The Importance of Network Topology in Local Contribution Games

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Abstract. We consider a model of content contribution in peer-to-peer networks with linear quadratic payoffs and very general interaction patterns. We find that Nash equilibria of this game always exist; moreover, they are computable by solving a linear complementarity problem. The equilibrium is unique when goods are strategic complements or weak substitutes and contributions are proportional to a network centrality measure called the Bonacich index. In the case of public goods, the equilibrium is non-unique and characterized by k-order maximal independent sets. The structure of optimal networks is always star-like when the game exhibits strict or weak complements. Under public good scenarios, while star-like networks remain optimal in the best case, they also yield the worst-performing equilibria. We also discuss a network-based policy for improving the equilibrium performance of networks by the exclusion of a single player.

1 Introduction

Peer effects, or the dependence of individual outcomes on group behaviour, is a characterizing feature of peer-to-peer systems. File-sharing systems rely on participants to provision the network with content. Participants can experience a marginal increase or decrease in utility from the kind of content contributed by others. We call such goods strategic complements and strategic substitutes, respectively. Following Ballester et al. (2006), we adopt a simple model for a contribution game in this paper. A player is modeled with a *linear-quadratic* utility function, that allows for utility-dependence on the contribution by other players. The utility structure provides for an individualized component, reflecting decreasing marginal-returns for a player's own contribution, in addition to a term that reflects *local interaction* that varies across pairs of players, meaning pairs of players can affect each other differently.

The model is appealing because Nash equilibria are always computable by solving a linear complementarity problem. Moreover, a unique Nash equilibrium of the contribution game can be readily computed as a metric of network centrality when the network exhibits complementarities. When substitutabilities are

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strong, equilibria are non-unique and the only stable equilibria are characterized by k-order maximal independent sets of optimally-contributing players, with the rest of the population free-riding completely.

We consider the problems of designing networks that maximize aggregate contribution and welfare, and find that the structure of optimal networks is star-like when the game exhibits strict or weak complements. Under public good scenarios, while star-like networks remain optimal in the best case, they also yield the worst-performing equilibria. We discuss a network-based policy aimed at improving the equilibrium performance of networks by the removal of a single *key* player.

This paper situates itself in a growing body of literature interested in games where endogenous play is susceptible to externalities passed along or represented by network links. Jackson (2008) and Kearns (2007) provide a good survey of the area. Demange (2007) and Bramoulle and Kranton (2006) study equilibrium profiles in a game with public (*substitutable*) good provisioning. Johari and Tsitsiklis (2005) and Roughgarden and Tardos (2004) investigate the effects of network architecture on the worst-case efficiency *(the price of anarchy)* of equilibria in routing games. Our model deals with a different payoff structure and allows for externalities to be either complementary or substitutable. We provide a partial characterization of equilibria in our game and relate a network's efficiency, in both the best and worst case, to its geometric properties.

2 The Model

Let $\mathcal{G}(v, e)$ denote the set of undirected and unweighted connected graphs without loops with v vertices and e edges.

Players are connected by a network $\mathbf{g} \in \mathcal{G}(v, e)$ with adjacency matrix $\mathbf{G} = [g_{ij}]$. This is a zero diagonal and non-negative square matrix, with $g_{ij} \in \{0, 1\}$ for all $i \neq j$.

Each player $i = 1, ..., n$ selects a contribution $x_i \geq 0$, and gets a payoff $u_i(x_1, \ldots, x_n)$. Letting $\mathbf{x} = (x_1, \ldots, x_n)$, we focus on *bilinear* utility functions of the form:

$$u_i(\mathbf{x}, \mathbf{g}) = x_i - \frac{1}{2}x_i^2 + a\sum_{j=1}^{n} g_{ij}x_i x_j, \tag{1}$$

The external effect of another agent on the utility of agent i is captured by the cross-derivatives $\frac{\partial^2 u_i}{\partial x_i x_j} = a g_{ij}$, for $i \neq j$. When $a > 0$, the effect on agent i of agent j's contribution is marginal-increasing if and only if i and j are connected in $\mathbf{g}$; when $a < 0$, the effect is marginal-decreasing. The network $\mathbf{g}$ reflects the pattern of existing payoff complementarities when a is positive, and substitutabilities, when a is negative, across all pairs of players. We use Σ to refer to the n-player game with payoffs given by Equation 1 and strategy space, the non-negative real line.

