Assumptions

DSIC
deterministic
rich/closed domains

Question

what can be truthfully implemented?

- extends Roberts to structured domains
- simple to use characterization
  - should be of use in solving "second-best" DA's
    - comput. feasible CA's
    - revenue-optimal CA's
- DSIC falls in "Wilson doctrine"

Argue Roberts not interesting
Rocket not useful

Motivation
Cyclic monotonicity

Necessary and sufficient characterization for any preference domain

\[ \text{Cyclic NON} \equiv \text{W-MON} \] on their pref. domains

"delineate boundaries"

Construct graph where vertices are \( v \in V \) for fixed \( v \).

\[ l_{vw} = w(f(w)) - w(f(v)) \]

\( f \) is DSIC \( \iff \) graph has no cycles of negative length

2-cycle: \( l_{vw} + l_{wv} \geq 0 \)

\[ w(f(w)) - w(f(v)) + v(f(v)) - v(f(w)) \geq 0 \]

\[ \equiv -v(f(v)) - v(f(\omega)) \geq w(f(v)) - w(f(\omega)) \]
"Positive association of differences" (PAD)

Roberts '79

Necessary and sufficient characterization on unrestricted preference domains

Equivalent to AM there

\[
\begin{align*}
\forall \ V = (V_1 \ldots V_N), \ V' = (V'_1 \ldots V'_N), \\
\forall \ v_i \in (f(v)) - v_i \notin (a) \geq v_i (f(v)) - v_i (a) \quad \forall a \neq f(v), \forall i \in N
\end{align*}
\]

Then \( f(v') = f(v) \)

AM

\[
f(v) \in \arg \max_{a \in A} \left( x(a) + \sum_{c=1}^{N} r_c \cdot v_c(a) \right)
\]

DSIC \( \iff \) PAD \( \iff \) AM

on unrestricted domain
\( A = \{a_1, a_2, \ldots, a_k\} \) outcomes

Quasilinear utility

\( v_i \in D \subseteq \mathbb{R}_+^k \quad f: D^n \to A \quad p: D^n \to \mathbb{R}^n \)

truthful:

\[
v_i(f(v_i, v_{-i})) - p_i(v_i, v_{-i}) \geq v_i(f(v'_i, v_{-i})) - p_i(v'_i, v_{-i})
\quad \forall v_i, v_{-i}, v'_i \neq v_i
\]

W-Mon:

\[
a_1 = f(v_i, v_{-i}) \neq f(v'_i, v_{-i}) = a_2
\]

\[
\Rightarrow v'_i(a_2) - v'_i(a_1) \geq v_i(a_2) - v_i(a_1)
\]

intuition?

S-Mon

\[
a_1 = f(v_i, v_{-i}) \neq f(v'_i, v_{-i}) = a_2
\]

\[
\Rightarrow v'_i(a_2) - v'_i(a_1) > v_i(a_2) - v_i(a_1)
\]

Equivalently, \( \forall a_2 \)

\[
v'_i(a_2) - v'_i(f(v_i, v_{-i})) \leq v_i(a_2) - v_i(f(v_i, v_{-i}))
\]

\[
\Rightarrow f(v'_i, v_i) = f(v_i, v_{-i})
\]
\[ v_i' (f(v_i, v_c)) - p_i(v_i, v_c) \geq v_i' (f(v_i', v_c)) - p_i(v_i', v_c) \quad (1) \]
\[ v_c' (f(v_c, v_c)) - p_i(v_c, v_c) \geq v_c' (f(v_c', v_c)) - p_i(v_c', v_c) \quad (2) \]

(1) + (2), rearrange

\[ v_i' (f(v_i', v_c)) - p_i(v_i', v_c) \geq v_c' (f(v_c', v_c)) - p_i(v_c', v_c) \]

**W-MON \implies DSIC ?**

\[ V^1 = (0, 55, 90) \quad V^2 = (0, 60, 85) \quad V^3 = (0, 40, 75) \]
\[ f(V^1) = a_1 \quad f(V^2) = a_2 \quad f(V^3) = a_3 \]
\[ p(V^1) = p_1 = 0 \quad p(V^2) = p_2 \quad p(V^3) = p_3 \]

**Truthful, need**
\[ p_2 \geq 55 \quad V^1 \rightarrow V^2 \]
\[ p_3 - p_2 \geq 25 \quad V^2 \rightarrow V^3 \]
\[ p_3 \geq 80 \quad \text{but then } V^3 \rightarrow V^1 \]
What preference domain is sufficient for \( W\text{-NON} \Rightarrow \text{DSIC} \) ?

**Rich domain \((A, \succeq)\)**

\[ a_k \succeq a_\ell \Rightarrow V(a_k) \succeq V(a_\ell) \]

unrestricted, partially-ordered, completely-ordered

special cases

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**D** is rich if every \( V \in \mathbb{R}^K_+ \) is consistent with \((A, \succeq)\) belongs to \( D \)
Examples

multi-object auction

\[ \preceq \text{ is partial order induced by set inclusion} \]

multi-unit

\[ \succeq \text{ is a complete order} \]

assignment model

\[ a_1 \text{ no object} \]
\[ a_2 \ldots a_k \text{ object} \]

\[ a_k \preceq a_1 \text{ for } k \geq 2 \]
\[ a_k \not\preceq a_1 \text{ for } k,k \geq 2, k \neq l \]
Theorem 1: A set on a rich domain is DSIC

$\Rightarrow$ W-MON.

Sufficiency:

$$S_{kh} = \inf \{ V(a_k) - V(a) \mid V \in Y(k) \}$$

smallest difference such that got k

$$p_k = -S_{kh} \text{ where } a_k \text{ is "maximal" outcome}$$

Example:

single-item auction

10
8
6

Note: proof was unbounded domain in a critical way
Completely ordered, bounded domains

\((A, \preceq)\) such that \(V(a_k) \geq V(a_k)\) or \(V(a_k) \geq V(a_k)\)
for all \(V \in \mathcal{D}\)

vs. single peaked preferences?

Example: multi-unit auction

Given complete-ordering \((A, \preceq)\) redefine valuation as defining **marginal values**.

**Bounded:**

(A) \(\mathcal{D} = \prod_{k=1}^{K} [0, \bar{V}_k]\)

(B) \(\mathcal{D}\) is convex hull of
\[
\begin{pmatrix}
0 & 0 & 0 \\
\bar{V}_1 & 0 & 0 \\
\bar{V}_1 & \bar{V}_2 & 0 \\
\bar{V}_1 & \bar{V}_2 & \bar{V}_3 \\
\end{pmatrix}
\]

captures marginal decreasing with \(\bar{V}_k \geq \bar{V}_{k+1}\)
Theorem 2 \[\text{A scf on a completely-ordered, bounded domain is DSIC } \Leftrightarrow \text{ W-Non}\]