Game Theory III

David C. Parkes
parkes@eecs.harvard.edu

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Outline

- Static games of Incomplete information
  - Dominant strategy equilibrium
  - Bayesian Nash equilibrium
- Dynamic games of Incomplete information
  - Perfect Bayesian Equilibrium
  - Sequential Equilibrium
Static Games of Incomplete Information

Model strategic-form games in which agents are uncertain of the payoffs of other agents.

Agent $i$ has type, $\theta_i \in \Theta_i$, which defines payoff $u_i(s; \theta_i) \in \mathbb{R}_{\geq 0}$.

Let $s_i(\theta_i) \in S_i$ denote the strategy that agent $i$ chooses, given type $\theta_i$: a strategy is a function, it maps type $\theta_i$ into a distribution on actions.

$s = (s_1, \ldots, s_n)$ denotes a strategy profile.

Def. A dominant-strategy equilibrium $s^*$ is one in which, for all agents $i$ and all $\theta_i$,

$$u_i(s_i^*(\theta_i), s_{-i}; \theta_i) \geq u_i(\hat{s}_i(\theta_i), s_{-i}; \theta_i)$$

for all $\hat{s}_i \neq s_i^*$ and all $s'_{-i} \in S_{-i}$.

Bayes-Nash Equilibrium

(Harsanyi 68)

Common prior: $p(\theta)$, over distr. of agent types; write the conditional prob. $p(\theta_{-i}|\theta_i)$.

Define the expected utility:

$$EU_i(s_i(\theta_i), s_{-i}; \theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} [p(\theta_{-i}|\theta_i) u_i(s_i(\theta_i), s_{-i}(\theta_{-i}); \theta_i)]$$

Def. A Bayesian-Nash equilibrium $s^*$, is one in which, for all agents $i$ and all $\theta_i$,

$$u_i(s_i^*(\theta_i), s_{-i}^*; \theta_i) \geq u_i(\hat{s}_i(\theta_i), s_{-i}^*; \theta_i)$$

for all $\hat{s}_i \neq s_i^*$.

i.e., best-response wrt beliefs about types of other agents, assuming they play a BR with respect to their beliefs.
Example 1: FPSB

Bidders 1, 2. Bidder $i$ has value $v_i$. Payoff is $v_i - p$. Values $v_i \sim [0, 1]$. Strategy space $S_i = [0, \infty)$. Strategy $b_i(v_i) \in S_i$, specifies a bid for each value. Payoff:

$$u_i(b_1, b_2, v_i) = \begin{cases} v_i - b_i & \text{if } b_i > b_j \\ (v_i - b_i)/2 & \text{if } b_i = b_j \\ 0 & \text{if } b_i < b_j \end{cases}$$

Strategies $(b_1^*, b_2^*)$ define a Bayesian-Nash eq. if $b_i(v_i)$ solves:

$$\max_{b_i} [(v_i - b_i)\Pr(b_i > b_j(v_j)) + 1/2(v_i - b_i)\Pr(b_i = b_j(v_j))]$$

for each $v_i \in [0, 1]$.

Simplify, assume a linear equilibrium, with $b_i(v_i) = a_i + c_i v_i$. Solve:

$$b_i^*(v_i, a_j, c_j) = \max_{b_i}(v_i - b_i)\Pr(b_i > a_j + c_j v_j)$$

Clearly, $a_j \leq b_i^*(v_i, a_j, c_j) \leq a_j + c_j$, and

$$\Pr(b_i > a_j + c_j v_j) = (b_i - a_j)/c_j.$$
Dynamic Games of Incomplete Information

Motivation.

(1) Agents with incomplete information about each others’ types interact in a repeated game over multiple time periods.

(2) Agents with incomplete information about each others’ types participate in a dynamic auction mechanism; e.g. bidding for a house.

Model as imperfect information extensive games: nature chooses type at root, information sets represent uncertainty of agents about each others’ types.

Will need a refinement of BNE to handle off-equilibrium beliefs; in the spirit of the subgame perfect refinement to NE for extensive games.

Example 2

Subgame perfect NE? (Careful: defined only from singleton information sets.)

