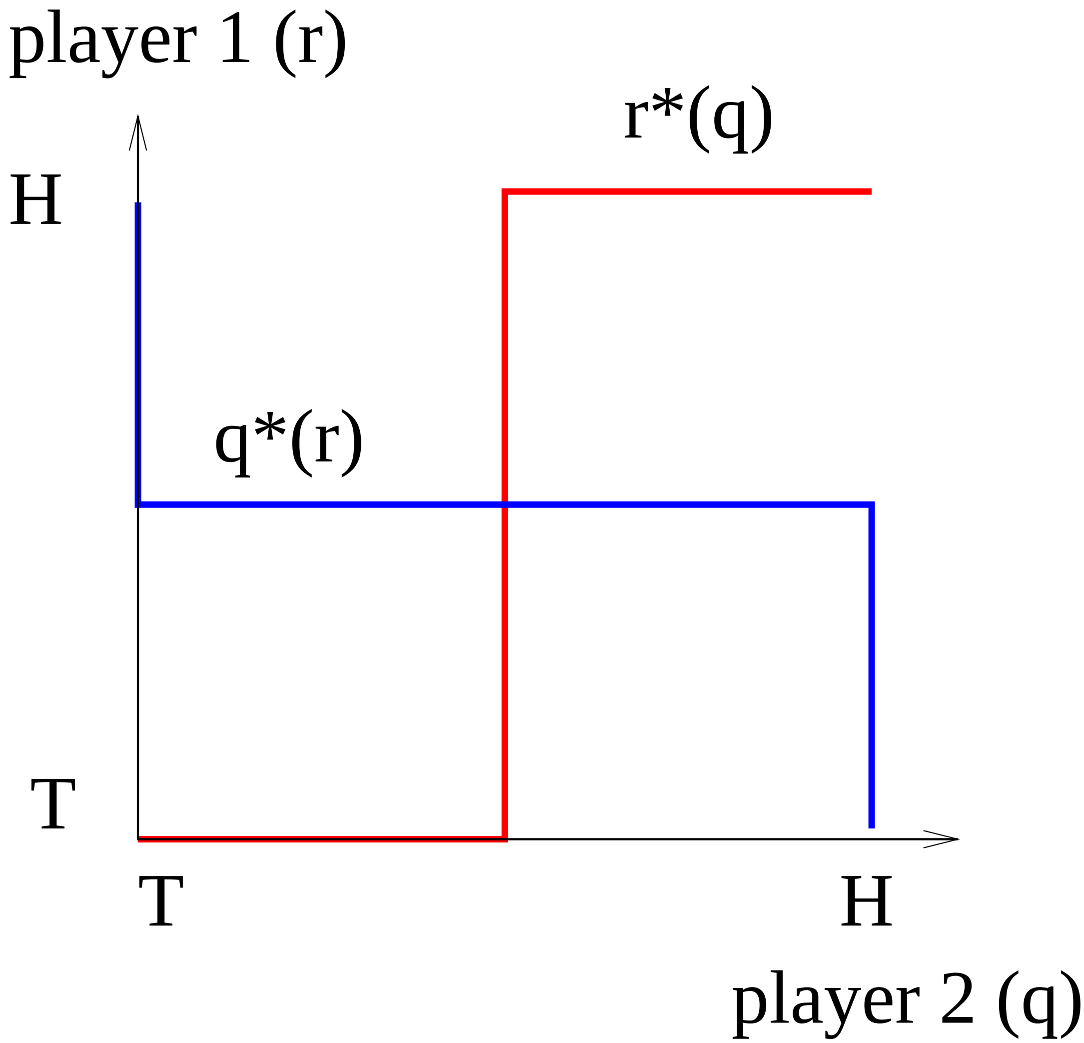
Let $(r, 1-r)$ denote the mixed strategy of player 1, and $(q, 1-q)$ denote the mixed strategy of player 2. Compute BR correspondences $r^*(q)$ and $q^*(r)$.

$$r(q) = r\left[q \cdot 1 + (1-q) - 1\right] + (1-r)\left[q \cdot -1 + (1-q)1\right]$$
$$= r(2q-1) + (1-r)(1-2q) = (1-2q) + r(4q-2)$$

Consider $q < 1/2, q = 1/2, q > 1/2$.

$$r^*(q) = \begin{cases} 0 & \text{, if } q < 1/2 \\ [0,1] & \text{, if } q = 1/2 \\ 1 & \text{, if } q > 1/2 \end{cases}$$

Solution Method: Graphical



Computational Problems

“players make their predictions of their opponents’ play by introspection and deduction, using knowledge of opponents’ payoffs, knowledge that opponents are rational, knowledge that each player knows that others know these things, etc.” [Fudenberg & Tirole]

Computing Nash eq. is a fundamental comput. problem, “complexity wide open”. Suppose that $|I| = 2$ and S_1 and S_2 are finite sets. *Is there a poly. time algorithm for computing a (mixed) Nash eq. in such a game?*

“Together with factoring, the complexity of finding a Nash eq. is in my opinion the most important concrete open question on the boundary of P today.” [Papadimitriou]

Note: *Kearns et al.* have an algorithm to compute Nash eq. over a “game tree” in polynomial time.

