Improving Allocations Through Revenue Redistribution in Auctions with Entry*

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Abstract

We consider a single-item private values auction setting with endogenous costly entry. It is known that for constant entry cost functions, the Vickrey auction has a unique symmetric Bayes-Nash "cutoff equilibrium" where only agents with value above some threshold enter. We extend this result to the case where cost is an arbitrary linear function of value. We then consider a class of strategyproof and no-deficit *redistribution auctions* that return Vickrey auction revenue back to participants [1; 3], demonstrating the existence of unique symmetric cutoff equilibria here as well, with increased participation and expected allocation value (gross of costs) over the Vickrey auction. We show that when individuals' costs are purely a function of value, the equilibrium symmetric cutoff strategy achieved by the Vickrey auction is more efficient (net of costs) than any other cutoff level. But in the case where costs decrease as participation increases (e.g., when there is a search cost associated with *obscure* auctions), the Vickrey auction is inefficient and a redistribution auction dominates.

1 Introduction

It is common in the study of auctions to assume a pre-determined set of participants. A richer model, by now also the object of significant attention, considers the set of participants to be a variable that itself is dependent on the properties of the auction to be specified. Making the model rich in this way is especially crucial for settings where agents incur a cost for participation ("entry costs"), since rational agents will not participate in an auction that does not produce expected payoff exceeding entry cost.[1]

There are several important facts known about auction design in this setting. Green and Laffont [6] consider entry costs that can vary arbitrarily across agents, showing that in the case where values and costs are both distributed uniformly, an equilibrium exists in which agents participate if and only if cost is less than some function of value. McAfee

*Versions of this work were presented at the SISHOO conference in Huntington Beach, CA (December 2010), and at the Second Conference on Auctions, Market Mechanisms and Their Applications in New York, NY (August 2011).

[1] For instance, if it costs an agent \$10 to drive to an auction and \$5 for the time spent at the auction, he must expect to gain at least \$15 from the auction or else participation is not rational.

and McMillan [11] adopt a different model, where values are learned only *after* deciding to participate and costs are constant across agents and known in advance;[2] they show that a first-price sealed bid auction elicits the number of bidders that is optimal with respect to expected revenue.

The second-price (Vickrey) auction receives particular attention in the literature, and with good reason. Most notably, Stegeman [12] shows that for arbitrary individualized costs for the various agents, the Vickrey auction has an equilibrium that is efficient in the sense of eliciting participation that, in expectation, maximizes allocation value minus participation costs.[3] It is common in much of the literature to assume valuations drawn from the same distribution (symmetric bidders) and participation costs that are *constant* across bidders. Tan and Yilankaya [14] show that in such settings the Vickrey auction has a unique equilibrium if the cumulative distribution function over values is concave. Even without this concavity assumption, there is always exactly one *symmetric* equilibrium, so their result amounts to a demonstration that with concavity there are no asymmetric equilibria. In general the efficient equilibrium of the Vickrey auction may be asymmetric, but the combination of the Stegeman and the Tan and Yilankaya results implies that with concave symmetric value distributions and a fixed cost the Vickrey auction has a unique equilibrium, and it is both symmetric and efficient. Equilibria in these settings all have the form of a "cutoff value", where an agent participates if and only if his value is above some critical level v_0; so in a symmetric cutoff equilibrium all agents with value above some v_0 will participate, and all others will not.

Symmetric equilibria are intuitively more compelling than asymmetric ones when there is a common distribution over values, since there is no a priori reason to distinguish one agent's expected behavior from another's. However, the assumption of constant costs is uncomfortably restrictive since it doesn't capture settings where, say, it is more costly to prepare a bid given a higher value, or where higher-valued agents bear more opportunity cost in terms of time spent at the auction. Thus in this paper we adopt a model that allows us to generalize some of the core results that assumed constant entry costs while maintaining enough structure to preserve a symmetric cutoff equilibrium; specifically we will consider costs that are linearly dependent on value.[4] While Stegeman's result [12] tells us that the Vickrey auction maximizes efficiency here, we will show that moreover the symmetric equilibrium it achieves is superior to all other symmetric cutoff values (regardless of whether there are asymmetric equilibria).

But the main focus of the paper is to consider cases in which alternatives to the Vickrey auction, despite Stegeman's striking result, may be preferred. Specifically, we consider settings where *increasing* participation over the Vickrey auction level is desirable. There are many scenarios that could generate this goal. For instance, consider the case where the agents' perception of cost is different than that of the social planner (think of "lazy" agents, or those whose behavior is unduly subject to inertia). Here, additional participation incentives are required to restore efficiency. Perhaps even more compelling is the case where agents and the social planner evaluate costs the same way, but where costs are dependent on the participation level. We will consider "obscurity costs", where an obscure (relatively sparsely attended) auction imposes greater search and logistical costs on participants than one in which participation is widespread. Here too we will show that the Vickrey auction is inefficient due to underparticipation.

This analysis leads us to consider alternatives to the Vickrey auction that, given a

[2] See [13] and [9] for other examples where this model is adopted. The model considered in this paper, as well as in all other references, assumes each agent knows his value and cost prior to making a participation decision.

[3] Stegeman also shows that the first-price sealed bid auction may have no efficient equilibrium.

[4] [8] and [2], though not addressing an auction context, are other examples where participation cost is taken to be a function of type.

fixed set of agents, yield greater expected agent utility, as this will increase participation. Such an alternative is presented by Bailey [1] and Cavallo [3] in the form of a *redistribution mechanism* (RM) that takes the revenue of the Vickrey auction and returns a large portion of it to the agents without distorting incentives or running a deficit.[5][6] We will expand the RM auction conception to define a parameterized spectrum of auctions (RM_ρ, for $\rho \in [0,1]$) that includes the Vickrey auction at one end ($\rho = 0$), the RM auction at the other ($\rho = 1$), and gradations of revenue redistribution in between. We will do equilibrium analysis and compare expectations of efficiency under the Vickrey auction and the redistribution auction variants. A summarized list of the main results is as follows:

- For an arbitrary cost function that is linearly dependent on value, the Vickrey auction and every redistribution auction with sufficiently small positive redistribution parameter ($\rho > 0$) has a unique symmetric cutoff equilibrium. (Theorem 2)
- The redistribution auction, with redistribution parameter ρ chosen arbitrarily from a non-empty interval bounded below by 0, yields strictly greater participation and expected allocation value than the Vickrey auction in its unique symmetric equilibrium. (Theorem 3)
- For many natural value distributions, adding a positive reserve price to the redistribution auction yields strictly *greater* participation and expected allocation value than the redistribution auction with no reserve price in the unique symmetric equilibrium. (Theorem 4)
- The Vickrey auction is efficient for arbitrary cost functions that are linear in agent valuations, but is *not* efficient for cost functions that depend on the level of participation. (Theorem 5)
- For any cost function that is linear in agent valuations and strictly decreasing in the expected number of participating agents, we can identify a redistribution auction that yields strictly greater efficiency than the Vickrey auction in its unique symmetric equilibrium. (Theorem 8)

In the rest of this section we formally define the setting, the notation to be used, and the auction mechanisms to be considered. In Section 2 we present results for linear cost functions. In Section 3 we consider a model where there is a cost associated with the obscurity of an auction. In Section 4 we look beyond cost functions that are linearly dependent on value, and we conclude in Section 5. About half of the proofs are deferred to the Appendix in the interest of readability.

1.1 Preliminaries

A single item is to be auctioned and there are N agents with independent private values for it. We will consider the possibility that only a subset of agents participate in the auction, and our convention will be to use $n \leq N$ to denote the number of bidders (participants). We will consider the following two auctions (where the second is an auction schema and reduces to the first if $\rho = 0$).

[5] Cavallo [3] provides a general mechanism that is truthful and efficient in dominant strategies, ex post individually rational, and no-deficit, for all non-negative value typespaces. The mechanism coincides with that of Bailey [1], proposed earlier, in the special case of single-item allocation if each agent's typespace admits the value 0 for the item.

[6] Note that simply applying fixed "entry bonuses" will not be a solution because they may lead to a deficit, which typically is unacceptable if not completely infeasible.

Definition 1. (Vickrey auction with reserve price r.) *Each agent communicates a bid. If the highest bid exceeds r, the item is allocated to the highest bidder, who pays an amount equal to the maximum of r and the second highest bid. No other agents make or receive payments.*

Definition 2. (redistribution auction RM_ρ, with parameter $\rho \in [0,1]$ and reserve price r). *Each agent communicates a bid. If the highest bid exceeds r, the item is allocated to the highest bidder, who pays an amount equal to the maximum of r and the second highest bid. Each agent i (including the high bidder) is then payed an amount equal to $\frac{\rho}{n}$ times the maximum of r and the second highest bid amongst agents other than i.*

RM_ρ (for $\rho > 0$) modifies the Vickrey auction by returning revenue to the agents. Because RM_ρ, for any value of ρ, falls within the class of Groves mechanisms,[7] it is efficient:

Proposition 1. *For any $\rho \in \Re$, RM_ρ yields truthful bidding and achieves the efficient allocation (given the set of bidders) in dominant strategies.*

We will be doing equilibrium analysis of the social welfare properties of these auctions and Proposition 1 will greatly simplify our analysis since, though there are two "stages" to agent strategies (choice to participate or not, and then bid choice), contingent on participation truthfulness is always a dominant strategy in the second stage.