Are they credible?
Introducing Beliefs into the Equilibrium

**Req. 1:** Any player with a move at an information set must assign *beliefs* (i.e. a prob.) to each node in the information set.

**Req. 2:** Strategies are *sequentially rational*, with respect to beliefs, in all continuation games (i.e. from all information sets forward.)

Bayesian Updating: PBE

“on the eq. path”: info set will be reached with +ve prob. if the game is played according to equil. strategies.

**Req 3.** Beliefs at information sets *on the equilibrium path* are determined by Bayes’ rule and equilibrium strategies.

**Req 4.** Beliefs are determined by Bayes’ rule *off the equilibrium path* — “where possible.” [Loosely: when the strategy is predictive.]

**Def.** (Rough) A *Perfect Bayesian Equilibrium* is a strategy profile $\sigma$ and a belief system $\mu$ that satisfies Req. 1 – 4.

Taken together, reqs 1–4 imply a BNE in every continuation game, and thus PBE extends the idea of SPNE to incomplete information games.
Example 4

Unique SPNE: \((D, L, R')\).

Reqs. 1–3 allow \(\{(A, L, L'), (p = 0)\}\).

Req. 4 rules this out.

Example 5

(For simplicity, ignore the payoffs.)

Now, if first two strategies are \((A, A')\) then req. 4 places no constraints on \(p\). But, what if agent 2’s strategy places \([q_1, q_2, 1 - q_1 - q_2]\) on actions \(L, R\) and \(A'\)?
Example 6: Further refinements?

![Game Tree](image)

PBE: \((L, L', p = 1)\) and \((R, R', p \leq 1/2)\).

One of these seems unreasonable. (Which one?)

Refinement I: Strictly-Dominated Strategies

**Req. 5.** Each player should place zero prob. (where possible) on nodes in info. sets off the equil. path reached only if another player plays strictly dominated actions.

**Def.** Strategy \(s\) is strictly-dominated by strategy \(s'\) if for every belief that \(i\) could hold at the info. set, and for all possible subsequent strategies by the other players, agent \(i\)'s expected payoff from \(s\) is strictly dominated by \(s'\).

Rules out \((R, R', p \leq 1/2)\) in example; since \(M\) is s.d. for player 1. [Note: if 1's payoff of 3 was 3/2 then \(R\) would s.d. both \(M\) and \(L\) and cannot use this rule.]
**Application: Bayesian Extensive Games with Observable Actions**

Defines a **Bayesian extensive game** \((\Gamma, \Theta, p, (u_i))\) where \(\Gamma = <N, H, P>\) and \(u_i : Z \times \Theta_i \rightarrow \mathbb{R}_{\geq 0}\). In stage 0, nature chooses a type for each agent from \(p\). Each agent learns its own type. Then, in every stage \(t\), all agents play an action; observe \(t - 1\) actions. Common prior \(p(\theta)\).

**From reqs.1–4, \(((s_i), (\mu_i))\) define a PBE as:**

1. Bayes’ rule used on equil. path
2. Bayes’ rule used off equil. path where possible, e.g. when updating from period \(t\) to \(t + 1\) even when \(h^t\) was prob-0; and for agent \(j\) when \(k\) played a prob-0 action.
3. Correct initial beliefs, with \(\mu_i(\emptyset) = p_i\).

+ A few technical conditions, e.g. to ensure agents retain the same beliefs and that actions by \(j \neq i\) cannot influence updating of beliefs on agent \(i\).

**Refinement II: Structural Consistency**

(Kreps and Wilson 82; Fudenberg and Tirole 91)

K&W'82 define **sequential equilibrium**. Place even more restrictions on beliefs about prob-0 events than PBE.

**Basic idea:** consider a small “tremble” (or mistake) at each information set; then Bayes’ can define beliefs everywhere.

- [mistakes assumed equally likely at \(x\) as \(x'\) ⇒ player 2 should have prob 1/3 on \(y\) and 2/3 on \(y'\).]

F&T add a new requirement to PBE (require consistency of relative probabilities in the tree); show extended-PBE == SE.
Next Lecture

- Mechanism Design: implementing socially-desirable outcomes by designing games with good properties
- the “revelation principle”
- Vickrey-Clarke-Groves mechanism
- Truthful mechanisms
- Comments on computational MD