

Mathematical tools for finance

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Abstract

This note deals with the basic mathematical tools commonly used in finance. Stochastic calculus and its applications will be presented. The best will be done to present both the *PDE* and the *probabilistic* approaches to finance. However, the focus will be mostly on (vanilla) option pricing and fixed income techniques won't really be presented. For additional information such as Girsanov's Theorem for instance, I recommend to read Schreve's book (Volume 2).

1 Introduction to stochastic calculus

How to model prices fluctuations ?

S : price of a stock.

The return $\frac{S_{t+\tau} - S_t}{S_t}$ is often assumed to be normally distributed with mean $\mu\tau$ and variance $\sigma^2\tau$.

$$\Rightarrow S_{t+\tau} - S_t = \delta S_t = \mu S_t \tau + \sigma S_t \epsilon_{t \rightarrow t+\tau}, \quad \epsilon_{t \rightarrow t+\tau} \sim \mathcal{N}(0, \tau)$$

We want to write that in continuous terms and we need to introduce the brownian motion.

Definition : A brownian motion is a continuous process W that verifies:

- $t \mapsto W_t$ is continuous
- $W_0 = 0$
- $\forall s < t, W_t - W_s \sim \mathcal{N}(0, t - s)$ and $W_t - W_s$ is independent of the past before s

An easy way to represent prices is therefore to write (this has *a priori* no meaning except perhaps formally):

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

More generally we can write:

$$dS_t = \mu(t, S_t) dt + \sigma(t, S_t) dW_t$$

To give a precise sense to that we need to speak about stochastic integral and that's not our purpose so we are going to give the basic rule of calculus with a process described with a drift and a noisy part modelled with a brownian motion.

The basic rule is the **Ito's formula** which states that for f sufficiently smooth we have :

$$df(t, S_t) = \partial_t f(t, S_t)dt + \partial_S f(t, S_t)\mu(t, S_t)dt + \partial_{SS}^2 f(t, S_t)\frac{\sigma(t, S_t)^2}{2}dt + \partial_S f(t, S_t)\sigma(t, S_t)dW_t$$

The basic idea behind that is that dW_t has a standard deviation equal to \sqrt{dt} and "therefore" $(dW_t)^2$ can be replaced by dt in any Taylor approximation.

Examples:

Solving the stochastic differential equation for the price:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

Idea: consider $\ln(S_t)$. If $f = \ln$ then:

$$d\ln(S_t) = \mu dt - \frac{\sigma^2}{2}dt + \sigma dW_t$$

$$\ln(S_t) = \ln(S_0) + (\mu - \frac{\sigma^2}{2})t + \sigma W_t$$

$$S_t = S_0 \exp((\mu - \frac{\sigma^2}{2})t + \sigma W_t)$$

Another example (try it by yourself): Mean reverting process.

$$dX_t = \theta(\alpha - X_t)dt + \sigma dW_t$$

2 Option pricing

2.1 Black-Scholes

There are different ways to price derivatives. The main one is certainly based on the hypothesis that there is no arbitrage in the market.

Consider an asset S (traded continuously) and a derivative C on that asset, typically a European call that will pay at date T the maximum between $(S_T - K)$ and 0. (Notation : $(S_T - K)^+$).

We will suppose that S follows a geometric brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

To price C , let's consider the following portfolio at time t :

- A call option of value C
- A short position in Δ stocks of value S

At time t , the value of the portfolio is:

$$C(t, S_t) - \Delta S_t$$

At time $t + dt$, the gain on that portfolio is:

$$\begin{aligned} & dC(t, S_t) - \Delta dS_t \\ = & \partial_t C(t, S_t)dt + \partial_S C(t, S_t)[\mu S_t dt + \sigma(t, S_t)dW_t] + \partial_{SS}^2 C(t, S_t) \frac{\sigma^2 S_t^2}{2} dt - \Delta dS_t \\ = & [\partial_t C(t, S_t) + \partial_{SS}^2 C(t, S_t) \frac{\sigma^2 S_t^2}{2}] dt + [\partial_S C(t, S_t) - \Delta] dS_t \end{aligned}$$

If we set $\Delta = \partial_S C(t, S_t)$ then our portfolio is not risky and must therefore have a return equal to $r dt$. Therefore we have :

$$\partial_t C + \partial_{SS}^2 C \frac{\sigma^2 S^2}{2} = r(C - \Delta S)$$

This is the Black Scholes partial differential equation:

$$\partial_t C + rS \partial_S C + \frac{\sigma^2 S^2}{2} \partial_{SS}^2 C = rC$$

Interestingly, there is no consideration about the drift μ : pricing derivatives is all about σ !!

The price of the call must verify this (parabolic) equation with the condition $\forall S > 0, C(T, S) = (S - K)^+$ corresponding to the payoff.

This equation has a unique solution:

$$\begin{aligned} C(t, S) &= SN(d_1) - K \exp(-r\tau)N(d_2) \\ \tau = T - t, \quad d_1 &= \frac{\ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau} \end{aligned}$$

(N stands for the cdf of a standard normal variable).

This is the famous **Black Scholes formula** for a call option.