We now formally describe the properties of our setting. Each of the N agents is aware of the form of the auction and the number of agents N. Each agent has a value and a cost of entry, where cost is defined by function $c : [0,1] \to (0,1)$ (the same for all agents) mapping value to cost, with $c(v)$ denoting the cost that an agent with value v bears for participation. Each agent privately observes his value (and thus his cost) and decides whether or not to participate. Values are independently and identically distributed over the interval $[0,1]$ according to density function f, with F denoting the cumulative distribution function. We define G to be the c.d.f. for the highest value amongst $N-1$ agents (so $G(x) = F(x)^{N-1}$) and g the p.d.f. We define H_m^z and h_m^z be the c.d.f. and p.d.f., respectively, for the z^{th} highest value amongst m agents, with $H_m^z(\cdot, v)$ and $h_m^z(\cdot, v)$ denoting the c.d.f. and p.d.f. conditioned on the m values all being at least v. We let $\mathcal{P}(N, n, v)$ denote the probability that exactly n out of N agents have value at least v, i.e.,

$$\mathcal{P}(N, n, v) = \binom{N}{n}(1 - F(v))^n F(v)^{N-n} \tag{1}$$

Finally, since we will be analyzing redistribution auctions, we define notation $R_\rho(v_0, r)$ to be the expected redistribution payment to each agent under RM_ρ with reserve price r, under the condition that every agent participates if and only if his value is at least v_0. In words, $R_\rho(v_0, r)$ is the expected value of: the second highest value at least as great as v_0 amongst $N-1$ agents (or r if there is none) divided by one plus the number of those agents with value at least v_0. That is, for a zero reserve price,

$$R_\rho(v_0, 0) = \sum_{m=2}^{N-1} \mathcal{P}(N-1, m, v_0) \cdot \frac{\rho}{m+1} \int_{v_0}^{1} h_m^2(x, v_0) \cdot x \, dx \tag{2}$$

[7]A Groves mechanism is any mechanism that chooses an efficient outcome based on agent reports and pays each agent i the value obtained by others minus a quantity independent of i's report. See, e.g., Chapter 2 of [4] for a discussion of this class of mechanisms and an intuitive proof that they are truthful and efficient in dominant strategies. The fact that RM_ρ, for arbitrary value ρ, is a Groves mechanism follows immediately from the fact that it modifies the Vickrey auction (well-known to be a Groves mechanism) by adding a payment to each agent that is independent of his bid.

And, more generally, for arbitrary $r \le v_0$,

$$R_\rho(v_0, r) = \mathcal{P}(N-1, 0, v_0) \cdot \rho r \ + \ \mathcal{P}(N-1, 1, v_0) \cdot \frac{\rho}{2} r + \tag{3}$$

$$\sum_{m=2}^{N-1} \mathcal{P}(N-1, m, v_0) \cdot \frac{\rho}{m+1} \int_{v_0}^{1} h_m^2(x, v_0) \cdot x \, dx \tag{4}$$

Since expected welfare depends on expected participation, and since expected participation depends on expected payoff for an agent, which in turn depends on the strategic participation decisions of other agents, we will form expectations based on what occurs in a game theoretic equilibrium. Specifically, we will be interested in outcomes that occur in Bayes-Nash equilibrium, where, given knowledge of the value density function f, no agent can expect to gain from deviating from the posited strategy given that other agents do not. We will find that equilibria have the form of a *cutoff level*, where agents participate if and only if they have values above the cutoff. We will use the term *Bayes-Nash cutoff equilibrium* to refer to a Bayes-Nash equilibrium that has this property. Formally:

Definition 3 (Bayes-Nash cutoff equilibrium). *Given a common-knowledge distribution over each agent's valuation, a vector of values $v_1, \ldots, v_n$ and bidding strategies $\sigma_1, \ldots, \sigma_n$, with $\sigma_i : \Re \rightarrow \Re$, $\forall i \in I$, constitutes a Bayes-Nash cutoff equilibrium if and only if: $\forall i \in I$, if every $j \in I \setminus \{i\}$ participates and bids $\sigma_j(v_j)$ if his value is at least v_j and otherwise does not participate, i maximizes his expected utility by participating and bidding $\sigma_i(v_i)$ if his value is at least v_i and otherwise not participating.*

In the context of the Vickrey auction or any redistribution auction, since truthful bidding is a dominant strategy contingent on participation, any Bayes-Nash cutoff equilibrium can be described simply by the cutoff levels. Accordingly we will say things like "value v_0 is a symmetric Bayes-Nash cutoff equilibrium" (with truthful bidding in the second stage left implicit). A cutoff equilibrium is *symmetric* if it specifies the same strategy (cutoff level v_0 and bidding strategy) for each agent.

As an example, consider $N = 4$ and values for agents 1, 2, 3, and 4 that are 0.1, 0.6, 0.75, and 0.8, respectively. Imagine, purely hypothetically, that all agents play cutoff level 0.5 and bid truthfully. Then agent 1 does not participate, and agents 2, 3, and 4 place truthful bids and the item is allocated to agent 4. Under the Vickrey auction agent 4 pays 0.75, the second highest bid. Under the redistribution auction RM_1, too, agent 4 makes this payment, but then additional redistribution payments are made. Agent 2 is payed $0.75/3 = 0.25$ and agents 3 and 4 are each payed $0.6/3 = 0.2$. Of the 0.75 "revenue" generated by the Vickrey auction, 0.65 has been redistributed by RM_1. Whether 0.5 or in fact any other cutoff value constitutes an *equilibrium* will depend on the entry costs and the distribution over values, but we will soon see that, given some restrictions on the nature of the entry cost function, a symmetric cutoff equilibrium practically always exists.

2 Linear cost functions

In this section we provide core extensions of the theory of auctions with entry, going beyond fixed costs and beyond analysis of Vickrey or first-price auctions. We consider the space of cost functions that are linear in agent valuations and have range lying in the open interval bounded by the minimum and maximum possible valuation.

Definition 4 (regular linear cost function). *A cost function $c : [0, 1] \rightarrow [0, 1]$ is termed regular linear if and only if $\exists b, k \in [0, 1]$ such that, $\forall v \in [0, 1]$, $c(v) = b + kv \in (0, 1)$.*

The following lemma, which is not specific to *linear* cost functions, will be useful as a first step towards identifying cutoff equilibria.

Lemma 1. *For arbitrary continuous cost function c, if cutoff level $v_0 \in (0,1)$ for each agent is a Bayes-Nash cutoff equilibrium for the RM_ρ auction with reserve price $r \in \Re$, then in that equilibrium any agent with value v_0 has expected utility 0 from participation.*

Proof. Assume the lemma fails to hold for the RM_ρ auction, for arbitrary $\rho \in [0,1]$ and $r \in \Re$, with cutoff equilibrium v_0. If playing cutoff level v_0 and truthful bidding yields negative expected utility for an agent, clearly setting a higher cutoff (e.g., 1, yielding non-participation and thus utility 0) would be a superior best-response strategy. Alternatively if the strategy yields expected utility $\gamma > 0$, consider the expected utility from participation of an agent with value $v_0 - \epsilon$, for some small $\epsilon > 0$. Noting that the agent will win the auction if and only if no other agent participates, given reserve price r his expected utility equals:

$$R_\rho(v_0, r) + (v_0 - r - \epsilon) \cdot G(v_0) - c(v_0 - \epsilon) \tag{5}$$

Regardless of the agent's value, in expectation he will obtain redistribution payment $R_\rho(v_0, r)$ from participation. The difference in expected utility for truthful participation between that of an agent with value v_0 and that of an agent with value $v_0 - \epsilon$ is:

$$[R_\rho(v_0, r) + (v_0 - r) \cdot G(v_0) - c(v_0)] - \tag{6}$$
$$[R_\rho(v_0, r) + (v_0 - r - \epsilon) \cdot G(v_0) - c(v_0 - \epsilon)] \tag{7}$$
$$= \epsilon \cdot G(v_0) - [c(v_0) - c(v_0 - \epsilon)] \tag{8}$$

Because cost function c is continuous, $\epsilon > 0$ can be chosen small enough such that this difference (Eq. (8)) is strictly less than γ. Then the expected utility of participating with value $v_0 - \epsilon$ is greater than 0, contradicting v_0 as a cutoff participation equilibrium since an agent with value $v_0 - \epsilon$ would gain from participation. □

A consequence of this lemma is that, for any putative equilibrium cutoff value v_0 of the RM_ρ auction with reserve price r,

$$R_\rho(v_0, r) + (v_0 - r)G(v_0) = c(v_0) \tag{9}$$

We can now precisely describe conditions for the existence of a symmetric cutoff equilibrium for RM_ρ.[8]

Theorem 1. *Consider the RM_ρ auction with reserve price $r \geq 0$ and regular linear entry costs. Consider the symmetric strategy profile where each agent enters and is truthful if his value is at least v_0 and otherwise does not enter, where v_0 is defined to solve $R_\rho(v_0, r) + (v_0 - r)G(v_0) = c(v_0)$. If such a $v_0 \in (r, 1)$ exists and $c(0) \geq R_\rho(v_0, r)$, then this is a Bayes-Nash equilibrium.*

Proof. For cost function $c(v) = b + kv$, assume existence of a $v_0 \in (r, 1)$ satisfying $R_\rho(v_0, r) + (v_0 - r)G(v_0) = c(v_0)$ and $c(0) = b \geq R_\rho(v_0, r)$. Assume all agents other than some $i \in I$ play the posited equilibrium strategy. If i decides to participate after learning his value, truthfulness will then be a dominant strategy, and so the theorem holds if i obtains expected net utility ≥ 0 if his value is at least v_0 and ≤ 0 otherwise.

[8]In writing the proof for Theorem 1 I was inspired by Matthews [10, Theorem 4.2], who analyzes symmetric equilibria of the Vickrey auction in a constant cost setting.

