1. (20 pts) Consider the design of a mechanism for a simple bilateral trading problem, in which there is a single seller (agent 1), with a single item, and a single buyer (agent 2). The outcome of the mechanism defines an allocation, \((x_1, x_2)\), where \(x_i \in \{0, 1\}\) and \(x_i = 1\) if agent \(i\) receives the item in the allocation, and defines payments \((p_1, p_2)\) by the agents to the mechanism. Let \(v_i\) denote the value of agent \(i\) for the item, and suppose quasilinear preferences, such that \(u_i(x_i, p_i) = x_i v_i - p_i\) is the utility of agent \(i\) for outcome \((x_1, x_2, p_1, p_2)\).

(a) (10 pts) Specify the Vickrey-Clarke-Groves mechanism for the problem; i.e. define the strategy space, the rule to select the allocation based on agent strategies, and the rule to select the payments based on agent strategies.

(b) (5 pts) Provide a simple example to show that the VCG mechanism for the exchange is not \((ex post)\) weak budget-balanced.

(c) (5 pts) Is it possible to build an exchange mechanism that leads to an efficient allocation in a dominant strategy equilibrium, and is also \((ex post)\) weak budget-balanced and interim individually-rational? What about in Bayes-Nash equilibrium? [Hint: Either refer to the appropriate impossibility theorem, or describe in brief terms the appropriate mechanism.]

2. (10 pts) Consider a problem in which the mechanism must make a choice \(k \in K\), and agents have all possible preference orderings across outcomes. Let \(a \succ_i b\), for \(a, b \in K\) denote a preference type in which agent \(i\) prefers \(a\) to \(b\). There are at least three possible outcomes. Explain (from first principles) why the following social-choice function cannot be implemented in a dominant-strategy equilibrium by any mechanism:

\[
    f(\theta) = \begin{cases} 
        a & \text{if for all } i \text{ we have } a \succ_i b \text{ for all } b \neq a \\
        a^* & \text{otherwise.} 
    \end{cases}
\]

where \(\theta\) denotes the preferences of agents and \(a^*\) is an arbitrary member of \(K\).
3. (30 pts) Consider a second-price sealed-bid (Vickrey) auction of one item, with bidders, i, with values, $v_i \in [0, \overline{v}]$, and quasilinear preferences, i.e. with $u_i(v_i, p) = v_i - p$, given price $p$.

(a) (10 pts) Show that bid $b_i(\theta_i) = v_i$ for all values, $v_i \in [0, \overline{v}]$, is a weakly dominant strategy for each bidder $i$. [Prove this from first principles, do not use the fact that the Vickrey auction is a special case of the Groves mechanism].

(b) (5 pts) Let $b_k$ denote the $k$th highest bid. Suppose that the seller introduces a reservation price, $r \in [0, 1]$, such that the item is only sold if $b_1(1) \geq r$, for price $p = \max[r, b_2(2)]$. Show that truthful bidding remains a weakly dominant strategy for bidders.

(c) (5 pts) Consider the special case of an auction with a single bidder, with a Uniformly distributed value $v_1 \sim U(0, \overline{v})$. In addition, suppose that the seller has value, $v_0$, for the item. Verify that strategies, $r^*(v_0) = (v_0 + \overline{v})/2$, $b_1^*(v_1) = v_1$ form a Bayesian-Nash eq. of this reserve-price Vickrey auction.

(d) (5 pts) In fact, $((v_0 + \overline{v})/2, v_1, \ldots, v_N)$, is also the Bayes-Nash eq. of the auction with $N$ bidders, each with value $v_i$. Assuming, $\overline{v} = 1$ and $v_0 = 0$, determine the seller’s expected revenue for the special case of two bidders. [Hint: construct an expression, by case analysis of the bids received, for the expected revenue to the seller. The fact, $E[v(2)|v(2) \geq 1/2] = 2/3$, where $v(2)$ is the second-highest value across two bidders, will be useful.]

(e) (5 pts) For this two-bidder case, compare the expected revenue in the reserve price Vickrey auction to that in the Vickrey auction with no reserve price, and provide an intuitive argument about the effect on allocative-efficiency. [Hint: The following fact is very helpful: the expected $k^{th}$ highest value among $n$ values independently drawn from the uniform distribution on $[v, \overline{v}]$ is $v + \frac{n+1-k}{n+1}(\overline{v} - v)$.

4. (20 pts) Consider a single-item auction problem and suppose all bidders’ values are uniform on $[0,1]$. Construct a revenue-maximizing auction (feel free to draw on any theorems presented in class). What is the reserve price? Provide a simple example to illustrate that this auction can have more revenue than the Vickrey auction.

5. (25 pts) Consider a sealed-bid auction that is defined in terms of:

(A1) an agent-independent price-function $p_i(v_i, x_i) \in \mathbb{R}_{\geq 0}$ for all $x_i \in X$ where $X$ is the set of possible allocations to agent $i$ and $v_{-i}$ is the reported values from agents $\neq i$.

(A2) an allocation rule $x_i(v) \in X$ that selects an allocation $x_i(v) \in \arg \max_{x \in X} \{v_i(x) - p_i(v_{-i}, x)\}$, for every agent $i$.

Assume that $X$ contains a “null” allocation, for which $v_i(x) = 0$ for all possible valuations $v_i$.

(a) (15 pts) Prove that an auction that satisfies A1 and A2 is truthful in a dominant-strategy equilibrium.

(b) (10 pts) Define the second-price Vickrey auction in these terms (i.e. exhibit an agent-independent price function $p_i(v_{-i}, x_i)$, and demonstrate that the allocation rule in the Vickrey auction satisfies (A2).)

6. (extra credit) Define the Vickrey-Clarke-Groves mechanism for a combinatorial allocation problem in these terms.