Non-Cooperative Games

John F. Nash, Jr.

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Presented by Andrew Hutchings

Bibliography

Nasar, Sylvia, A Beautiful Mind (The basis for the possible movie)

Morgenstern and Von Neumann, Theory of Games and Economic Behaivor

Poundstone, William, The Prisoner's Dilemma (Von Neumann biography)

Selten, John, Generalizations of the Nash Equilibrium

Brief History of Game Theory

0 - 500 CE Talmudic Law

1700's Waldegrave proposes first minimax mixed solution to a 2 person game

1800's First results on Chess, Darwin's thoughts on biology

1928 Von Neumann proves minimax theorem

1944 <u>Theory of Games and Economic Behaivor</u>

1948 RAND founded

1950 Nash's PhD Dissertation.

Biography

Born 1928, Bluefield, WV

Undergraduate at Carnegie Tech (now CMU)

PhD in 1950 under Tucker

RAND Consultant 1950-1951

MIT 1951-59

1959-1984? Departure from "Scientifically Rational Thinking"

1994 Hansanyi, Nash, Selten recipients of Nobel Prize in Economics

2000 John Nash - Extraordinary Person?

Definitions

A *finite n-person game* is a set of n players, each with an associated finite set of pure strategies. Each player also has an associated payoff function p_i which maps from the set of all n-tuples of pure strategies to the reals.

A *mixed strategy* of player i is a convex combination of pure strategies.

$s_i = \sum_\alpha c_{i\alpha}\pi_{i\alpha}$

Where the $\pi_{i\alpha}$'s are i's pure strategies.

Mixed strategies have a simple geometric interpretation as points on a simplex.

i, j, k will denote players.

α, β, γ will indicate pure strategies of a player

s_i, t_i, r_i will indicate mixed strategies of i

$\pi_{i\alpha}$ will indicate i's α^{th} pure strategy

The payoff function for pure strategies has a unique extension to the n-tuples of mixed strategies which is linear in the mixed strategy of each player.

Use $\mathbf{S}$ or $\mathbf{T}$ to denote an n-tuple of mixed strategies. These can be thought of as a point in "strategy space", the product of the simplices for the players.

If $\mathbf{S} = (s_1, s_2, \ldots, s_n)$ then $(\mathbf{S}; t_i)$ indicates the use of t_i instead of s_i

Equilibrium Points

A n-tuple $\mathbf{S}$ is an equilibrium point if and only if for every i

$$p_i(\mathbf{S}) = \max_{all r_i' s}[p_i(\mathbf{S}; r_i]$$

From linearity of p_i we have

$$\max_{\alpha}[p_i(\mathbf{S}; \pi_{i\alpha})] = \max_{all r_i' s}[p_i(\mathbf{S}; r_i)]$$

Define $p_{i\alpha}(\mathbf{S}) = p_i(\mathbf{S}; \pi_{i\alpha})$.

Then we have that $\mathbf{S}$ is an equilibrium point if

$$p_i(\mathbf{S}) = \max_{\alpha} p_{i\alpha}(\mathbf{S})$$

We say $\mathbf{S}$ uses $\pi_{i\alpha}$ if $c_{i\alpha}$ is positive.

For the previous characterization of equilibrium to hold, we must have $p_{i\alpha}(\mathbf{S}) = \max_{\beta}(p_{i\beta}(\mathbf{S}))$.

Theorem 1 *Every finite game has an equilibrium point.*

Proof: If $\mathbf{S}$ is an n-tuple of mixed strategies. Define $p_{i\alpha}$ as above. Also define a set of continuous functions os $\mathbf{S}$

$$\phi_{i\alpha}(\mathbf{S}) = \max(0, p_{i\alpha}(\mathbf{S}) - p_i(\mathbf{S}))$$

Define a function T by the following transformation:

$$s_i' = \frac{s_i + \sum_\alpha \phi_{i\alpha}(\mathbf{S})\pi_{i\alpha}}{1 + \sum_\alpha \phi_{i\alpha}(\mathbf{S})}$$

This clearly maps onto the "strategy space", now we only need show that equilibrium points are exactly the points fixed under this mapping.

We must use some $\pi_{i\alpha}$, and one of the ones we use must have $p_{i\alpha}(\mathbf{S}) \leq p_i(\mathbf{S})$, so $\phi_{i\alpha}(\mathbf{S}) = 0$.

Now if an n-tuple is fixed by T, (positive) amount of $\pi_{i\alpha}$ can't be decreased. But it's usage isn't increased in the numerator, so denominator must be 0.

This says all the $\phi_{i\beta}(\mathbf{S}) = 0$. So no pure strategy $\pi_{i\beta}$ can improve the payoff, which means that it is an equilibrium point.

Then, by Brouwer Fixed Point theorem, we must have some equilibrium point.

A *symmetry* or *automorphism* of a game will is a permutation of the pure strategies that ensures that strategies belonging to one player are still owned by one player, and preserves the payoff value of the “block” of each player’s strategies.

A very similar method is used to prove the existence of a symmetric equilibrium point, that is, one that is preserved under all autmorphisms of the game.

Solutions to Games

A game is solvable if any players strategy r_i at some equlibrium point, can be substituted into any other equilibrium point and still be at equilibrium. The set of equilibrium points is the *solution* of the game.

A game is strongly solvable if it is solvable and if any strategy substitution at equilibrium which preserves the payoff also preserves the equilibrium property.

When they exist, these solutions are unique.

A property that games always have is that of sub-solutions, which are subsets of the set of equilibrium points that are maximal relative to the equilibrium property.

The factor sets of a sub-solution (S), the the i^{th} factor set is the set of all s_i's such that $\mathcal{S}$ contains $(\mathbf{T}; s_1)$ for some $\mathbf{T}$.

Theorem 2 *Geometrically, $\mathcal{S}$ is the product of its factor sets.*

Theorem 3 *The factor sets of a sub-solution are closed and convex as subsets of the mixed strategy spaces.*

Theorem 4 *Sets $S_1, S_2, \ldots, S_n$ of equilibrium strategies in a solvable games are polyhedral convex subsets of the respective mixed strategies*

This is a generalization of a known result for two-player games.

Discussion

1. What about actually finding Nash equilibrium points?

2. What if the opponent plays “imperfectly”?

3. How do you choose one equilibrium when there are potentially many?

4. What if the Nash Equilibrium is “unsatisfactory”?

5. What refinements, generalizations can be made based on the ideas of “equilibrium”, “solution” as presented here?

6. Pick a favorite application (economics, biology, international relations, philosophy, etc.) and talk about that.