If i's value equals v_0, if he participates his expected utility is:

$$u_i(v_0) = R_\rho(v_0, r) + (v_0 - r)G(v_0) - c(v_0) = 0, \tag{10}$$

Bearing in mind that $G(v_0) = \frac{c(v_0) - R_\rho(v_0,r)}{v_0 - r}$, if i's value is $\underline{v} < v_0$ and he participates his expected utility is:

$$\begin{aligned}
u_i(\underline{v}) &= R_\rho(v_0, r) + (\underline{v} - r)G(v_0) - c(\underline{v}) & (11)\\
&= R_\rho(v_0, r) + \frac{\underline{v} - r}{v_0 - r}(c(v_0) - R_\rho(v_0, r)) - c(\underline{v}) & (12)\\
&= \frac{\underline{v} - r}{v_0 - r}(b + kv_0 - R_\rho(v_0, r)) - (b + k\underline{v} - R_\rho(v_0, r)) & (13)\\
&\leq \frac{\underline{v}}{v_0}(b + kv_0 - R_\rho(v_0, r)) - (b + k\underline{v} - R_\rho(v_0, r)) & (14)\\
&= \frac{\underline{v}}{v_0}(b - R_\rho(v_0, r)) - (b - R_\rho(v_0, r)) \leq 0 & (15)
\end{aligned}$$

We use the fact that $v_0 - r > 0$ and $b = c(0) \geq R_\rho(v_0, r)$. So i cannot gain by participating for any $\underline{v} < v_0$.

Now consider the case where i's value is $\overline{v} > v_0$. Let $p(y)$ equal y if $y \geq v_0$ and r otherwise. If i participates then his expected utility is:

$$\begin{aligned}
u_i(\overline{v}) &= R_\rho(v_0, r) + \int_0^{\overline{v}} [\overline{v} - p(y)]g(y)\, dy - c(\overline{v}) & (16)\\
&= R_\rho(v_0, r) + \int_0^{v_0} [\overline{v} - p(y)]g(y)\, dy - c(\overline{v}) + \int_{v_0}^{\overline{v}} [\overline{v} - p(y)]g(y)\, dy & (17)\\
&= R_\rho(v_0, r) + (\overline{v} - r)G(v_0) - c(\overline{v}) + \int_{v_0}^{\overline{v}} [\overline{v} - p(y)]g(y)\, dy & (18)\\
&\geq R_\rho(v_0, r) + (\overline{v} - r)G(v_0) - c(\overline{v}) & (19)\\
&= R_\rho(v_0, r) + \frac{\overline{v} - r}{v_0 - r}(c(v_0) - R_\rho(v_0, r)) - c(\overline{v}) & (20)\\
&= \frac{\overline{v} - r}{v_0 - r}(b + kv_0 - R_\rho(v_0, r)) - (b + k\overline{v} - R_\rho(v_0, r)) & (21)\\
&\geq \frac{\overline{v}}{v_0}(b + kv_0 - R_\rho(v_0, r)) - (b + k\overline{v} - R_\rho(v_0, r)) & (22)\\
&= \frac{\overline{v}}{v_0}(b - R_\rho(v_0, r)) - (b - R_\rho(v_0, r)) \geq 0 & (23)
\end{aligned}$$

Again we use $v_0 - r > 0$ and $b = c(v_0) > R_\rho(v_0, r)$. The move from Eq. (18) to Eq. (19) holds because $p(y) < \overline{v}$ for all $y < \overline{v}$. So i cannot gain from *not* participating when his value is greater than v_0, which completes the proof. □

Since the Vickrey auction is a special case of the RM_ρ auction class with $\rho = 0$, the following is a direct consequence of Theorem 1.

Corollary 1. *Consider the Vickrey auction with regular linear entry costs and reserve price r. Consider the symmetric strategy profile where each agent enters and is truthful if his value is at least v_0 and otherwise does not enter, where v_0 is defined to solve $(v_0 - r)G(v_0) = c(v_0)$. If such a $v_0 \in (0, 1)$ exists, this is a Bayes-Nash equilibrium.*

We can now use the combination of Lemma 1 and Corollary 1 to demonstrate that there always exists exactly *one* such equilibrium for the Vickrey auction or a redistribution auction with sufficiently small redistribution parameter.

Theorem 2. *For arbitrary regular linear entry cost function and values with arbitrary continuous density function with support* $[0,1]$*, for arbitrary* $r \in (0, 1-c(1))$*, there exists strictly positive* ρ^* *such that* RM_ρ *with any* $\rho \in [0, \rho^*]$ *and reserve price* r *has a unique symmetric Bayes-Nash cutoff equilibrium.*

Proof. Given cost function $c(v) = b + kv$, given Lemma 1 and Theorem 1, it is sufficient to show there exists a $\rho^* > 0$ such that for all $\rho \in [0, \rho^*]$, $\pi(v_0) = R_\rho(v, r) + (v_0 - r)G(v_0)$ equals $c(v_0)$ at exactly one point $v_0 \in (0, 1)$. Note that $\pi(v) < 0$ for all $v < r$, $\pi(r) = 0 < c(r)$, $\pi(1) = R_\rho(1, r) + (1-r)G(1) = 1 - r > c(1)$, and $\forall v \in (r, 1)$, $\pi(v) \in (0, 1-r)$. Therefore $\pi(v_0) = c(v_0)$ at at least one point v_0, and so a symmetric Bayes-Nash cutoff equilibrium exists.

Now assume for contrdiction that there exist $v', v'' \in (r, 1)$ with $v'' > v'$ such that $\pi(v') = c(v')$ and $\pi(v'') = c(v'')$. This entails that $R_\rho(v', r) + (v' - r)G(v') - c(v') = R_\rho(v'', r) + (v'' - r)G(v'') - c(v'') = 0$. This in turn implies that $G(v') = \frac{b+kv'-R_\rho(v',r)}{v'-r}$ and $G(v'') = \frac{b+kv''-R_\rho(v'',r)}{v''-r}$. Let $\gamma = G(v'') - G(v')$. Since G is strictly increasing over $[0, 1)$ (a consequence of the fact that $G(v) = F(v)^{N-1}$ and F is a cumulative distribution with positive support over $[0, 1]$), $\gamma > 0$. Now:

$$G(v'') - G(v') = \frac{b + kv'' - R_\rho(v'', r)}{v'' - r} - \frac{b + kv' - R_\rho(v', r)}{v' - r} \tag{24}$$

$$= b\Big(\frac{1}{v'' - r} - \frac{1}{v' - r}\Big) + k\Big(\frac{v''}{v'' - r} - \frac{v'}{v' - r}\Big) + \frac{R_\rho(v', r)}{v' - r} - \frac{R_\rho(v'', r)}{v'' - r} \tag{25}$$

$$< k\Big(\frac{v''}{v'' - r} - \frac{v'}{v' - r}\Big) + \frac{R_\rho(v', r)}{v' - r} - \frac{R_\rho(v'', r)}{v'' - r} \tag{26}$$

$$< \frac{R_\rho(v', r)}{v' - r} - \frac{R_\rho(v'', r)}{v'' - r} \tag{27}$$

But regardless of the value of γ, given that it is strictly greater than zero, a value of $\rho^* > 0$ can be selected such that for all $\rho \in [0, \rho^*]$, $\frac{R_\rho(v',r)}{v'-r} - \frac{R_\rho(v'',r)}{v''-r} < \gamma$ (as ρ approaches 0 this quantity approaches 0). Thus we have a contradiction with the assumption that there are two possible equilibria v' and v''. So there must be only one. ☐

In practice, the range of ρ values that yield a unique symmetric cutoff equilibrium for RM_ρ will be quite large, often encompassing the complete $[0, 1]$ interval. The fact that the equilibrium in the Vickrey auction is unique directly follows from this theorem.

Corollary 2. *For arbitrary regular linear entry cost function and values with arbitrary continuous density function with support* $[0, 1]$*, the Vickrey auction with reserve price* $r \in (0, 1 - c(1))$ *has a unique symmetric Bayes-Nash cutoff equilibrium.*

Figure 1 illustrates the unique symmetric equilibria of the Vickrey and RM_1 auctions with no reserve. For the Vickrey auction, it is the point at which the cost line intersects the $vG(v)$ curve; for RM_1 it is somewhat lower, at the point where the cost line minus the redistribution curve intersects the $vG(v)$ curve. Figure 2 illustrates the range of ρ values that maintain a unique symmetric equilibrium, for a setting with 5 agents, uniformly distributed values, and constant costs ($c(v) = b + 0 \cdot v$). The maximum ρ value is plotted as a function of the cost. For most cost levels (all above threshold 0.125 in the example), all $\rho \in [0, 1]$ yield a unique symmetric equilibrium.

Given that the Vickrey auction and an RM_ρ auction with $\rho > 0$ each have a unique symmetric equilibrium, because RM_ρ yields higher expected utility for every agent (regardless of valuation), more agents will participate and a higher expected allocation value will result.

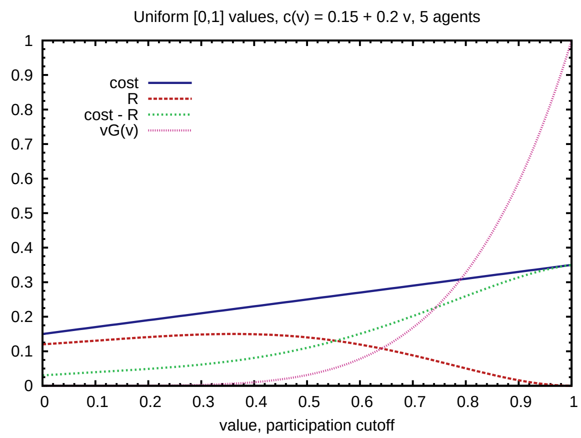

Figure 1: For cost function $c(v) = 0.15 + 0.2v$, with $\rho = 1$ and 5 agents. Note that the cost minus redistribution curve intersects the $vG(v)$ curve at exactly one point, which is the cutoff equilibrium value for RM_ρ; and this point is less than the intersection point of $vG(v)$ and the cost curve, which is the cutoff equilibrium value for the Vickrey auction.

