Mechanism Design II

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Positive & Negative Results

• We have seen two positive results:
  – median-choice, single-peaked preferences [Pareto optimal; Dominant]
  – Groves; VCG mechanism [Efficient; Dominant]

• Much of the mechanism design literature considers a class of agent types, and asks which SCF’s can be implemented in a particular solution concept.
Outline

• Negative:
  – Gibbard-Satterthwaite; Green-Laffont; Myerson-Satterthwaite

• Positive:
  – Groves; Median; Shapley/Cost-share; Optimal auctions.

• Could go either way: Roberts’ theorem
Gibbard-Satterthwaite Impossibility

[Arrow 51, Gibbard & Satterthwaite 73, 75]

Consider SCF, \( f(\theta) \), and an outcome space \( \mathcal{O} \). Let \( R_f \subseteq \mathcal{O} \) denote the range of \( f \), i.e.
\[
R_f = \{ o \in \mathcal{O} : \exists \theta \in \Theta \text{ s.t. } o = f(\theta) \}.
\]

Let \( o^*_i \in \mathcal{O} \) denote the outcome that maximizes the value, \( u_i(o, \theta_i) \), over \( o \in R_f \).

Def. [Dictatorial] SCF \( f(\theta) \) is dictatorial if there is an agent, \( h \), s.t. \( f(\theta) = o^*_h \), for all \( \theta \).

[Gibbard-Satterthwaite Impossibility] Suppose that the types include all possible strict orderings over \( \mathcal{O} \). A SCF, \( f(\theta) \), with \( |R_f| > 2 \), is implementable in dominant strategies (strategyproof) if and only if it is dictatorial.
Implications


Introducing Transfers

[Groves; special-case in which non-dicatorial strategyproof is possible]

Define the outcome space, $\mathcal{O} = \mathcal{K} \times \mathbb{R}^N$, such that an outcome rule, $o = (k, t_1, \ldots, t_N)$, defines a choice, $k(s) \in \mathcal{K}$, and a transfer, $t_i(s) \in \mathbb{R}$ from agent $i$ to the mechanism, given strategy profile $s \in S$.

Assume \textbf{quasilinear} preferences,

$$u_i(o, \theta_i) = v_i(k, \theta_i) - t_i$$

where $v_i(k, \theta_i)$ is the valuation function of agent $i$.

General/No-transfer $\supset$ Quasi-linear/Transfer

$\rightarrow$

easier
Roberts’ Theorem

TO DO!
But, what about “Budget Balance”? 

Introduce constraints over the total transfers made from agents to the mechanism. Let $s^*(\theta)$ denote the equilibrium strategy of a mechanism. **Flavors:**

- **no-deficit (or “weak”)**
  - *ex post:* $\sum_i t_i(s^*(\theta)) \geq 0$, for all $\theta$
  - *ex ante:* $E_{\theta \in \Theta} \left[ \sum_i t_i(s^*(\theta)) \right] \geq 0$

- **strong BB**
  - *ex post:* $\sum_i t_i(s^*(\theta)) = 0$, for all $\theta$
  - *ex ante:* $E_{\theta \in \Theta} \left[ \sum_i t_i(s^*(\theta)) = 0 \right]$

- ex ante weak $\supset$ ex post weak
  - U
    - ex ante weak $\supset$ ex post weak
    - ex ante strong $\supset$ ex post strong
  - harder
Efficiency & Budget-balance Tension

[Hurwicz 75; Green & Laffont 79]

**Def. [Efficiency]** A choice rule, $k^* : \Theta \rightarrow \mathcal{K}$, is (ex post) efficient if for all $\theta \in \Theta$, $k^*(\theta)$ maximizes $\sum_{k \in \mathcal{K}} v_i(k, \theta_i)$.

**Thm. [Green-Laffont Impossibility]** If $\Theta$ allows all valuation functions from $\mathcal{K}$ to $\mathbb{R}$, then no mechanism can implement an efficient and ex post strong budget-balanced SCF in dominant strategy.

$\Rightarrow$ impossible to implement “fully ex post efficient” SCFs in dominant strategy in general case.

**Approaches:** (a) restrict space of preferences; (b) weaken budget-balance or efficiency requirement; (c) weaken implementation concept.
Bayesian-Nash Implementation

Idea: drop dominant-strategy implementation, try to achieve budget-balance. Introduce interim IR.

**Bilateral trading problem:** single seller, single buyer. One good. Values drawn from $v_1 \in [0, 1], v_2 \in [0, 1]$.

**Thm.** [Myerson-Satterthwaite 83] In the bilateral trading problem, no mech. can implement an efficient, interim IR, and *ex post* (weak) budget-balanced SCF, even in Bayes-Nash eq.

**Proof.** via VCG mechanism.

*** VCG provides a useful unification here! **
Expected Externality Mechanism

[Arrow79,d’Aspremont&Gerard-Varet79] **Retain** Bayesian-Nash, but **relax** interim IR to ex ante IR.

The d’AGVA mechanism (or *expected-Groves* mechanism), uses the same allocation as the Groves, but computes a transfer term averaged across all possible types of agents.

