1. (30 pts) Consider a double auction (DA), with \( m \) buyers and \( n \) sellers, each trading a single item. Buyers and sellers submit bids and asks, and the DA determines the trade, and agents’ payments. Let \( b_1, \ldots, b_m \) denote the bid prices from buyers, and assume \( b_1 \geq b_2 \geq \ldots \geq b_m \geq 0 \). Let \( s_1, \ldots, s_n \) denote the ask prices from sellers, and assume \( 0 \leq s_1 \leq s_2 \leq \ldots \leq s_n \). In addition, define \( b_{m+1} = 0 \) and \( s_{n+1} = \infty \). Later we refer to the following examples: (i) buyer values 9, 8, 7, 4, seller values 2, 3, 4, 5; (ii) buyer values 9, 8, 7, 4, seller values 2, 3, 4, 12.

(a) (10 pts) Define the VCG mechanism for this problem, and show that the mechanism is not ex post weak BB for example (i). [Hint: it is useful to interpret a bid, or an ask, as an agent’s claim about its value for the item. Define the trades implemented, payment by each buyer, payment to each seller.]

Consider the following modified trading mechanism, the McAfee-DA:

1. select \( k \), s.t. \( b_k \geq s_k \) and \( b_{k+1} < s_{k+1} \).

2. compute candidate trading price, \( p_0 = 1/2(b_{k+1} + s_{k+1}) \).

3. if \( s_k \leq p_0 \leq b_k \), then the buyers/sellers from 1 to \( k \) trade at price \( p_0 \); otherwise, the buyers/sellers from 1 to \( k - 1 \) trade, and each buyer pays \( b_k \), each seller gets \( s_k \).

(b) (15 pts) Prove that the McAfee-DA is strategy-proof, and ex post weak budget-balanced.
(c) (5 pts) Run the McAfee-DA on examples (i) and (ii). Is the DA efficient?

2. (5 pts) Formulate the following as linear programs:
   (i) \( u = \min \{ x_1, x_2 \} \), assuming that \( 0 \leq x_j \) for \( j = 1, 2 \)
   (ii) \( v = |x_1 - x_2| \), assuming that \( 0 \leq x_j \), for \( j = 1, 2 \)

3. (10 pts)
   (i) Show that
   \[
   X = \{ x \in \{0, 1\}^4 : 93x_1 + 49x_2 + 37x_3 + 29x_4 \leq 111 \}
   \]
   \[
   = \{ x \in \{0, 1\}^4 : 2x_1 + x_2 + x_3 + x_4 \leq 2 \}
   \]
   \[
   = \{ x \in \{0, 1\}^4 : 2x_1 + x_2 + x_3 + x_4 \leq 2; x_1 + x_2 \leq 1; x_1 + x_3 \leq 1; x_1 + x_4 \leq 1 \}
   \]
   \[\text{[Hint: first enumerate the feasible solutions defined for the first formulation.]}\]

   Now consider solving the problem \( \max \{ c^T x : x \in X \} \) as a linear program, i.e.
   with \( x \in \{0, 1\}^4 \) replaced by \( x \in \mathbb{R}^4_+ \).

   (ii) Will the LP relaxation of the integer program provide an upper- or lower-bound?

   (iii) Which formulation of the contraints would you expect to provide the tightest
   bound? Why?

4. (20 pts) John Harvard is attending a summer school where he must take four
   courses per day. Each course lasts an hour, but because of the large number
   of students, each course is repeated several times per day by different teachers.
   Section \( i \) of course \( k \) denoted \( (i, k) \) meets at the hour \( t_{ik} \), where courses start on
   the hour between 10am and 7pm. Let \( T \) denote the set of start times, and suppose
   there are \( m \) courses, and that each course has \( n \) sections. John’s preferences for
   when he takes courses are influenced by the reputation of the teacher, and also
   the time of day. Let \( v_{ik} \) be his value for section \( (i, k) \).

   (i) Carefully formulate an integer program to choose a feasible course schedule
   that maximizes the sum of John’s preferences.

   (ii) Modify the formulation in (i), so that John never has more than two consec-
   utive hours of classes without a break.

   (iii) Modify the formulation in (i), so that John chooses the schedule in which
   he starts his day as late as possible (ignoring the \( v_{ik} \) preferences). \[\text{[Hint: you’ll need to introduce a new decision variable.]}\]

   (iv) How might you formulate the problem to select the latest schedule that also
   maximizes John’s preferences?