2.1 The Linear Complementarity Problem

We analyze the set of pure strategy Nash equilibria of the game introduced above. We note that an equilibrium exists if and only if $\frac{\partial u_i}{\partial x_i}(x^*) \leq 0, \forall i \in N$. In matrix notation, this necessary and sufficient condition for a Nash equilibrium becomes:

$$\begin{aligned} \mathbf{x}^* &>= \mathbf{0}, \\ -\mathbf{a} + \mathbf{\Sigma}\mathbf{x}^* &>= \mathbf{0}, \\ -\mathbf{x}^{\mathbf{t}*}(-a + \mathbf{\Sigma}\mathbf{x}^*) &= \mathbf{0}. \end{aligned} \tag{2}$$

The problem of finding a vector $\mathbf{x}^*$ such that the above conditions hold is known as the linear complementarity problem $LCP(-a, -\mathbf{\Sigma})$. We can therefore state the following:

Theorem 1. *The set of pure strategy Nash equilibria of the contribution game with parameters α and $\mathbf{\Sigma}$ are given by the set of solutions to $LCP(-\alpha, -\mathbf{\Sigma})$.*

The linear complementarity problem is a well-studied problem and we borrow from this literature to address existence of the Nash equilibrium in our game, as well as in our empirical studies to characterize optimally-designed networks. In the next sections we study the current model under strict complementarities, when $a > 0$, and under substitutabilities, when $a < 0$. The local interaction graph connecting agents becomes irrelevent when $a = 0$, as the contribution levels of other agents does not impact an agent's utility. In this case, the network-independent optimal contribution level for each agent is 1.

3 Complementary Goods

We first study the game under local complementarities, i.e. $a > 0$. Before turning to the equilibrium analysis, we define a network centrality measure due to Bonacich (1987) that proves useful for this analysis.

3.1 The Bonacich Network Centrality Measure

Given the network $\mathbf{g} \in \mathcal{G}(v, e)$, denote by $\lambda_1(\mathbf{g})$ its largest eigenvalue, also called the *index* of $\mathbf{g}$. This index is always well-defined and $\lambda_1(\mathbf{g}) > 0$.

Definition 1. *Let $\mathbf{B}(\mathbf{g}, a) = [\mathbf{I} - a\mathbf{G}]^{-1}$, which is well-defined and non-negative if and only if $a\lambda_1(\mathbf{G}) < 1$. The vector of Bonacich centralities of parameter a in $\mathbf{g}$ is $\mathbf{b}(\mathbf{g}, a) = \mathbf{B}(\mathbf{g}, a) \cdot \mathbf{1}$.*

Since $\mathbf{B}(\mathbf{g}, a) = \sum_{k=0}^{+\infty} a^k \mathbf{G}^k$, its coefficients $b_{ij}(\mathbf{g}, a)$ count the number of paths in $\mathbf{g}$ starting at i and ending at j, where paths of length k are weighted by a^k.

Theorem 2. *For $a\lambda_1(\mathbf{G}) < 1$, the game Σ has a unique Nash equilibrium $\mathbf{x}^*(\Sigma)$ given by $\mathbf{x}^*(\Sigma) = \mathbf{b}(\mathbf{g}, a)$, where the utility of player i at equilibrium is $u_i(\mathbf{x}^*, \mathbf{g}) = \frac{1}{2} {x_i^*}^2 = \frac{1}{2} b_i(\mathbf{g}, a)^2$.*

The correspondence between the Bonacich centrality indices of a graph and its equilibrium when $a > 0$ establishes the uniqueness and interiority of equilibria when $a\lambda_1(\mathbf{G}) < 1$. When $a\lambda_1(\mathbf{G}) > 1$, an equilibrium fails to exist because the positive feedback from other agents' contributions is too high and contributions increase without bound.