**Thm.** The d’AGVA mechanism is efficient, *ex post* (strong) budget-balanced, but only *ex ante* IR.

**Demonstrates:** (a) *ex ante* IR makes MD easier than *interim* IR (compare Myerson-Satterthwaite with d’AGVA)

(b) Bayes-Nash implementation makes MD easier than DSE (compare Green-Laffont with d’AGVA).
## Summary

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<th>Name</th>
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<th>Solution</th>
<th>Possible</th>
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<tbody>
<tr>
<td>Median</td>
<td>no transfers</td>
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<td>Parto opt.</td>
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<td>Eff</td>
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<td>dAGVA</td>
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<td>Bayesian-Nash</td>
<td>Eff, BB, <em>ex ante</em> IR</td>
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<td>Eff &amp; IR</td>
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<td>Non-dictatorial</td>
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<td>MyerSat</td>
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<td>Bayesian-Nash</td>
<td>Eff &amp; weak BB &amp; IR</td>
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*Eff: ex post efficiency; BB: ex post strong budget-balance; IR: interim IR.*
Goal: maximize expected payoff of one agent. Allocation rule \( g : \Theta \rightarrow \Delta(K) \), payment rule \( p : \Theta \rightarrow \mathbb{R}^n \). By revelation principle, formulate MD problem as:

\[
\max_{g, p} V_0(g, p) + \sum_{i} m_i(p)
\]

s.t \( U_i(g, p, \theta_i|\theta_i) \geq U_i(g, p, \hat{\theta}_i|\theta_i), \ \forall i, \forall \theta_i, \forall \hat{\theta}_i \neq \theta_i \)

(additional constraints)

where

\[
m_i(p) = \mathbb{E}_{\theta} p_i(\theta) \quad \text{(ex ante payment)}
\]

\[
U_i(g, p, \hat{\theta}_i|\theta_i) = \mathbb{E}_{\theta_{-i}} v_i(g(\hat{\theta}_i, \theta_{-i}), \theta_i) - \mathbb{E}_{\theta_{-i}} p_i(\hat{\theta}_i, \theta_{-i}) \quad \text{(interim payoff)}
\]

\[
V_0(g, p) = \mathbb{E}_{\theta} v_0(g(\theta)) \quad \text{(ex ante seller value)}
\]
Solving the Problem

• **Decompose** into a subproblem and a masterproblem.
  – **subproblem**: take a particular allocation rule $g'$ and compute the optimal payment rule given $g'$ subject to IC and other constraints.
  – **masterproblem**: determine an allocation rule to maximize the value of the subproblem.

• But, set of allocation rules need not be finite or countable.

• Solutions known only for special cases: single-item allocation problems (Myerson 81), simple multiattribute allocation problems (Che 93).
Optimal Single Item Auction

\[
\max_g E_\theta \left[ \sum_i (J_i(\theta_i) - \theta_0) \pi_{g,i}(\theta) \right]
\]

s.t. \( Q_i(g, \theta'_i) \leq Q_i(g, \theta''_i), \ \forall i, \theta'_i < \theta''_i \)

where

\[
J_i(\theta_i) = \theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)}
\]

(virtual val.)

and \( \pi_{g,i}(\theta) \) is the prob that \( i \) gets the item, and \( Q_i(g, \theta'_i) \) is the conditional prob that \( i \) will get the item when reporting type \( \theta'_i \) in equilibrium, and \( f_i \) is the p.d.f. for type of agent \( i \), \( F_i \) is the c.d.f.
Solving this...

- The seller should sell the item to the agent with the highest $J_i(\theta_i)$ whenever that is larger than $\theta_0$.
  - a technical condition (regularity) ensures that the monotonicity constraint is automatically satisfied.

- The buyer pays the smallest $\hat{\theta}_i$ it could have bid and still won the auction (i.e. s.t. $J_i(\hat{\theta}_i) \geq \theta_0$ and $J_i(\hat{\theta}_i) \geq \max_{j \neq i} J_j(\theta_j)$).

**Comments:** (a) if agents symmetric, this is a Vickrey auction with reserve price $p_0 = J^{-1}(\theta_0)$; (b) the optimal auction is NOT efficient; (c) in the general asymmetric case the auction is biased in favor of agents that *a priori* are expected to have lower values; (d) the seller needs to have information about the distribution over agent types.
Additional Stability Properties

- **Group strategyproofness.** No coalition of agents can usefully deviate (one must be worse off).

- Consider a setting in which agents either receive a service or not, and the decision problem is to select the receiver set $R \subseteq I$ and share the cost $C(R)$. Agents announce values. If the cost function, $C(S)$, is *submodular*, i.e.
  \[ C(i \cup T) - C(T) \leq C(i \cup S) - C(S) \]
  for all $S \subseteq T \subseteq I$ and $i \neq S$, then the Shapley value defines a GSP and BB mechanism together with the Moulin-Shenker (99) cost-sharing mechanism.

- **Core.** A mechanism satisfies the *core* conditions if there is no incentive for a *subset* of agents to break away from the mechanism and work amongst themselves. [[“group individual-rationality constraint”]]
What is Missing?

- No computational constraints
- Focus on efficiency (social-welfare), little considerations of alternative objectives (e.g. fairness, max-min, make-span, etc.)
- Little discussion of special preference structure in resource allocation (beyond quasilinear preferences, some concavity assumptions)
- No use of randomization in the mechanism itself
- Revelation principle is the central paradigm, and there is no attention to *indirect* mechanisms