# Voting Protocols:
# Characterization and Strategic Manipulation

Yiling Chen

September 19, 2011

## Roadmap

- Last time
  - Several voting rules
  - Arrow's and Muller-Satterthwaite's impossibility results.
  - A bit characterization using the axiomatic approach: May's theorem and Young's theorem
- Today
  - More characterizations
    - Distance to consensus
    - Voting as information aggregation
  - Strategic manipulation
    - Gibbard-Stterthwaite theorem
    - Single peaked preferences

## Distance to Consensus

Voting rules as a compromise?

## Charles Lutwidge Dodgson

- 1832 – 1898
- English author, mathematician, logician, and photographer
- Better known as Lewis Carroll, the authors of *Alice in Wonderland*
- Suggested choosing an alternative "as close as possible" to a Condorcet winner

## Dodgson Rule

- Score of alternative $a$ = minimum number of swaps between adjacent alternatives needed to make $a$ a Condorcet winer.
- The alternative with the lowest score wins

## Dodgson Score Example

Voter 1: $a \succ b \succ c \succ d \succ e$
Voter 2: $e \succ c \succ d \succ a \succ b$
Voter 3: $b \succ a \succ c \succ d \succ e$
Voter 4: $b \succ e \succ d \succ a \succ c$
Voter 5: $e \succ b \succ c \succ a \succ d$

- $a \succ b$: 2
- $a \succ c$: 3
- $a \succ d$: 3
- $a \succ e$: 2

## Dodgson Score Example

Voter 1: $a \succ b \succ c \succ d \succ e$
Voter 2: $e \succ c \succ d \succ a \succ b$
Voter 3: $a \succ b \succ c \succ d \succ e$
Voter 4: $b \succ e \succ d \succ a \succ c$
Voter 5: $e \succ b \succ c \succ a \succ d$

- $a \succ b$: 3
- $a \succ c$: 3
- $a \succ d$: 3
- $a \succ e$: 2

## Dodgson Score Example

Voter 1: $a \succ b \succ c \succ d \succ e$
Voter 2: $e \succ c \succ d \succ a \succ b$
Voter 3: $a \succ b \succ c \succ d \succ e$
Voter 4: $b \succ a \succ e \succ d \succ c$
Voter 5: $e \succ b \succ c \succ a \succ d$

- $a \succ b$: 3
- $a \succ c$: 3
- $a \succ d$: 3
- $a \succ e$: 3

Dodgson score for $a$ is 3.
In general, computing Dodgson score and determining Dodgson winner is computationally hard.

## Intuition of Dodgson Rule

- Consensus = existence of a Condorcet winner
- Distance = number of adjacent swaps
- Minimize distance to the concensus

## Rationalization via Consensus and Distance

We can define (rationalize) a voting protocol by:

- Fixing a class of consensus profiles, each of which has a clear winner (or a set of winners)
- Choosing a metric to measure the distance between two preference profiles
- Finding the closest consensus profile and elect the corresponding winner

## Notions of Consensus

Some natural notions of a consensus preference profile $[\succ]$ (vector of preferences)

- Condorcet winner: the preference profile $[\succ]$ has a Condorcet winner
- Majority winner: there exists an alternative that is ranked first by an absolute majority of the voters
- Unanimous winner: there exists an alternative that is ranked first by all voters
- Unanimous ranking: all voters report the same preference ranking

## Distance Metrics

Two natural distance metrics between preference profiles $[\succ]$ and $[\succ']$

- Swap distance: minimum number of swaps between adjacent alternatives needed to get from $[\succ]$ to $[\succ']$

  This is equivalent to the number of pairwise disagreement between $[\succ]$ and $[\succ']$
- Discrete distance: the number of preference rankings that are different in $[\succ]$ and $[\succ']$

## Kemeny Rule

- Recall: Kemeny rule outputs the overall ordering that is inconsistent with as few votes on pairs of alternatives as possible.

## Kemeny Rule

- Recall: Kemeny rule outputs the overall ordering that is inconsistent with as few votes on pairs of alternatives as possible.

Kemeny Rule = Unanimous Ranking + Swap Distance

# Borda Rule

- Recall: Borda rule is the position scoring rule with scoring vector $< m-1, m-2, \ldots, 0 >$

# Borda Rule

- Recall: Borda rule is the position scoring rule with scoring vector $< m-1, m-2, \ldots, 0 >$

## Plurality Rule

- Recall: plurality rule is the position scoring rule with scoring vector $< 1, 0, \ldots, 0 >$

Borda Rule = Unanimous Winner + Swap Distance

Dodgson, Kemeny, and Borda are all characterized using the same distance metric.

## Plurality Rule

- Recall: plurality rule is the position scoring rule with scoring vector $< 1, 0, \ldots, 0 >$

Plurality Rule = Unanimous Winner + Discrete Distance

## Voting as Information Aggregation

- So far, we have interpreted voting as preference aggregation — there isn't a "correct" ranking of alternatives
- Alternatively, assume that there exists an objective, "correct" ranking of alternatives
- Voters want to identify the correct ranking or winner, but they have uncertainty on the correct ranking.
- The ranking of a voter what he believes to be the correct ranking.

Can we identify the most likely ranking using voting?

## An Example with Two Alternatives

- Two alternatives: $a$ and $b$
- Either $a \succ b$ or $b \succ a$ (but we don't know which one is the correct ranking)
- 10 voters, with probability 0.6 each of them independently gets it right

Suppose 7 out of 10 voters report $a \succ b$.