4 Substitutable Goods

When $a < 0$ we have a substitutability effect between players' contributions, i.e. we have a public good game. Contrary to the case when $a > 0$, an equilibrium now always exists. The best-response function is continuous from the compact convex set $\{x \in \mathcal{R}^n : \forall i, 0 \leq x_i \leq x^*\}$ to itself and so Brouwer's Fixed Point Theorem applies. We study the game under two separate conditions: when subtitutabilities are weak and when they are strong, i.e. the case of pure public goods.

4.1 Weak Substitutes

We define the complement network $\overline{\mathbf{G}} = \mathbf{J} - \mathbf{I} - \mathbf{G}$, where $\mathbf{J}$ is the all-ones matrix, i.e., $\overline{g_{ij}} = 1 - g_{ij}$, for all $i \neq j$. In words, two vertices are linked in $\overline{\mathbf{G}}$ if and only if they are not linked in $\mathbf{G}$. We write:

$$\mathbf{\Sigma} = (1 + a)\,\mathbf{I} + a\overline{\mathbf{G}} - a\mathbf{J}.$$

Suppose first that $-1 < a < 0$. Solving for the Nash equilibrium is then equivalent to solving $LCP(-\frac{1}{1+a}\mathbf{e}, \mathbf{I} + \frac{a}{1+a}\overline{\mathbf{G}} - \frac{a}{1+a}\mathbf{J})$. The solution can be equivalently written in terms of the Bonacich index of nodes on the complement network.

Theorem 3. *Consider a game on G where $a < 0$ and let $\overline{\mathbf{G}} = \mathbf{J} - \mathbf{I} - \mathbf{G}$ as before. There exists a unique equilibrium if and only if $-a\lambda_1\left(\overline{\mathbf{G}}\right) < 1 + a$. Then, the equilibrium is unique, interior and proportional to Bonacich, that is,*

$$x_i^* = \frac{1}{1 + a + a\sum_{j=1}^{n} b_j\left(\frac{-a}{1+a}, \overline{\mathbf{G}}\right)} b_i\left(\frac{-a}{1+a}, \overline{\mathbf{G}}\right), \text{ for all } i = 1, ..., n$$

Recall that we are dealing with the case $-1 < a < 0$. Notice that $-a\lambda_1\left(\overline{\mathbf{G}}\right) < 1 + a$ is equivalent to $-\frac{1}{1+\lambda_1(\overline{\mathbf{G}})} < a$. Therefore, the interior unique equilibrium is obtained on $-\frac{1}{1+\lambda_1(\overline{\mathbf{G}})} < a < 0$.

4.2 Pure Public Goods

When substitutabilities are large, i.e. when $a < -\frac{1}{1+\lambda_1(\overline{\mathbf{G}})}$, the above transformations fail to work. In these circumstances, agents' free-riding on others' contributions is severe enough that some agents do not contribute at all. Agents whose equilibrium contribution levels are non-zero either contribute the optimum (i.e. in this case 1) or some value less than optimum.

Partially Corner Equilibria. Precisely, a partially corner equilibrium profile $\mathbf{x}^*$ on the network $\mathbf{G}(\mathbf{v}, \mathbf{e})$ is one such that there exists some $i, j \in v$ such that $x_i = 0$ and $0 < x_j < 1$. We remark that equilibrium contributions of non-corner agents, i.e. all agents $j \in v$ such that $0 < x_j < 1$, are related to the Bonacich centrality index on the subnetwork joining them.

Lemma 1. *Let $a < 0$. Given a partially corner equilibrium profile $\mathbf{x}$ on the network $\mathbf{G}(\mathbf{v}, \mathbf{e})$, the contribution levels of all nodes i such that $0 < x_i < 1$ is given by the expression in Theorem 3.*

Corner Equilibria. As mentioned earlier, when local substitutabilities are large, corner equilibria, with agents either free-riding completely or contributing optimally, also exist. For such a situation to be an equilibrium, both free-riders and contributors must gain by doing so. Given the graph $\mathbf{G} = (v, e)$, let $N_G(v')$ designate the set of neighbors of node $v' \in v$. We find that all corner equilibria for $a \leq -1$ are described by maximal independent sets of contributors.