Other Issues

- Multiple Nash equilibria (“declarative concept”)
  – *focal points* (Schelling 60)
  – e.g. names, past experiences, Pareto-dominance, etc.
- Learning dynamics
  – adjustments, convergence, etc.
- Common knowledge
  – of payoffs, of rationality, of beliefs, etc.
  – def. as knowledge *ad infinitum*
- Robustness
  – to perturbations, uncertain information, irrational agents, etc.

Bayesian-Nash equilibrium

Agent i has *type*, $\theta_i \in \Theta_i$, s.t. the payoff function is $u_i(s, \theta_i)$ for agent i.

Agents have *common prior*, $p(\theta_1, \ldots, \theta_I)$, where $\theta_i \in \Theta_i$, about the distribution of agent types; with conditional prob. $p(\theta_{-i}|\theta_i)$.

Strategy spaces, payoff functions, possible types, prior distributions are *common knowledge*.

Let $\sigma_i(\theta_i) \in \Sigma_i$ denote the strategy that agent i chooses, given type θ_i. Let $\sigma(\cdot) = (\sigma_1(\cdot), \ldots, \sigma_I(\cdot))$ denote a strategy profile.

Expected utility:

$$u_i(\sigma_i(\theta_i), \sigma_{-i}(\cdot), \theta_i) = \sum_{\theta_{-i}} p(\theta_{-i}|\theta_i) u_i(\sigma_i(\theta_i), \sigma_{-i}(\theta_{-i}), \theta_i)$$

Def. Strategy-profile, $\sigma^*(\cdot)$, is in Bayesian-Nash eq. if, for every agent i, and every $\theta_i \in \Theta_i$,

$$u_i(\sigma_i^*(\theta_i), \sigma_{-i}^*(\cdot), \theta_i) \geq u_i(\sigma_i(\theta_i), \sigma_{-i}^*(\cdot), \theta_i) \quad \forall \sigma_i(\cdot) \neq \sigma_i^*(\cdot)$$

The strategy, $\sigma_i^*(\cdot)$, of agent i must be expected-utility maximizing, w.r.t. the distribution over strategies of other agents.

Example: FPSB

Bidders 1, 2. Bidder i has value v_i. Payoff is $v_i - p$. Values $v_i \sim [0,1]$. Strategy space $S_i = [0, \infty)$. Strategy $b_i(v_i) \in S_i$, specifies a bid for each value. Payoff:

$$u_i(b_1, b_2, v_i) = \begin{cases} v_i - b_i & \text{, if } b_i > b_j \\ (v_i - b_i)/2 & \text{, if } b_i = b_j \\ 0 & \text{if } b_i < b_j \end{cases}$$

Strategies $(b_1^*(\cdot), b_2^*(\cdot))$ define a Bayesian-Nash eq. if $b_i(v_i)$ solves:

$$\max_{b_i}(v_i - b_i)\Pr(b_i > b_j(v_j)) + 1/2(v_i - b_i)\Pr(b_i = b_j(v_j))$$

for each $v_i \in [0,1]$.

Simplify, assume a *linear equilibrium*, with $b_i(v_i) = a_i + c_i v_i$. Solve:

$$b_i^*(v_i, a_j, c_j) = \max_{b_i}(v_i - b_i)\Pr(b_i > a_j + c_j v_j)$$

Clearly, $a_j \leq b_i^*(v_i, a_j, c_j) \leq a_j + c_j$, and $\Pr(b_i > a_j + c_j v_j) = (b_i - a_j)/c_j$.

Bayes-Nash: Comput. Issues

- More reasonable assumptions about agent information than Nash
- Remaining problems:
  - existence of multiple equilibria
  - common prior
  - rationality assumptions
  - common-knowledge assumptions

Dominant strategy equilibria much more desirable!

What's Missing?

- **Extensive-form games**
  – multiple-stages, dynamic structure
  – explicit order, information made explicit
  – Subgame-perfect Nash equil.
- **Repeated games**
  – strategic-form (or "stage game") in each round
  – weighted average of payoffs in each stage
  – "Folk theorems"
- **Equilibrium refinements**
  – perfect equil., perfect Bayesian equil, etc.