CS286r Multi-Agent Learning
Homework 2: Mechanism Design and Implementation

Spring Term 2006
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Feb 6, 2006

Due: Monday 2/27/2006, at the beginning of class. You may use any sources that you want, but you must cite the sources that you use. You can also work in a group, just list off the people you're working with. **If you took the class last year, or are an economics student please talk to me or the TFs about do the reading assignment instead.** *Work hard on making the proofs clear, concise, and easy to read.* Total points: 120

1. (20 pts) Consider a problem in which the outcome space, $\mathcal{O} \subset \mathbb{R}$, and each agent i, with type θ_i, has *single-peaked* preferences, $u_i(o, \theta_i)$ over outcomes. In particular, each agent, i, with type θ_i, has a *peak*, $p_i(\theta_i) \in \mathcal{O}$, such that $p(\theta_i) \geq d > d'$ or $d' > d \geq p(\theta_i)$ imply that $u_i(d, \theta_i) > u_i(d', \theta_i)$.

   (a) Show that the "median selection" mechanism, in which each agent declares its peak and the mechanism selects the median (with a tie break in the case of an even number of agents) is *strategyproof*, and implements a *Pareto Optimal* outcome.

   (b) Let N denote the number of agents. Suppose, in addition, that the mechanism can position its own $N-1$ "phantom peaks", before the peaks from the agents are received. Show that the median selection mechanism applied to the combined, $2N-1$, peaks remains strategyproof.

   (c) In combination with the phantom peaks, the median selection mechanism can implement a rich variety of outcomes. Describe a method to position the peaks to implement the kth order statistic *of the peaks announced by agents*, for some $1 \leq k \leq N$. (i.e. implement the outcome at the kth largest peak)

2. (20 pts) Consider the design of a mechanism for a simple bilateral trading problem, in which there is a single seller (agent 1), with a single item, and a single buyer (agent 2). The outcome of the mechanism defines an *allocation*, (x_1, x_2), where $x_i \in \{0, 1\}$ and $x_i = 1$ if agent i receives the item in the allocation, and defines *payments* (p_1, p_2) by the agents to the mechanism. Let v_i denote the value of agent i for the item, and suppose quasilinear preferences, such that $u_i(x_i, p_i) = x_i v_i - p_i$ is the utility of agent i for outcome (x_1, x_2, p_1, p_2).

   (a) Specify the Vickrey-Clarke-Groves mechanism for the problem; i.e. define the strategy space, the rule to select the allocation based on agent strategies, and the rule to select the payments based on agent strategies.

   (b) Provide a simple example to show that the VCG mechanism for the exchange is not (*ex post*) weak budget-balanced.

   (c) Is it possible to build an exchange mechanism that leads to an efficient allocation in a *dominant strategy* equilibrium, and is also *ex post* weak budget-balanced and *interim* individual-rational? What about in *Bayes-Nash* equilibrium? [**Hint:** Either refer to the appropriate impossibility theorem, or describe in brief terms the appropriate mechanism.]

3. (5 pts) (Easy!) Show that if $f : \Theta \to \mathcal{O}$ is truthfully implementable in dominant strategies when the set of possible types is Θ_i for $i = 1, \ldots, N$ [i.e. the direct revelation mechanism, $\mathcal{M} = (\Theta, f)$, is strategyproof], then when each agent i's set of possible types is $\hat{\Theta}_i \subset \Theta_i$ (for $i = 1, \ldots, N$) the social choice function $\hat{f} : \hat{\Theta} \to \mathcal{O}$ satisfying $\hat{f}(\theta) = f(\theta)$ for all $\theta \in \hat{\Theta}$ is truthfully implementable in dominant strategies.

4. (15 pts) Consider a problem in which the mechanism must make a choice $k \in \mathcal{K}$, and agents have all possible preference orderings across outcomes. Let $a \succ_i b$, for $a, b \in \mathcal{K}$ denote a preference type in which agent i prefers a to b. There are at least three agents. Explain (from first principles) why the following social-choice function cannot be implemented in a dominant-strategy equilibrium by any mechanism:
$$f(\theta) = \begin{cases} a & \text{, if for all } i \text{ we have } a \succ_i b \text{ for all } b \neq a \\ a^* & \text{, otherwise.} \end{cases}$$
where θ denotes the preferences of agents and a^* is an arbitrary member of $\mathcal{K}$.

5. (20 pts) Suppose an object is to be assigned to an agent in the set $\{1, \ldots, n\}$. Assume $n \geq 3$. Let $v = (v_1, \ldots, v_n)$ denote the agents' values for the object. Assume for all v there is exactly a single agent with value for the object (the others have value zero.) The goal is to assign the object to this agent.

   (a) Show that the choice function is monotonic.

   (b) Instantiate the game form on p.669 on the "A crash course in implementation theory" handout and prove that *every* Nash equilibrium implements the choice function.

6. (15 pts) Consider a variation of the same problem in Q#5 where there are two players who prefer to have the object than to not have it, and that we always want to give the object to one of these two players.

   (a) Construct a choice function satisfying this requirement that is monotonic.

   (b) Instantiate the game form on p.669 on the "A crash course in implementation theory" handout and prove that *every* Nash equilibrium implements the choice function.

7. (25 pts) ("Solomon's predicament"; adapted from Osborne and Rubinstein) An iPod is found in Maxwell Dworkin. It belongs to one of two agents, 1 and 2. Each of them knows who is the true owner, but neither can prove that they own it. Queen Seltzer (QS) wants to design a mechanism to return the iPod to the legitimate owner. QS may assign the iPod to one of the agents, or to neither, and can also impose fines.

   The set of outcomes is the set of triples (x, m_1, m_2) where $x = 0$ (neither agent gets the iPod) or $x \in \{1, 2\}$ (the iPod is given to agent x) and m_i is a fine imposed on agent i. Agent i's payoff if he gets the iPod is $v_H - m_i$ if he is the legitimate owner and $v_L - m_i$ if he is not, where $v_H > v_L > 0$; it is $-m_i$ if he does not get the iPod. There are two possible preference profiles, θ^1 denotes that agent 1 is the legitimate owner and θ^2 denotes agent 2. QS wishes to implement the choice function f for which $f(\theta^1) = (1, 0, 0)$ and $f(\theta^2) = (2, 0, 0)$.

   (a) Show that this function is monotonic.

   (b) Instantiate the game form on p.688–689 in "A crash course in implementation theory" handout and prove that every subgame perfect Nash equilibrium implements the choice function.