Definition 2. *A set $S \subseteq v$ is called a k-order maximal independent set if and only if it is a maximal independent set such that each node not in the set is connected to at least k nodes in the set, i.e. $\forall v' \notin S$, $\left|N_G(v') \cap S\right| \geq k$.*

Theorem 4. *Let $\lceil 1/a \rceil$ be the smallest integer greater than or equal to $\left|1/a\right|$. For $a \leq -\frac{1}{1+\lambda_1(\overline{\mathbf{G}})}$, a corner profile is a Nash equilibrium if and only if the set of contributing players, i.e. $\left\{i \in v : x_i = 1\right\}$, is a $\lceil 1/a \rceil$-order maximal independent set of the graph $\mathbf{G}$.*

Maximal independent sets correspond to maximal independent sets of order 1. Every graph has a maximal independent set, therefore there always exists a corner equilibrium for $a \leq -1$. However, for $k \geq 2$, k-order maximal independent sets need not always exist. Therefore, when $-1 < a \leq -\frac{1}{1+\lambda_1(\overline{\mathbf{G}})}$, we may not have a corner equilibrium, though we may still have a partially corner equilibrium. Recall that an equilibrium is guaranteed to exist for all $a \leq 0$.

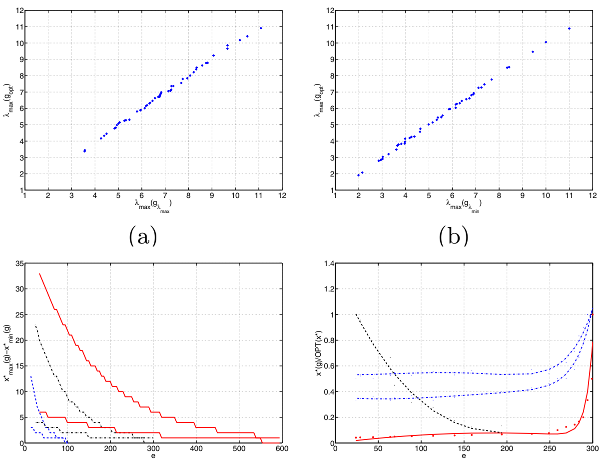
Stable Equilibria. We use a simple notion of stability based on Nash tâtonnement (e.g. Fudenberg 1991). We find that corner equilibria are the only stable equilibria to this perturbed best-response procedure.

Theorem 5. *For any network $\mathbf{G}$, an equilibrium is stable if and only if it is a corner equilibrium.*

The result is convenient because it helps to mitigate the problem of multiple equilibria when $a \leq -\frac{1}{1+\lambda_1(\overline{\mathbf{G}})}$, where $\overline{\mathbf{G}}$ is given as the complement of the network $\mathbf{G}$, as before. The correspondence of corner equilibria to maximal independent sets of order k also gives us insight into the computational complexity of computing equilibria under substitutabilities. These results are discussed in the full paper (Corbo et al. 2007) and are leveraged in this paper's empirical studies to solve for a graph's contribution-maximizing (best-case) and contribution-minimizing (worst-case) equilibria, across all a when equilibria are non-unique.

5 Optimal Network Design

The problem of optimal network design consists of arranging a network's v vertices and e edges in such a way that some objective function is maximized. In the first problem, the social planner wants to maximize aggregate activity (or contribution) at equilibrium. In the second problem, the social planner wants



(a) (b)

(c) (d)

Fig. 1. (a) shows that the first eigenvalue of aggregate contribution- and utility-maximizing equilibrium graphs corresponds to the largest first eigenvalue possible for v, e. When equilibria are non-unique, (a) includes the best case equilibrium performance of the graph, while (b) plots the first eigenvalue of networks with best worst-case stable equilibrium performance (against the smallest first eigenvalue possible for v, e). (a), (b) refer to networks with varying number of edges while keeping $v = 12$ and $-3 < a < 0.5$. (c) shows the difference in aggregate contribution for the best- and worst-case equilibria, for graphs with the best best-case and best worst-case performance, fixing $a = -1$. $v = 15$ **(dash blue)**, 25 **(dot-dash blue)**, 35 **(solid red)**. (d) shows the best-case and worst-case equilibrium performance of graphs with best worst-case performance **(dot-dash blue)** compared to graphs with best best-case performance **(solid red)**, for $v = 12$, $a = -1$. The best best-case performing graphs have worse worst-case performance. The **dash black** curve in (d) gives the worst-case performance of graphs with the *key player* removed, starting from the best best-case performing graphs **(solid red)**.

