

CS286r Electronic Market Design

Homework 2: Mechanism Design

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Due: Thursday 2/20/2003, in the beginning of class. You may use any sources that you want, but you must cite the sources that you use. Teamwork is not allowed. **If you took the class last year you must choose your own topic to work on for this initial part of the course.**¹ Please work hard on making the proofs clear, concise, and easy to read.

1. Consider a problem in which the outcome space, $\mathcal{O} \subset \mathbb{R}$, and each agent i , with type θ_i , has *single-peaked* preferences, $u_i(o, \theta_i)$ over outcomes. In particular, each agent, i , with type θ_i , has a *peak*, $p_i(\theta_i) \in \mathcal{O}$, such that $p(\theta_i) \geq d > d'$ or $d' > d \geq p(\theta_i)$ imply that $u_i(d, \theta_i) > u_i(d', \theta_i)$ (p.10–11, M.Jackson “Mechanism Theory” handout).
 - (a) (10 pts) Show that the “median selection” mechanism, in which each agent declares its peak and the mechanism selects the median (with a tie break in the case of an even number of agents) is *strategyproof*, and implements a *Pareto Optimal* outcome.
 - (b) (5 pts) Let N denote the number of agents. Suppose, in addition, that the mechanism can position its own $N - 1$ “phantom peaks”, before the peaks from the agents are received. Show that the median selection mechanism applied to the combined, $2N - 1$, peaks remains strategyproof.
 - (c) (5 pts) In combination with the phantom peaks, the median selection mechanism can implement a rich variety of outcomes. Describe a method to position the peaks to implement the k th order statistic *of the peaks announced by agents*, for some $1 \leq k \leq N$. (i.e. implement the outcome at the k th largest peak)
2. Consider a generalized multi-cast routing problem, with a network $G = (N, E)$ (that may or may not have a tree structure). The server is located at node $\alpha_S \in N$, and cost $c_e \geq 0$ is incurred for sending data along edge $e \in E$. Users, \mathcal{I} ,

¹This can involve choosing and answering questions from a GT text, or writing a review paper of some area of GT/MD about which you would like to learn more. Come talk to me!

each with value $v_i \geq 0$, are located at nodes $\alpha_i \in N$. Define the *receiver set*, $R \subseteq \mathcal{I}$, as the set of users that will receive service. The *efficient* outcome, R^* , solves:

$$\max_{R \subseteq \mathcal{I}} \sum_{i \in R} v_i - C(R)$$

where $C(R) \geq 0$ is the cost of the minimal-cost tree connecting the server node, α_S , to all receiver nodes, $\{\alpha_i : i \in R\}$.

(a) (15 pts) Consider that the users *and* the network are self-interested (i.e. rational) agents. Define the VCG mechanism for the problem. **[Hint:** the choice rule must define both the receiver set and the multi-cast tree, the transfer rule must define both the payment by each user and the payment to the network. It is not necessary (although it would be nice) to provide a formal mathematical definition of the minimal cost tree to server users in set R . If you prefer, you can simply introduce notation to denote the *solution* to this problem.]

(b) (5 pts) We know that Groves mechanisms are strategyproof and efficient. Prove that the VCG mechanism is *ex post* individual-rational for both the users and the network. Assume that neither the users nor the network have any outside options.

(c) (10 pts) Either prove that the VCG mechanism is *ex post* weak budget-balanced, or construct a simple counter-example for this multi-cast problem.

Consider a modified mechanism, the *marginal cost* mechanism. The mechanism is unchanged from the VCG, except that the payment to the network by the mechanism is simply equal to its reported cost, $C(R^*)$, for providing service to receiver set R^* .

(d) (5 pts) Assume for the moment that the true costs of the network are already known to the mechanism (or equivalently, that the network is cooperative). Either prove that this marginal cost mechanism is *ex post* weak BB, or provide a simple counterexample.

(e) (5 pts) Is the marginal cost mechanism strategyproof for the network itself? Either prove, or provide a simple counterexample.

3. Consider the design of a mechanism for a simple bilateral trading problem, in which there is a single seller (agent 1), with a single item, and a single buyer (agent 2). The outcome of the mechanism defines an *allocation*, (x_1, x_2) , where $x_i \in \{0, 1\}$ and $x_i = 1$ if agent i receives the item in the allocation, and defines *payments* (p_1, p_2) by the agents to the mechanism. Let v_i denote the value of agent i for the item, and suppose quasilinear preferences, such that $u_i(x_i, p_i) = x_i v_i - p_i$ is the utility of agent i for outcome (x_1, x_2, p_1, p_2) .

(a) (10 pts) Specify the Vickrey-Clarke-Groves mechanism for the problem; i.e. define the strategy space, the rule to select the allocation based on agent strategies, and the rule to select the payments based on agent strategies.

(b) (5 pts) Provide a simple example to show that the VCG mechanism for the exchange is not (*ex post*) weak budget-balanced.

- (c) (5 pts) Is it possible to build an exchange mechanism that leads to an efficient allocation in a *dominant strategy* equilibrium, and is also *ex post* weak budget-balanced and *interim* individual-rational? What about in *Bayes-Nash* equilibrium? [**Hint:** Either refer to the appropriate impossibility theorem, or describe in brief terms the appropriate mechanism.]
4. (10 pts) Show that if $f : \Theta \rightarrow \mathcal{O}$ is truthfully implementable in dominant strategies when the set of possible types is Θ_i for $i = 1, \dots, N$ [i.e. the direct revelation mechanism, $\mathcal{M} = (\Theta, f)$, is strategyproof], then when each agent i 's set of possible types is $\hat{\Theta}_i \subset \Theta_i$ (for $i = 1, \dots, N$) the social choice function $\hat{f} : \hat{\Theta} \rightarrow \mathcal{O}$ satisfying $\hat{f}(\theta) = f(\theta)$ for all $\theta \in \hat{\Theta}$ is truthfully implementable in dominant strategies.
5. (10 pts) Consider a problem in which the mechanism must make a choice $k \in \mathcal{K}$, and agents have all possible preference orderings across outcomes. Let $a \succ_i b$, for $a, b \in \mathcal{K}$ denote a preference type in which agent i prefers a to b . There are at least three agents. Explain, without making reference to the Gibbard-Satterthwaite impossibility theorem, why the following social-choice function cannot be implemented in a dominant-strategy equilibrium by any mechanism:

$$f(\theta) = \begin{cases} a & , \text{ if for all } i \text{ we have } a \succ_i b \text{ for all } b \neq a \\ a^* & , \text{ otherwise.} \end{cases}$$

where θ denotes the preferences of agents and a^* is an arbitrary member of \mathcal{K} .