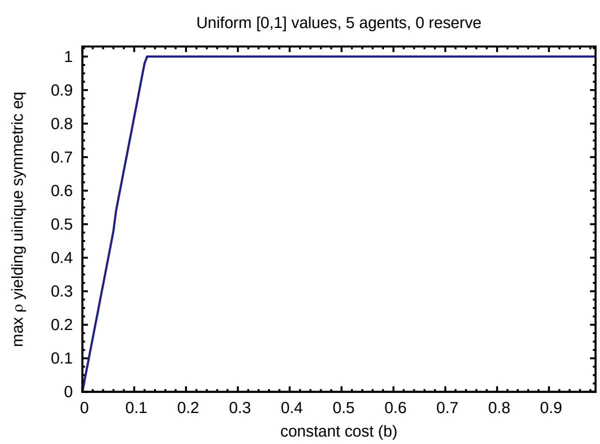

Figure 2: For constant costs, 5 agents, and uniformly distributed values, the maximum ρ yielding a unique symmetric equilibrium is plotted as a function of cost. For costs above 0.125, all ρ values yield a unique symmetric cutoff equilibrium.

Theorem 3. *For arbitrary regular linear entry cost function and values with arbitrary density function with support* $[0, 1]$*, there exists a* $\rho > 0$ *such that* RM_ρ *with no reserve price yields strictly higher expected allocation value than the Vickrey auction in a unique symmetric Bayes-Nash equilibrium.*

Proof. For RM_ρ with any positive ρ, if there is a unique symmetric Bayes-Nash cutoff equilibrium, the cutoff value will be lower than that of the Vickrey auction (since $R_\rho(v_0, 0) > 0$ for all $v_0 \in (0, 1)$ and $\rho > 0$, combined with Eq. (9)). By Theorem 2 we know a positive ρ yielding a unique symmetric cutoff equilibrium always exists. Given two mechanisms with unique cutoff equilibria, any agent that participates in the one with higher cutoff value will participate in the one with lower cutoff, and there exist agents (more precisely, realizations of values) that will participate in the one with lower cutoff but not the one with higher. Moreover, there is positive probability that there will be agents with value above the lower cutoff but none with value above the higher cutoff. Thus, since both the Vickrey auction and the RM_ρ auction choose efficient allocations given the set of participating agents, the expected allocation value of RM_ρ will be strictly higher. □

2.1 Increasing participation through reserve prices

We will now demonstrate what, at first, seems to be a quite counterintuitive fact: adding a reserve price to the RM_ρ auction can lead to increased participation. In the Vickrey auction this will never be the case: adding a reserve price only decreases expected payoff for the agents, since its only effect is to introduce the possibility that agents will pay the entry cost but none will obtain any surplus, or that an agent will win the item but have to pay an increased price. In the case of RM_ρ there is a countervailing factor: a reserve price provides an agent-independent guarantee on revenue. For the purpose of redistribution, a reserve price plays the role of an "extra bid", so that even in the case where there are just two bidders, the reserve price provides a basis for redistribution, whereas without it no redistribution could occur.

Theorem 4. *Consider an arbitrary regular linear entry cost function. Let v_0 be the unique symmetric Bayes-Nash cutoff equilibrium of RM_ρ with no reserve price (where ρ is restricted to choices that ensure such an equilibrium exists and is unique). There exists a positive reserve price leading to increased expected allocation value in a symmetric Bayes-Nash cutoff equilibrium if $F(v_0) < \frac{N-1}{N-3+\frac{2}{\rho}}$.*

Proof. See Appendix. □

Figure 3 illustrates the effect a reserve price can have on increasing redistribution levels. As the participation cutoff level grows, the amount by which a reserve price increases redistribution also grows. The figure also illustrates how the resulting equilibrium participation level is lowered with a reserve price. Figure 4 illustrates how the allocation value improves (a consequence of lower equilibrium cutoff levels) as a function of reserve price.

2.2 Efficiency of the Vickrey auction

We now shift our focus from allocation value to efficiency, i.e., allocation value minus the aggregate costs of participation. When our goal is maximizing the aggregate social welfare that results from the auction, efficiency is the appropriate metric to consider. Stegeman [12] proved that the Vickrey auction has an efficient equilibrium, which may be asymmetric. Here we will show something different, but highly related: for regular linear entry costs, the unique symmetric equilibrium cutoff strategy profile of the Vickrey auction is at least as efficient as any other symmetric cutoff strategy profile. The following lemma will be central to the proof:

Lemma 2. *Consider arbitrary $v_0, v' \in \Re$. In the Vickrey auction with entry, if all agents playing participation cutoff level v' yields greater efficiency than all playing $v_0 \neq v'$, then one agent playing strategy v' and all others playing v_0 yields greater efficiency than all playing v_0.*

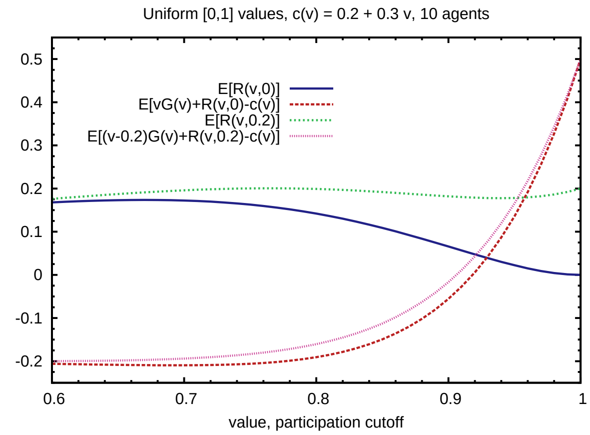

Figure 3: Comparison of redistribution and cutoff equilibrium levels for RM_1 with no reserve price and RM_1 with reserve price of 0.2. The cutoff equilibria can be identified as the points where the bottom two curves cross the 0 line. The cutoff equilibrium for reserve price 0.2 is smaller and thus in equilibrium there will be greater participation and greater allocation value.

Proof. See Appendix. □

Theorem 5. *For arbitrary regular linear entry cost function and values with arbitrary density function with support* $[0, 1]$*, the cutoff strategy profile achieved by the Vickrey auction in its unique symmetric Bayes-Nash equilibrium yields strictly greater expected social welfare net of costs than all other symmetric cutoff strategies.*

Proof. Given cost function c, let v_0 be the unique symmetric cutoff equilibrium achieved by the Vickrey auction in Bayes-Nash equilibrium (we know it exists and is unique by Corollary 2). Assume for contradiction that the theorem does not hold. Then there is some value $v \in (0, 1)$ not equal to v_0 such that, if all agents played cutoff strategy v, greater expected social welfare net of costs (efficiency) would result. By Lemma 2, this in turn implies that if all but one agent played cutoff strategy v_0 and one played v, greater expected efficiency would result.

Consider all agents playing v_0 as a baseline. Now consider that a single agent changes to strategy $\underline{v} < v_0$. We want to show that for all such $\underline{v}$, the "efficiency gain" is negative, i.e., that:

$$\forall \underline{v} \in [0, v_0), \quad F(v_0)^{N-1} \int_{\underline{v}}^{v_0} f(x)x \, dx - \int_{\underline{v}}^{v_0} f(x)c(x) \, dx < 0 \tag{28}$$

The first term is the increased expected allocation value from the change, and the second term is the increased cost. By Lemma 1, since v_0 is the unique symmetric equilibrium of the Vickrey auction, we know that $F(v_0)^{N-1} = G(v_0) = \frac{c(v_0)}{v_0}$. So for arbitrary $\underline{v} < v_0$,

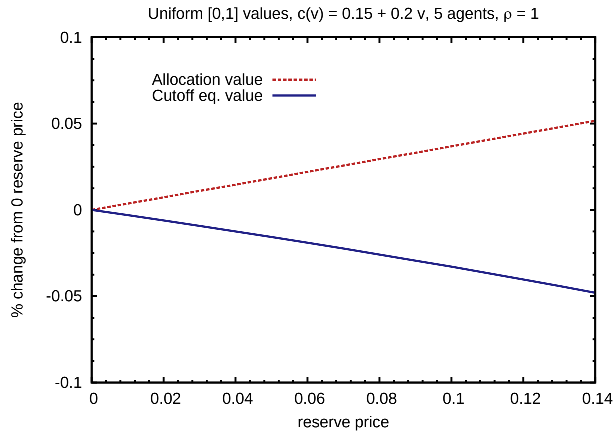

Figure 4: For cost function $c(v) = 0.15 + 0.2v$, with $\rho = 1$ and 5 agents. The cutoff equilibrium value decreases and the allocation value increases linearly as the reserve price increases. At a certain reserve price threshold (0.14, here), a unique equilibrium is no longer sustained without decreasing ρ.

we have:

$$F(v_0)^{N-1}\int_{\underline{v}}^{v_0} f(x)x\,dx - \int_{\underline{v}}^{v_0} f(x)c(x)\,dx \tag{29}$$

$$= \frac{c(v_0)}{v_0}\int_{\underline{v}}^{v_0} f(x)x\,dx - \int_{\underline{v}}^{v_0} f(x)c(x)\,dx \tag{30}$$

$$= \int_{\underline{v}}^{v_0} f(x)\Big[\frac{x}{v_0}\cdot c(v_0) - c(x)\Big]\,dx \tag{31}$$

$$= \int_{\underline{v}}^{v_0} f(x)\Big[\frac{x}{v_0}\cdot (b+kv_0) - (b+kx)\Big]\,dx \tag{32}$$

$$= \int_{\underline{v}}^{v_0} f(x)\Big(\frac{x}{v_0} - 1\Big)b\,dx \tag{33}$$

$$< \int_{\underline{v}}^{v_0} f(x)\Big(\frac{v_0}{v_0} - 1\Big)b\,dx = 0 \tag{34}$$

Now consider a single agent playing a $\overline{v} > v_0$. We want to show that for all such $\overline{v}$, the "efficiency loss" is positive, i.e., that:

$$\forall \overline{v} \in (v_0, 1], \quad F(v_0)^{N-1}\int_{v_0}^{\overline{v}} f(x)x\,dx - \int_{v_0}^{\overline{v}} f(x)c(x)\,dx > 0 \tag{35}$$