- $\mathrm{P}(7 \text{ out of } 10 \text{ report } a \succ b | a \succ b) = C_{10}^{7} \times 0.6^7 \times 0.4^3$
- $\mathrm{P}(7 \text{ out of } 10 \text{ report } a \succ b | b \succ a) = C_{10}^{3} \times 0.6^3 \times 0.4^7$

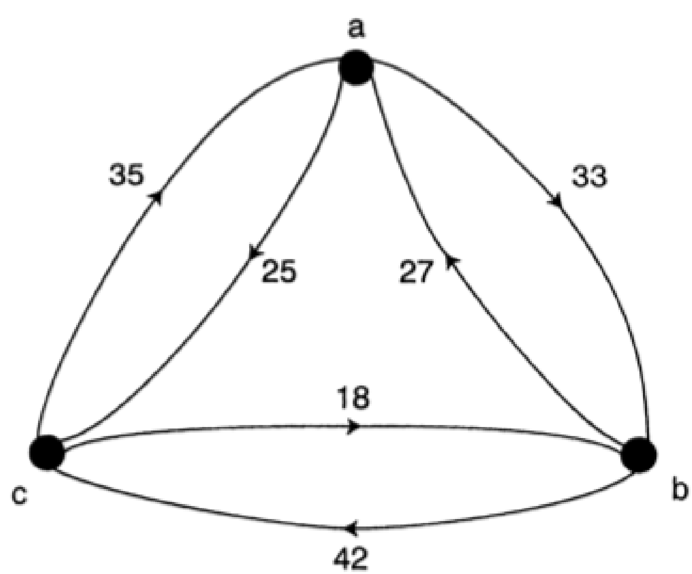
$$\frac{\mathrm{P}(7 \text{ out of } 10 \text{ report } a \succ b | a \succ b)}{\mathrm{P}(7 \text{ out of } 10 \text{ report } a \succ b | b \succ a)} = 0.6^4 / 0.4^4 = 5.06$$

Given the votes, $a$ is more likely to be better.

# The Condorcet Jury Theorem (Condorcet 1785)

- A jury of $n$ voters need to select the better of two alternatives. Each voter independently makes the correct decision with probability $p \geq 1/2$. Then, the probability that the plurality rule returns the correct decision increases in $n$ and approaches 1 as $n$ goes to $\infty$.

# Three or More Alternatives




- If the probability of any voter to rank any pair correctly is $p$, the voting rule outputting the most likely ranking coincides with the Kemeny rule. (Young 1995)

# Voting Rules as Maximum Likelihood Estimators

Conitzer and Sandholm (2005) asked a general question: for a giving voting rule, is there a noisy model such that the voting rule is a maximum likelihood estimator for the winner?

- Borda rule: if each voter independently ranks the true winner at position $k$ with probability $\frac{2^{m-k}}{2^m-1}$, then the maximum likelihood estimator is the Borda rule.

## Strategic Manipulation

We only deal with resolute voting rules (unique winner).

## Social Choice Function: Manipulability

- A social choice function is manipulable if some voter can be better off by lying about his preference
- An example with plurality voting

$$\begin{array}{c} 1 \text{ agent: } a \succ b \succ c \\ 2 \text{ agents: } b \succ c \succ a \\ 2 \text{ agents: } c \succ b \succ a \end{array}$$

## Social Choice Function: Onto

- A social choice function $C$ is onto if for each $a \in A$ there is a preference profile $[\succ] \in L^n$ such that $C([\succ]) = a$.

- Onto means that every alternative can be a winner under some preference profile.

## Gibbard-Satterthwaite's Impossibility Results (1973, 1975)

- If $|A| \geq 3$, any social choice function can not simultaneously satisfy
  - Nonmanipulable
  - Onto
  - Nondictatorship

## Using Muller-Satterthwaite to Prove Gibbard-Satterthwaite

- Muller-Satterthwaite (1977): If $|A| \geq 3$, any social choice function $C$ can not simultaneously satisfy
  - Weak Pareto efficiency (unanimity)
  - Strong monotonicity
  - Nondictatorship
- Strategy proofness $\Longrightarrow$ strong monotonicity (refer to reading)
- Strategy proofness + onto $\Longrightarrow$ weak Pareto efficiency (refer to reading)

## Restricting the Preference Domain: Single-Peaked Preferences

- Alternatives are a linear order (e.g. ordered on real line)
- Single-peaked preference: every voter has his most-preferred alternative and prefers alternatives that are closer to his favorite alternative
- Ask the voters to only report his favorite alternative
- The social choice function chooses the median voter's favorite alternative as the winner
- Onto and nondictatorial
- The winner is a Condorcet winner
- Nonmanipulable!

## Single Peaked Preferences: Median Voter = Condorcet Winner

For simplicity, assume the number of voters is odd.

- Let $a^*$ be the winner's reported value.
- Compare $a^*$ to $b$ to the left of $a^*$. Because $a^*$ is median, more than half of the voters $a^*$ is between their favorite value and $b$. They prefer $a^*$.

## Single Peaked Preferences: Strategy Proofness

For simplicity, assume the number of voters is odd.

- Without loss of generality, suppose a voter's favorite alternative is to the right of the median.
- He can nominate some other candidate to the right of the current winner. The median doesn't change.
- He can nominate some other candidate to the left of the current winner. Then the new median will be to the left of the old median. He is worse off by single-peakedness assumption.