Due: Tuesday 2/24/2004, in the beginning of class. You may use any sources that you want, but you must cite the sources that you use. You can also work in a group, just list off the people you’re working with. If you took the class last year please turn in a brief paper review on one of the papers that will be distributed later this week instead. Please work hard on making the proofs clear, concise, and easy to read.

1. Consider a problem in which the outcome space, $\mathcal{O} \subset \mathbb{R}$, and each agent $i$, with type $\theta_i$, has single-peaked preferences, $u_i(o, \theta_i)$ over outcomes. In particular, each agent, $i$, with type $\theta_i$, has a peak, $p_i(\theta_i) \in \mathcal{O}$, such that $p(\theta_i) \geq d > d'$ or $d' > d \geq p(\theta_i)$ imply that $u_i(d, \theta_i) > u_i(d', \theta_i)$ (p.10–11, M.Jackson “Mechanism Theory” handout).

(a) (10 pts) Show that the “median selection” mechanism, in which each agent declares its peak and the mechanism selects the median (with a tie break in the case of an even number of agents) is strategyproof, and implements a Pareto Optimal outcome.

(b) (5 pts) Let $N$ denote the number of agents. Suppose, in addition, that the mechanism can position its own $N - 1$ “phantom peaks”, before the peaks from the agents are received. Show that the median selection mechanism applied to the combined, $2N - 1$, peaks remains strategyproof.

(c) (5 pts) In combination with the phantom peaks, the median selection mechanism can implement a rich variety of outcomes. Describe a method to position the peaks to implement the $k$th order statistic of the peaks announced by agents, for some $1 \leq k \leq N$. (i.e. implement the outcome at the $k$th largest peak).

2. Consider the design of a mechanism for a simple bilateral trading problem, in which there is a single seller (agent 1), with a single item, and a single buyer (agent 2). The outcome of the mechanism defines an allocation, $(x_1, x_2)$, where $x_i \in \{0, 1\}$ and $x_i = 1$ if agent $i$ receives the item in the allocation, and defines payments $(p_1, p_2)$ by the agents to the mechanism. Let $v_i$ denote the value of
agent $i$ for the item, and suppose quasilinear preferences, such that $u_i(x_i, p_i) = x_i v_i - p_i$ is the utility of agent $i$ for outcome $(x_1, x_2, p_1, p_2)$.

(a) (10 pts) Specify the Vickrey-Clarke-Groves mechanism for the problem; i.e. define the strategy space, the rule to select the allocation based on agent strategies, and the rule to select the payments based on agent strategies.

(b) (5 pts) Provide a simple example to show that the VCG mechanism for the exchange is not (ex post) weak budget-balanced.

(c) (5 pts) Is it possible to build an exchange mechanism that leads to an efficient allocation in a dominant strategy equilibrium, and is also ex post weak budget-balanced and interim individual-rational? What about in Bayes-Nash equilibrium? [Hint: Either refer to the appropriate impossibility theorem, or describe in brief terms the appropriate mechanism.]

3. (10 pts) Show that if $f : \Theta \rightarrow O$ is truthfully implementable in dominant strategies when the set of possible types is $\Theta_i$ for $i = 1, \ldots, N$ [i.e. the direct revelation mechanism, $\mathcal{M} = (\Theta, f)$, is strategyproof], then when each agent $i$’s set of possible types is $\hat{\Theta}_i \subset \Theta_i$ (for $i = 1, \ldots, N$) the social choice function $\hat{f} : \hat{\Theta} \rightarrow O$ satisfying $\hat{f}(\theta) = f(\theta)$ for all $\theta \in \hat{\Theta}$ is truthfully implementable in dominant strategies.

4. (10 pts) Consider a problem in which the mechanism must make a choice $k \in \mathcal{K}$, and agents have all possible preference orderings across outcomes. Let $a \succ_i b$, for $a, b \in \mathcal{K}$ denote a preference type in which agent $i$ prefers $a$ to $b$. There are at least three agents. Explain (from first principles) why the following social-choice function cannot be implemented in a dominant-strategy equilibrium by any mechanism:

$$f(\theta) = \begin{cases} a & \text{if for all } i \text{ we have } a \succ_i b \text{ for all } b \neq a \\ a^* & \text{otherwise.} \end{cases}$$

where $\theta$ denotes the preferences of agents and $a^*$ is an arbitrary member of $\mathcal{K}$. 