2.2 The Greeks

We used in our derivation of the BS formula a replication portfolio which consists in having at all times a number of stocks equal to $\Delta = \partial_S C(t, S_t)$.

This quantity, called Δ (Delta) is utterly important for practitioners and can be written as (it's not as obvious as it seems):

$$\Delta = N(d_1)$$

Other Greeks are important to practitioners. The Gamma Γ is defined by $\partial_{SS}^2 C$. The Vega ν is defined by $\partial_\sigma C$ and is positive. Other Greeks can be used (Θ, ρ, \dots).

2.3 Probabilistic representation

There are two main approaches to pricing: PDE's and probability theory and there is a complete equivalence between the two (see Feynman-Kac theorem for the deep reasons below these results).

The probabilistic approach needs the introduction of a new probability called risk-neutral probability \mathbb{Q} for which discounted prices are martingales that is:

$$\forall s < t, \mathbb{E}_s^{\mathbb{Q}}[e^{-rt} S_t] = e^{-rs} S_s$$

(This is linked to the no-arbitrage hypothesis)

The idea is that under this probability:

$$dS_t = rS_t dt + \sigma S_t dW_t$$

where the brownian motion is a brownian motion under \mathbb{Q} (see Girsanov's Theorem for the change of probability).

The idea is just that the pricing has to be done with the probability \mathbb{Q} with just a discounting reasoning and we have the formula :

$$C_t = \mathbb{E}_t^{\mathbb{Q}}[e^{-r(T-t)}(S_T - K)^+]$$

This allows to derive the Black Scholes formula directly and to show for example that $N(d_2)$ is just the probability (using \mathbb{Q}) that the stock price is above the strike.

3 Limits of Black Scholes. Models beyond Black & Scholes.

The BS Formula allowed us to price a call given (S_t, K, r, T, σ) .

All these parameters are data except the volatility σ . We have however seen that the Greek ν was positive and therefore that the call price was an increasing function of σ .

It's thus easy to consider several options actually traded in the market (with the same maturity T and different strikes K , say) and to reverse the formula.

That is we can obtain for various K what is the volatility parameter implied by the market.

If the BS formula were correct, the function $K \mapsto \sigma^{imp}(K)$ would be constant. Unfortunately, it is never the case and we commonly observe a skewed smile instead of a straight horizontal line.

Since $\nu = \partial_{\sigma} C > 0$ this means that call options for very low or very large strikes K can seem relatively expensive compared to other options for *reasonable* strikes. In economic terms, this means that extreme events are valued by the market and indeed it's reasonable to think that the normal hypothesis underlying the brownian model is a bad approximation: we need to consider fat/heavy tails !

3.1 Levy Flights

The discussion above have shown that it was necessary to take into account the fat tails in the distribution of returns.

The tails, in the gaussian case, decrease very rapidly, like $e^{-\frac{x^2}{2}}$. A better idea could be to introduce processes with fat tails such that the pdf decreases as a polynomial : $\frac{1}{|x|^{1+\alpha}}$. This is not completely easy and requires a lot of technical work (The basic ideas for the introduction of the stable Levy's distribution is rooted in generalizations of the CLT for r.v. without variance).

We can see on Figure 5.10 in the readings (Voit) that these tails are too fat so the problem is not solved.

3.2 A stochastic volatility model : The Heston Model

Another way to introduce fat tails is to consider stochastic volatility models. This can be done in discrete time using ARCH and GARCH models (this is useful to model volatility clustering) and in continuous time, this is often done using a model after Heston.

The idea is simple and consists in modelling prices like this:

$$dS_t = rS_t dt + \sigma_t S_t dW_t$$

where σ_t is a random variable driven by the following mean reverting stochastic differential equation:

$$dv_t = \theta(\alpha - v_t)dt + \eta\sqrt{v_t}d\tilde{W}_t, \quad v_t = \sigma_t^2$$

where the two brownian motions can either be independent or correlated.

This model is more complex and requires the calibration of three parameters (θ, α, η) using data on the markets (let's consider the two brownians are independent to simplify).

This can be done using the probabilistic approach and Monte Carlo simulations.

Remarkably however, it's *easy* to derive from this model a *nearly closed form* pricing formula for the call that resembles BS one:

$$C(t, S, v) = Sf(t, \ln(S), v) - Ke^{-r(T-t)}g(t, \ln(S), v)$$

where f and g can easily be computed numerically:

$$\begin{aligned} f(t, x, v) &= \mathbb{Q}[X_T \geq \ln(K) | X_t = x, v_t = v] \\ dX_t &= (r + \frac{1}{2}v_t)dt + \sqrt{v_t}dW_t \\ dv_t &= \theta(\alpha - v_t)dt + \eta\sqrt{v_t}d\tilde{W}_t \end{aligned}$$

and

$$\begin{aligned} g(t, x, v) &= \mathbb{Q}[X'_T \geq \ln(K) | X'_t = x, v_t = v] \\ dX'_t &= (r - \frac{1}{2}v_t)dt + \sqrt{v_t}dW_t \\ dv_t &= \theta(\alpha - v_t)dt + \eta\sqrt{v_t}d\tilde{W}_t \end{aligned}$$