to maximize aggregate equilibrium welfare. When equilibria are non-unique, we consider both the best case and worst case, respectively contribution- or utility-maximizing and contribution- or utility-minimizing, equilibria of networks. We study the relationship between the best-case and worst-case equilibria of a given network, and particularly how contribution- or utility-maximizing networks in the best case perform in the worst case, as well as how contribution- or utility-maximizing networks in the worst case perform in the best case.

5.1 Optimizing Under Complementarities

Let $a > 0$. We observe that contribution- and welfare-maximization correspond to maximizing the L_1 and L_2 norms of the Bonacich index vector, i.e. $\max_g\{\mathbf{b}(\mathbf{g}, a) \cdot \mathbf{1} : \mathbf{g} \in \mathcal{G}(v, e)\}$ and $\max_g\{\mathbf{u}(\mathbf{x}^*(\mathbf{g}, a)) : g \in \mathcal{G}(v, e)\}$, respectively.

The relationship suggests a way to characterize optimal equilibria using spectral graph theory.

Lemma 2. *Let* $\mathbf{g} \in \mathcal{G}(v, e)$*, and* $\lambda_1(\mathbf{G})$ *its index. As* $a \uparrow \frac{1}{\lambda_1 \mathbf{G}}$*, the welfare- and contribution-maximizing graphs problems are equivalent and reduce to* $\max\{\lambda_1(\mathbf{g}) : \mathbf{g} \in \mathcal{G}(v, e)\}$*.*

These asymptotic results reveal a great deal about how the optimal networks change with the level of externalities. Figure (a) illustrates precisely this across a large range of a values, for graphs with varying numbers of edges. The graph shows that the first eigenvalue of aggregate contribution- and utility-maximzing graphs (L_1- and L_2-maximizing graphs in the case of $a > 0$) corresponds to the largest first eigenvalue possible for graphs with given v, e. The largest eigenvalue of a graph is a measure of its regularity. A higher eigenvalue corresponds to an irregular star-like structure, whereas a lower eigenvalue refers to a more regular network.

5.2 Optimizing Under Substitutabilities

While equilibria are interior, aggregate contribution- and utility-maximizing graphs still coincide; these networks are maximal index graphs. When substitutabilities are strong enough, we lose interiority and have both partially corner and corner equilibria. Corner equilibria being the only stable equilibria, we only consider these.

Figures (c), (d) illustrate the tension between optimal networks in the best and worst cases. Networks that yield the highest contribution in the best case also exhibit worse worst case performance. These networks are maximal index graphs. Networks that yield the best worst case performance are minimal index graphs, as shown in Figure (b). Minimal index graphs also exhibit the smallest spread between best and worst case equilibrium performance, as illustrated in Figure (d).

5.3 Excluding the Key Player: A Network-Based Policy

We investigate a policy aimed at mitigating the discrepancy between best- and worst-case equilibrium performance. We denote by $\mathbf{G}^{-i}$ (respectively $\mathbf{\Sigma}^{-i}$) the new adjacency matrix (respectively the matrix of cross-effects), obtained from $\mathbf{G}$ (respectively from Σ) by setting to zero all of its i-th row and column coefficients. The resulting network is $\mathbf{g}^{-i}$. We want to solve $\max_{i \in N}\{\mathbf{x}^*(\mathbf{\Sigma}^{-i}) - \mathbf{x}^*(\mathbf{\Sigma})\}$

This is a finite optimization problem, that admits at least one solution. A good heuristic for the solution of this problem is the the removal of the highest degree node, since a node with highest degree imposes the largest number of constraints on the independent set construction. Figure (d) illustrates the policy's effectiveness in reconciling best- and worst- case equilibrium performance. Worst-case performance of graphs can be dramatically improved and even match best-case performance when the network graph is sparse. Figure (d) also shows that the policy becomes less effective as graphs grow dense.

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