For arbitrary $\overline{v} > v_0$, we have:

$$F(v_0)^{N-1} \int_{v_0}^{\overline{v}} f(x)x\,dx - \int_{v_0}^{\overline{v}} f(x)c(x)\,dx \tag{36}$$

$$= \frac{c(v_0)}{v_0} \int_{v_0}^{\overline{v}} f(x)x\,dx - \int_{v_0}^{\overline{v}} f(x)c(x)\,dx \tag{37}$$

$$= \int_{v_0}^{\overline{v}} f(x)\Big(\frac{x}{v_0} - 1\Big)b\,dx \tag{38}$$

$$> \int_{v_0}^{\overline{v}} f(x)\Big(\frac{v_0}{v_0} - 1\Big)b\,dx = 0 \tag{39}$$

We have thus demonstrated that for any $v' \neq v_0$, the change in efficiency from cutoff v_0 is negative, and so the proof is complete. □

3 An obscurity costs model

In the linear cost setting we've been considering, and more generally when costs are independent of the behavior of other agents, the Vickrey auction is known to be optimal with respect to maximizing efficiency [12]; it elicits exactly the right amount of participation, so that the marginal expected improvement in allocation value of more participation doesn't outweigh the added costs. But this model is restrictive and doesn't allow for the possibility that the costs of participating in an auction may be dependent on the equilibrium properties of the mechanism, such as expected participation. In many cases participation costs will not come in the form of an entry fee, but rather will be "search costs" related, for instance, to finding the auction and going through the bureaucracy of preparing for participation. It is natural to expect that such costs fall in proportion to the popularity of the auction.

Consider this example: a new marketplace is attempting to establish itself as a competitor to the auction site Ebay. The barriers to participation in the upstart marketplace will be significant, as a result of its relative obscurity: becoming aware of the auction, finding out how to sign-up for participation, overcoming the psychological concerns about reliability of the company's privacy and enforcement claims, etc. There is a cost of obscurity. Let oc denote an *obscurity cost* function that maps equilibrium participation rates to costs. If we're concerned with what happens in a unique symmetric cutoff equilibrium, the equilibrium cutoff value is a completely sufficient stand-in for "equilibrium participation rate"; so $oc(v_0)$ denotes the obscurity cost of an auction in which all and only agents with value above v_0 participate. The obscurity cost is independent of the agent's value, but it may come *in addition to* a standard (potentially value-dependent) entry cost of the sort we've been considering up until now.

We will show that in such settings the Vickrey auction is inefficient and is dominated by an RM_ρ auction. We start, with Theorem 6, by describing the conditions that hold in a symmetric equilibrium of RM_ρ. Then in Theorem 7 we demonstrate that, given mild conditions on the cost functions, there is a unique symmetric equilibrium. Finally, in Theorem 8 we demonstrate that a redistribution mechanism is always more efficient than the Vickrey auction, for a wide space of obscurity cost functions.

Theorem 6. *Consider arbitrary regular linear entry cost function c, arbitrary obscurity cost function oc, and the RM_ρ auction with reserve price $r \geq 0$. Consider the symmetric strategy profile where each agent enters and is truthful if his value is at least v_0 and* *otherwise does not enter, where* v_0 *is defined to solve* $R_\rho(v_0, r) + (v_0 - r)G(v_0) = c(v_0) + oc(v_0)$*. If such a* $v_0 \in (r, 1)$ *exists and* $c(0) \geq R_\rho(v_0, r)$*, this is a Bayes-Nash equilibrium.*

Proof. See Appendix. □

Theorem 7. *Consider arbitrary regular linear entry cost function* c *and arbitrary continuous obscurity cost function* oc *such that* $c(v) + oc(v)$ *is continuous, monotonic, and concave or convex in* v*, and,* $\forall v, v' \in [0, 1]$*,* $c(v) + oc(v') \in (0, 1)$*. For values with arbitrary continuous density function with support* $[0, 1]$*, for arbitrary* $r \in (0, 1 - c(1) - oc(1))$*, there exists strictly positive* ρ^* *such that* RM_ρ *with any* $\rho \in [0, \rho^*]$ *and reserve price* r *has a unique symmetric Bayes-Nash cutoff equilibrium.*

Proof. See Appendix. □

Since the Vickrey auction is a special case of the RM_ρ auction class, Theorem 7 tells us that the Vickrey auction as well as the redistribution auction, for some space of redistribution parameters ρ, each have unique symmetric equilibria. This sets up a natural comparison. We can now address the question of whether redistribution is helpful or harmful to efficiency in an obscurity costs setting. Theorem 8 demonstrates that it is helpful.

Theorem 8. *Consider arbitrary regular linear entry cost function* c *and arbitrary continuous, strictly increasing obscurity cost function* oc *such that* $c(v)+oc(v)$ *is monotonic and concave or convex in* v*, and,* $\forall v, v' \in [0, 1]$*,* $c(v) + oc(v') \in (0, 1)$*. For values with arbitrary density function with support* $[0, 1]$*, there exists a* $\rho > 0$ *such that* RM_ρ *has a unique symmetric Bayes-Nash cutoff equilibrium and yields greater expected social welfare than the Vickrey auction in its unique symmetric equilibrium.*

Proof. We will show that there exists a $\rho > 0$ such that RM_ρ has a unique cutoff equilibrium v_0 with cutoff less than that of the unique Vickrey auction equilibrium (call it $\overline{v}$), and that expected allocation value minus costs (efficiency) is greater with cutoff v_0 than with $\overline{v}$.

The expected efficiency in equilibrium v_0 equals the efficiency of an auction with no obscurity costs minus: $N \cdot oc(v_0) \cdot \int_{v_0}^1 f(x)\,dx$. So expected efficiency in a cutoff equilibrium v_0, given N agents and cost functions c and oc, equals:

$$1 - v_0 F(v_0)^N - \int_{v_0}^1 F(x)^N\,dx - N\int_{v_0}^1 f(x)c(x)\,dx - N \cdot oc(v_0) \cdot \int_{v_0}^1 f(x)\,dx \tag{40}$$

$$= 1 - v_0 F(v_0)^N - \int_{v_0}^1 F(x)^N\,dx - N\int_{v_0}^1 f(x)[c(x) + oc(v_0)]\,dx \tag{41}$$

The first three terms of Eq. (41) represent the expected allocation value and the last is the expected social cost. Examining the partial derivative with respect to the cutoff value, we get:

$$\frac{\partial}{\partial v_0}\left(1 - v_0 F(v_0)^N - \int_{v_0}^1 F(x)^N\,dx - N\int_{v_0}^1 f(x)[c(x) + oc(v_0)]\,dx\right) \tag{42}$$

$$= -N v_0 F(v_0)^{N-1} f(v_0) + N f(v_0) c(v_0) - \left(N \cdot oc'(v_0)\int_{v_0}^1 f(x)\,dx - N f(v_0) oc(v_0)\right) \tag{43}$$

$$= N f(v_0) \cdot \left(c(v_0) + oc(v_0) - v_0 F(v_0)^{N-1} - \frac{oc'(v_0)}{f(v_0)}\int_{v_0}^1 f(x)\,dx\right) \tag{44}$$

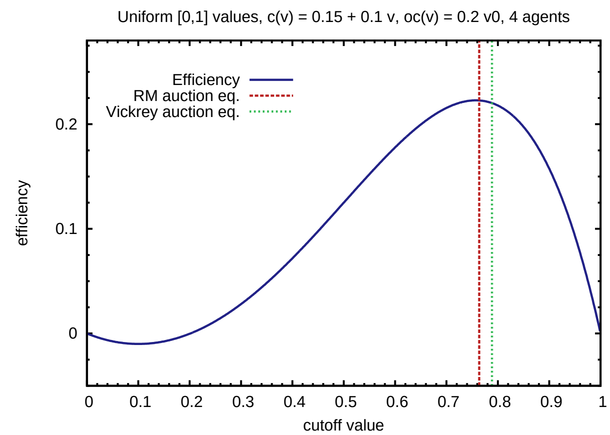

Figure 5: Plot of the expected efficiency (allocation value minus cost) that will result, as a function of symmetric cutoff value played by all agents. For cost functions $c(v) = 0.15 + 0.1v$ and $oc(v_0) = 0.2v_0$, with 4 agents. The RM_1 auction equilibrium cutoff level is significantly lower than that of the Vickrey auction, which, in this obscurity costs setting, makes it more efficient.

For the cutoff equilibrium $\overline{v}$ of the Vickrey auction, we know, from an argument completely analogous to Lemma 1, that $\overline{v}F(\overline{v})^{N-1} = c(\overline{v}) + oc(\overline{v})$. So, evaluating the partial derivative at $v_0 = \overline{v}$, we have:

$$N f(\overline{v}) \cdot \left(c(\overline{v}) + oc(\overline{v}) - \overline{v}F(\overline{v})^{N-1} - \frac{oc'(\overline{v})}{f(\overline{v})} \int_{\overline{v}}^{1} f(x)\, dx \right) \tag{45}$$

$$= -N f(\overline{v}) \cdot \frac{oc'(\overline{v})}{f(\overline{v})} \int_{\overline{v}}^{1} f(x)\, dx \tag{46}$$

$$= -N \cdot oc'(\overline{v}) \cdot \int_{\overline{v}}^{1} f(x)\, dx \tag{47}$$

N is positive, $\int_{\overline{v}}^{1} f(x)\, dx$ is positive, and, given the assumption that obscurity costs are strictly increasing in equilibrium cutoff value, $oc'(\overline{v})$ is positive; thus the derivative at $\overline{v}$ is negative, indicating that there is an $\epsilon > 0$ such that any cutoff in the interval $[\overline{v} - \epsilon, \overline{v})$ yields greater efficiency than $\overline{v}$.

It only remains to show that RM_ρ for some $\rho \in (0, 1]$ achieves unique symmetric cutoff equilibrium $v_0 \in [\overline{v} - \epsilon, \overline{v})$. Note that for any $\delta^* > 0$, we can select $\rho \in (0, 1]$ such that $R_\rho(v, 0) \in (0, \delta^*)$, $\forall v \in [0, 1)$. Since values have density function with support $[0, 1]$ and $c(v) + oc(v)$ is continuous and $\in (0, 1)$, $\forall v \in [0, 1]$, the value v'_0 that solves $v'_0 G(v'_0) = c(v'_0) + oc(v'_0) - \delta$ is strictly less than v_0 for any $\delta > 0$ and converges to v_0 in the limit as δ goes to 0. Choose δ^* such that the solution to $v'_0 G(v'_0) = c(v'_0) + oc(v'_0) - \delta^*$ is in the interval $[\overline{v} - \epsilon, \overline{v})$ and choose ρ such that RM_ρ has a unique symmetric Bayes-Nash cutoff equilibrium and $R_\rho(v, 0) \in (0, \delta^*)$, $\forall v \in [0, 1)$. Then the solution v_0 to $v_0 G(v_0) = c(v_0) + oc(v_0) - R_\rho(v_0, 0)$ is in the interval $[\overline{v} - \epsilon, \overline{v})$, which completes the proof. □

Figure 5 illustrates the efficiency gain of RM_1 over the Vickrey auction for one particular linear cost function and linear obscurity cost function. The Vickrey auction equilibrium is to the right of the peak of the efficiency curve (under-participation), and the RM_1 equilibrium moves closer to the peak, increasing efficiency through increased participation.

So this demonstrates that we can always improve the Vickrey auction outcome by moving to a redistribution auction. But can we achieve perfect efficiency this way, and, if not, can we do so with an alternative auction? RM_1 comes close to redistributing all revenue, and so it comes commensurately close to maximizing participation without running a deficit. The following theorem outlines constraints on achieving the efficient participation level in a setting with obscurity costs.

Consider the cutoff equilibrium point for the Vickrey auction v_{vic} which satisfies:

$$v_{vic}G(v_{vic}) - c(v_{vic}) - oc(v_{vic}) = 0 \tag{48}$$

Now consider an auction that redistributes *more* than the revenue to the agents, specifically, one that redistributes $1/n$ times the highest bid amongst the other agents to each of the n bidders; and consider (counterfactually) that it is strategyproof with respect to bids. Here each agent's expected utility from truthful participation is strictly *greater* than we can achieve with a strategyproof, efficient, and no-deficit auction, since redistribution payments always meet or exceed revenue, so we can use this idea as a device to establish a bound on the symmetric participation level that *is* achievable. In any putative cutoff equilibrium v^* of this hypothetical auction, the expected redistribution payment to any agent contingent on participation is the expected value of: $1/n$ times the maximum bid amongst those of the other $N-1$ agents who have value at least v^*. Denote this quantity $Y(v^*)$. By extension of Lemma 1, we know that if v^* is in fact an equilibrium, the following will hold:

$$Y(v^*) = v^*G(v^*) - c(v^*) - oc(v^*) \tag{49}$$

Since we know that the Vickrey auction yields under-participation in an increasing obscurity costs setting (Theorem 8), and since v^* is a lower-bound on the symmetric equilibrium cutoff level we can achieve with a no-deficit and incentive compatible mechanism, if the slope of the efficiency curve is negative at every point between v^* and v_{vic}, then even this ideal (and unattainable) auction that redistributes more than the Vickrey auction revenue will not yield an efficient participation level, and thus neither will any auction that redistributes an intermediary amount. As we show with the following theorem, this condition is satisfied if the slope of the obscurity cost function is greater than the hazard rate times Y at each v in this range.

Theorem 9. *In an arbitrary setting with strictly increasing obscurity cost function and values with continuous density function with support* $[0, 1]$*, if for all values* $v \in [v^*, v_{vic}]$*,*[9] *the slope of the obscurity cost function at* v *is greater than the hazard rate at* v *times* $Y(v)$*, no efficient dominant strategy incentive compatible auction achieves an efficient participation level in a symmetric cutoff equilibrium without running a deficit in expectation.*

Proof. See Appendix. □

The theorem says that efficiency is unattainable if $oc'(v_0) - \frac{f(v_0)}{1-F(v_0)} \cdot Y(v_0) > 0$, $\forall v \in [v^*, v_{vic}]$. As a demonstration, consider a cost function $c(v) = 0.15 + 0.1v$ and obscurity cost function $oc(v_0) = 0.4v_0$, and assume there are 4 agents with uniformly distributed values. Here $v^* = 0.853$ and $v_{vic} = 0.876$. Noting that $Y(0.853) = 0.042$ and

[9] Note that v_{vic} and v^* are defined as the solutions to Eqs. (48) and (49), respectively.

$Y(0.876) = 0.035$, the theorem tells us that efficiency is impossible since $0.4 - \frac{1}{1-0.853} \cdot 0.042 = 0.1143 > 0$, $0.4 - \frac{1}{1-0.876} \cdot 0.035 = 0.1145 > 0$, and also—though ¿ we don't demonstrate this here—all intermediary values exceed 0.

4 Beyond linear cost functions

The redistribution payments made by RM_ρ impose a need for the linear structure of cost functions in proving the equilibrium results of the previous sections. In the case of the Vickrey auction, we can go well beyond linear costs and still demonstrate existence of a unique symmetric cutoff equilibrium.

Definition 5. *(*REGULAR COST FUNCTION WITH NEGATIVE PROPORTIONAL GRADIENT*) A cost function $c : [0,1] \to [0,1]$ is termed regular if and only if $\forall v \in [0,1]$, $c(v) \in (0,1)$, and has negative proportional gradient if and only if, $\forall v \in [0,1]$:*

$$\left(\frac{c(v)}{v}\right)' \leq 0 \tag{50}$$

Theorem 10. *Consider the Vickrey auction with arbitrary regular entry costs with negative proportional gradient. Consider the symmetric strategy profile where each agent enters and is truthful if his value is at least v_0 and otherwise does not enter, where v_0 is defined to solve $v_0G(v_0) = c(v_0)$. This is a Bayes-Nash equilibrium for any such $v_0 \in [0,1]$, and is the only symmetric cutoff equilibrium if $c(v)$ and $vG(v)$ are distinct functions of v and each is either concave or convex.*

Proof. See Appendix. □

A wide array of cost functions have negative proportional gradient. Important subclasses are all decreasing functions and all linear functions, decreasing or increasing.

Proposition 2. *All regular monotonically decreasing cost functions and all regular linear cost functions have negative proportional gradient.*

Proof. Taking the derivative of $\frac{c(v)}{v}$, we can see that negative proportional gradient is equivalent to the following condition:

$$\frac{vc'(v) - c(v)}{v^2} \leq 0, \tag{51}$$

which holds if and only if:

$$vc'(v) \leq c(v) \tag{52}$$

For any decreasing function $c'(v)$ is negative, and then since c is a function from $[0,1]$ to $[0,1]$, the condition holds. If c is a regular linear function, then there exists some $b, k \in \Re$ such that $c(v) = b + kv$, $\forall v \in [0,1]$. Plugging this into Eq. (52), we have:

$$v \cdot (b + kv)' \leq b + kv, \text{ i.e.,} \tag{53}$$

$$kv \leq b + kv \tag{54}$$

This is guaranteed to hold by regularity of c. □

So, for instance, an agent's cost of entry might rise proportionally with his value, e.g., $c(v) = 0.1 + \frac{v}{2}$, or decrease sharply, e.g., $c(v) = 0.9 - v^2$. A unique cutoff equilibrium will obtain for all such functions under the Vickrey auction.

5 Conclusion

In most real-world auction settings, individuals are not captives, simply trying to do their best in a situation they have no choice but to engage with. More often if they participate it is by choice, with the auction providing expected utility that is greater than what they could obtain by spending their time and efforts elsewhere. Modeling the foregoing of this outside option as a "cost" (and there may be other more direct costs associated with participation as well), we find that the form of the costs is central to predicting the circumstances under which agents will participate and the efficiency level that will result.

In this paper we started by extending previous known results, showing that for costs that are an arbitrary linear function of value (rather than simply fixed), a broad space of auctions has a unique symmetric equilibrium where each agent will participate if and only if his value is above some threshold. Within this space, the Vickrey auction is superlative when costs are solely a function of value and do not depend on the participation rate. But another natural cost model takes into account the fact that participating in "obscure" auctions brings additional costs: costs to find the auction, to prepare a bid, to engage in a system lacking the social seal of approval that mass participation brings, etc. In this case the Vickrey auction is inefficient because it yields underparticipation; we demonstrated that redistribution auctions improve efficiency without ever running a deficit.

While we formally considered only a one-shot setting, perhaps some broader lessons for market design can be drawn. Our results can be interpreted to suggest that for "upstart" markets it may be advisable to choose a mechanism that channels much of the social surplus to buyers (e.g., a redistribution auction); as the market matures and obscurity costs decrease, the impact on efficiency of switching to a higher-revenue mechanism will be less dramatic. In this paper we were concerned only with efficiency while meeting a hard budget-balance constraint, but future work may consider the revenue impact of using redistribution to spur extra participation. Although redistribution very explicitly subtracts revenue from the seller, the extra competition that increased participation brings provides a countervailing factor that increases revenue, which may significantly mitigate the overall revenue loss.

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Appendix

Theorem 4. *Consider an arbitrary regular linear entry cost function. Let v_0 be the unique symmetric Bayes-Nash cutoff equilibrium of RM_ρ with no reserve price (where ρ is restricted to choices that ensure such an equilibrium exists and is unique). There exists a positive reserve price leading to increased expected allocation value in a symmetric Bayes-Nash cutoff equilibrium if $F(v_0) < \frac{N-1}{N-3+\frac{2}{\rho}}$.*

Proof. Any cutoff equilibrium for the RM_ρ auction with no reserve price is a value v_0 such that $\pi(v_0, 0) = R_\rho(v_0, 0) + v_0 G(v_0) - c(v_0) = 0$. With reserve price $r > 0$, a cutoff equilibrium is v_0' such that $\pi(v_0', r) = R_\rho(v_0', r) + (v_0' - r)G(v_0') - c(v_0') = 0$.

Since with no reserve price there is a unique cutoff equilibrium, by supposition, $r > 0$ can be chosen sufficiently small such that the equilibrium with reserve price r is also unique. Essentially, r can be chosen sufficiently small such that the slope of $\pi(\cdot, 0)$ is sufficiently undisturbed to ensure that $\pi(\cdot, r) = 0$ also at exactly one point. Then adding reserve price r leads to a *smaller* unique cutoff equilibrium value if $\pi(v_0, r) > \pi(v_0, 0)$, so to prove the theorem it is sufficient to show that $\pi(v_0, r) > \pi(v_0, 0)$ for all $r \in (0, v_0)$. We see that $\pi(v_0, r) - \pi(v_0, 0) =$

$$
\begin{aligned}
&[R_\rho(v_0, r) + (v_0 - r)G(v_0) - c(v_0)] - [R_\rho(v_0, 0) + v_0 G(v_0) - c(v_0)] && (55)\\
&= R_\rho(v_0, r) - R_\rho(v_0, 0) - rG(v_0) && (56)\\
&= \mathcal{P}(N-1, 0, v_0) \cdot \rho r + \mathcal{P}(N-1, 1, v_0) \cdot \frac{\rho}{2} r && (57)\\
&\quad + \sum_{m=2}^{N-1} \mathcal{P}(N-1, m) \cdot \frac{\rho_{m+1}}{m+1} \int_{v_0}^{1} h_m^2(x, v_0) \cdot x \, dx && (58)\\
&\quad - \sum_{m=2}^{N-1} \mathcal{P}(N-1, m) \cdot \frac{\rho_{m+1}}{m+1} \int_{v_0}^{1} h_m^2(x, v_0) \cdot x \, dx \; - \; rF(v_0)^{N-1} && (59)\\
&= r \cdot \left(\mathcal{P}(N-1, 0, v_0) \cdot \rho + \mathcal{P}(N-1, 1, v_0) \cdot \frac{\rho}{2} - F(v_0)^{N-1} \right) && (60)
\end{aligned}
$$

So $\pi(v_0, r) > \pi(v_0, 0)$ if and only if $\mathcal{P}(N-1, 0, v_0) \cdot \rho + \mathcal{P}(N-1, 1, v_0) \cdot \frac{\rho}{2} - F(v_0)^{N-1} > 0$, which can be reduced via the following sequence of equivalent statements:

$$
\binom{N-1}{0}(1 - F(v_0))^0 F(v_0)^{N-1}\rho + \binom{N-1}{1}(1 - F(v_0))^1 F(v_0)^{N-2} \cdot \frac{\rho}{2} > F(v_0)^{N-1} \qquad (61)
$$

$$
F(v_0)^{N-1}\rho + (N-1) \cdot (1 - F(v_0))F(v_0)^{N-2} \cdot \frac{\rho}{2} > F(v_0)^{N-1} \qquad (62)
$$

$$
1 + (N-1) \cdot (1 - F(v_0))F(v_0)^{-1} \cdot \frac{1}{2} > \frac{1}{\rho} \qquad (63)
$$

$$
F(v_0)^{-1} - 1 > \frac{2(\frac{1}{\rho} - 1)}{N-1} \qquad (64)
$$

$$
F(v_0) < \frac{1}{1 + \frac{2}{N-1}(\frac{1}{\rho} - 1)} \qquad (65)
$$

$$
F(v_0) < \frac{N-1}{N-3+\frac{2}{\rho}} \qquad (66)
$$

This completes the proof. □

Lemma 2. *Consider arbitrary $v_0, v' \in \Re$. In the Vickrey auction with entry, if all agents playing participation cutoff level v' yields greater efficiency than all playing $v_0 \neq v'$, then one agent playing strategy v' and all others playing v_0 yields greater efficiency than all playing v_0.*

Proof. Consider a $\underline{v} < v_0$. The expected efficiency gain from all agents moving from v_0 to $\underline{v}$ is:

$$\int_{\underline{v}}^{v_0} Nf(x)F(x)^{N-1}x\,dx - N\int_{\underline{v}}^{v_0} f(x)c(x)\,dx \tag{67}$$

$$< \int_{\underline{v}}^{v_0} Nf(x)F(v_0)^{N-1}x\,dx - N\int_{\underline{v}}^{v_0} f(x)c(x)\,dx \tag{68}$$

$$= N\Big[F(v_0)^{N-1}\int_{\underline{v}}^{v_0} f(x)x\,dx - \int_{\underline{v}}^{v_0} f(x)c(x)\,dx\Big] \tag{69}$$

So if the first expression is positive then the last expression is positive as well. But the last expression is exactly N times the expected efficiency gain from one agent moving from cutoff v_0 to $\underline{v}$ with the others remaining at v_0.

Now consider a $\overline{v} > v_0$. The expected efficiency gain from all agents moving from v_0 to $\overline{v}$ is:

$$-\int_{v_0}^{\overline{v}} Nf(x)F(x)^{N-1}x\,dx + N\int_{v_0}^{\overline{v}} f(x)c(x)\,dx \tag{70}$$

$$< -\int_{v_0}^{\overline{v}} Nf(x)F(v_0)^{N-1}x\,dx + N\int_{v_0}^{v_0} f(x)c(x)\,dx \tag{71}$$

$$= N\Big[-F(v_0)^{N-1}\int_{v_0}^{\overline{v}} f(x)x\,dx + \int_{v_0}^{\overline{v}} f(x)c(x)\,dx\Big] \tag{72}$$

Again, if the first expression is positive then the last expression is too. The last expression is N times the expected efficiency gain for one agent moving to strategy $\overline{v}$. This completes the proof. □

Theorem 6. *Consider arbitrary regular linear entry cost function c, arbitrary obscurity cost function oc, and the RM_ρ auction with reserve price $r \geq 0$. Consider the symmetric strategy profile where each agent enters and is truthful if his value is at least v_0 and otherwise does not enter, where v_0 is defined to solve $R_\rho(v_0, r) + (v_0 - r)G(v_0) = c(v_0) + oc(v_0)$. If such a $v_0 \in (r, 1)$ exists and $c(0) \geq R_\rho(v_0, r)$, this is a Bayes-Nash equilibrium.*

Proof. For linear cost function $c(v) = b + kv$ and obscurity cost function oc, assume a $v_0 > r$ satisfying $R_\rho(v_0, r) + (v_0 - r)G(v_0) = c(v_0) + oc(v_0)$ and $c(0) = b \geq R_\rho(v_0, r)$ exists. Assume all agents other than some i play the posited equilibrium strategy. If i decides to participate after learning his value, truthfulness will then be a dominant strategy, and so the theorem holds if i obtains net utility ≥ 0 if his value is at least v_0 and utility ≤ 0 otherwise.

If i's value equals v_0, if he participates his expected utility is:

$$u_i(v_0) = R_\rho(v_0, r) + (v_0 - r)G(v_0) - c(v_0) - oc(v_0) = 0, \tag{73}$$

If i's value is $\underline{v} < v_0$, if he participates his expected utility is:

$$
\begin{aligned}
u_i(\underline{v}) &= R_\rho(v_0, r) + (\underline{v} - r)G(v_0) - c(\underline{v}) - oc(v_0) && (74)\\
&= R_\rho(v_0, r) + \frac{\underline{v} - r}{v_0 - r}(c(v_0) + oc(v_0) - R_\rho(v_0, r)) - c(\underline{v}) - oc(v_0) && (75)\\
&= \frac{\underline{v} - r}{v_0 - r}(b + kv_0 + oc(v_0) - R_\rho(v_0, r)) - (b + k\underline{v} + oc(v_0) - R_\rho(v_0, r)) && (76)\\
&\leq \frac{\underline{v}}{v_0}(b + kv_0 + oc(v_0) - R_\rho(v_0, r)) - (b + k\underline{v} + oc(v_0) - R_\rho(v_0, r)) && (77)\\
&= \frac{\underline{v}}{v_0}(b - R_\rho(v_0, r)) - (b - R_\rho(v_0, r)) - \Big(1 - \frac{\underline{v}}{v_0}\Big)oc(v_0) \leq 0 && (78)
\end{aligned}
$$

We use the fact that $v_0 - r > 0$ and $b = c(0) \geq R_\rho(v_0, r)$. So i cannot gain by participating for any $\underline{v} < v_0$.

Now consider the case where i's value is $\overline{v} > v_0$. Let $p(y)$ equal y if $y \geq v_0$ and r otherwise. If i participates then his expected utility is:

$$
\begin{aligned}
u_i(\overline{v}) &= R_\rho(v_0, r) + \int_0^{\overline{v}} [\overline{v} - p(y)]g(y)\,dy - c(\overline{v}) - oc(v_0) && (79)\\
&= R_\rho(v_0, r) + \int_0^{v_0} [\overline{v} - p(y)]g(y)\,dy - c(\overline{v}) - oc(v_0) + \int_{v_0}^{\overline{v}} [\overline{v} - p(y)]g(y)\,dy && (80)\\
&= R_\rho(v_0, r) + (\overline{v} - r)G(v_0) - c(\overline{v}) - oc(v_0) + \int_{v_0}^{\overline{v}} [\overline{v} - p(y)]g(y)\,dy && (81)\\
&\geq R_\rho(v_0, r) + (\overline{v} - r)G(v_0) - c(\overline{v}) - oc(v_0) && (82)\\
&= R_\rho(v_0, r) + \frac{\overline{v} - r}{v_0 - r}(c(v_0) + oc(v_0) - R_\rho(v_0, r)) - c(\overline{v}) - oc(v_0) && (83)\\
&= \frac{\overline{v} - r}{v_0 - r}(b + kv_0 + oc(v_0) - R_\rho(v_0, r)) - (b + k\overline{v} + oc(v_0) - R_\rho(v_0, r)) && (84)\\
&\geq \frac{\overline{v}}{v_0}(b + kv_0 + oc(v_0) - R_\rho(v_0, r)) - (b + k\overline{v} + oc(v_0) - R_\rho(v_0, r)) && (85)\\
&= \frac{\overline{v}}{v_0}(b - R_\rho(v_0, r)) - (b - R_\rho(v_0, r)) + \Big(\frac{\overline{v}}{v_0} - 1\Big)oc(v_0) \geq 0 && (86)
\end{aligned}
$$

Again the final inequality makes use of the fact that $b = c(0) \geq R_\rho(v_0, r)$. The move from Eq. (81) to Eq. (82) holds because $p(y) < \overline{v}$ for all $y < \overline{v}$, and Eq. (82) to Eq. (83) holds by the fact that v_0 is defined such that $R_\rho(v_0, r) + (v_0 - r)G(v_0) = c(v_0) + oc(v_0)$. The final two moves hold since $v_0 - r > 0$ and $\overline{v} > v_0$. So i cannot gain from *not* participating when his value is greater than v_0, which completes the proof. □

Theorem 7. *Consider arbitrary regular linear entry cost function c and arbitrary continuous obscurity cost function oc such that $c(v) + oc(v)$ is continuous, monotonic, and concave or convex in v, and, $\forall v, v' \in [0, 1]$, $c(v) + oc(v') \in (0, 1)$. For values with arbitrary continuous density function with support $[0, 1]$, for arbitrary $r \in (0, 1 - c(1) - oc(1))$, there exists strictly positive ρ^* such that RM_ρ with any $\rho \in [0, \rho^*]$ and reserve price r has a unique symmetric Bayes-Nash cutoff equilibrium.*

Proof. First note that the argument of Lemma 1 extends immediately to the case with continuous obscurity costs we consider here. Now, given cost function $c(v) = b + kv$ and obscurity cost function oc, for arbitrary $v \in [0, 1]$, let $\pi(v) = R_\rho(v, r) + (v - r)G(v) -$

$c(v) - oc(v)$. To show existence, it is sufficient to show that there exists a $\rho > 0$ such that $\pi(v_0) = 0$ has a solution $v_0 \in (r, 1)$ such that $R_\rho(v_0, r) \le c(0)$. Let positive ρ be defined arbitrarily to satisfy $R_\rho(v, r) < c(0)$, $\forall v \in [0, 1]$ and $R_\rho(r, r) < c(r) + oc(r)$. Such a ρ exists since $c(v) + oc(v) > 0$, $\forall v \in [0, 1]$ and $R_\rho(\cdot, \cdot)$ decreases to 0 as ρ approaches 0. We have:

$$\pi(r) = R_\rho(r, r) + (r - r)G(r) - c(r) - oc(r) \tag{87}$$
$$= R_\rho(r, r) - c(r) - oc(r) < 0, \tag{88}$$

and:

$$\pi(1) = R_\rho(1, r) + (1 - r)G(1) - c(1) - oc(1) \tag{89}$$
$$= R_\rho(1, r) + 1 - r - c(1) - oc(1) \tag{90}$$
$$= R_\rho(1, r) + (1 - c(1) - oc(1)) - r > 0 \tag{91}$$

The last inequality holds because $c(1) + oc(1) < 1$ and $r < 1 - c(1) - oc(1)$. Now since y is continuous there must therefore be a $v_0 \in (r, 1)$ such that $\pi(v_0) = 0$. In light of Theorem 1, this demonstrates existence of the equilibrium.

To show uniqueness, in light of Lemma 1 (extended to the case of obscurity costs) it is sufficient to show that $\pi(\cdot) = 0$ at exactly one point, v_0. For $\rho = 0$ (the Vickrey auction), since $c(v) + oc(v)$ is monotonic and concave or convex, $c(v) + oc(v)$ intersects $vG(v)$ and thus $\pi(v) = 0$ at exactly one point $v \in [0, 1]$. Then the result will hold also if we choose ρ small enough such that $R(v, r) < c(v) + oc(v)$, $\forall v \in [0, 1]$ and, whenever $(v - r)G(v) - c(v) - oc(v)$ is within $\max_v R_\rho(v, r)$ of 0, the slope of $(v - r)G(v) - c(v) - oc(v)$ is greater than negative one times the slope of $R_\rho(v, r)$ (or 0 if $R_\rho(v, r)$ is increasing). As ρ goes to 0 the minimum slope of $R_\rho(v, r)$ at any $v \in (0, 1)$ goes to 0, which completes the proof. □

Theorem 9. *In a setting with a strictly increasing obscurity cost function and values with arbitrary continuous density function with support* $[0, 1]$*, if for all values* $v \in [v^*, v_{vic}]$,[10] *the slope of the obscurity cost function at* v *is greater than the hazard rate at* v *times* $Y(v)$*, no efficient dominant strategy incentive compatible auction achieves an efficient participation level in a symmetric cutoff equilibrium without running a deficit in expectation.*

Proof. Consider any dominant strategy incentive compatible auction that is efficient in the sense of allocating the item to the highest bidder and does not run a deficit in expectation, and assume that, for all $v \in [0, 1]$, the slope of the obscurity cost function at v is greater than the hazard rate at v times $Y(v)$. We know from [5; 7] that the auction can be described as a Vickrey auction that makes a payment (possibly negative) to each agent that is independent of that agent's bid. If value v_0 is a symmetric cutoff equilibrium value for the auction, by a trivial extension of Lemma 1, letting $X(v_0)$ be the expected Vickrey-modifying payment made to each participant, we have that:

$$X(v_0) = c(v_0) + oc(v_0) - v_0 F(v_0)^{N-1} \tag{92}$$

Following the analysis in the proof of Theorem 8, the expected efficiency (net of costs) of any such auction will be:

$$1 - v_0 F(v_0)^N - \int_{v_0}^{1} F(x)^N \, dx - N \int_{v_0}^{1} f(x)[c(x) + oc(v_0)] \, dx \tag{93}$$

[10]Note that v_{vic} and v^* are defined as the solutions to Eqs. (48) and (49), respectively.

Examining the partial derivative with respect to the cutoff value, we get:

$$\frac{\partial}{\partial v_0}\left(1 - v_0 F(v_0)^N - \int_{v_0}^1 F(x)^N \, dx - N \int_{v_0}^1 f(x)[c(x) + oc(v_0)] \, dx\right) \tag{94}$$

$$= Nf(v_0) \cdot \Big(c(v_0) + oc(v_0) - v_0 F(v_0)^{N-1} - \frac{oc'(v_0)}{f(v_0)} \int_{v_0}^1 f(x) \, dx\Big) \tag{95}$$

$$= Nf(v_0) \cdot \Big(X(v_0) - \frac{oc'(v_0)}{f(v_0)} \int_{v_0}^1 f(x) \, dx\Big) \tag{96}$$

$$= \; - Noc'(v_0) \int_{v_0}^1 f(x) \, dx + Nf(v_0)X(v_0) \tag{97}$$

$$= N\Big(f(v_0)X(v_0) - oc'(v_0)(1 - F(v_0))\Big) \tag{98}$$

This quantity is negative whenever:

$$oc'(v_0) > \frac{f(v_0)}{1 - F(v_0)} X(v_0) \tag{99}$$

Recall that $Y(v_0)$ is defined to be the expected value of: allocation value divided by number of bidders, given that all play cutoff strategy v_0. By the supposition that the auction does not run a deficit in expectation, we know that $X(v_0) \leq Y(v_0)$. Combining this fact with Eq. (99), the derivative at v_0 is negative whenever:

$$oc'(v_0) > \frac{f(v_0)}{1 - F(v_0)} \cdot Y(v_0) \tag{100}$$

This holds by the supposition of the theorem. Negative slope of the derivative at v_0 implies that the auction yields underparticipation and is thus sub-optimal, which completes the proof. □

Theorem 10. *Consider the Vickrey auction with arbitrary regular entry costs with negative proportional gradient. Consider the symmetric strategy profile where each agent enters and is truthful if his value is at least v_0 and otherwise does not enter, where v_0 is defined to solve $v_0 G(v_0) = c(v_0)$. This is a Bayes-Nash equilibrium for any such $v_0 \in [0, 1]$, and is the only symmetric cutoff equilibrium if $c(v)$ and $vG(v)$ are distinct functions of v and each is either concave or convex.*

Proof. For regular cost function c with decreasing proportional gradient, assume a $v_0 > 0$ satisfying $v_0 G(v_0) = c(v_0)$ exists. Decreasing proportional gradient implies that:

$$\forall \underline{v} \leq v_0 \leq \overline{v} \in (0, 1), \quad \frac{c(\underline{v})}{\underline{v}} \geq \frac{c(v_0)}{v_0} \geq \frac{c(\overline{v})}{\overline{v}} \tag{101}$$

Assume all agents other than some $i \in I$ play the posited equilibrium strategy. If i decides to participate after learning his value, truthfulness will then be a dominant strategy, and so the theorem holds if i obtains expected net utility ≥ 0 if his value is at least v_0 and utility ≤ 0 otherwise.

If i's value equals v_0, if he participates his expected utility is:

$$u_i(v_0) = v_0 G(v_0) - c(v_0) = 0, \tag{102}$$

If i's value is $\underline{v} < v_0$, if he participates his expected utility is:

$$u_i(\underline{v}) = \underline{v}G(v_0) - c(\underline{v}) \tag{103}$$
$$= \underline{v}\frac{c(v_0)}{v_0} - c(\underline{v}) \leq 0, \tag{104}$$

where the inequality holds by Eq. (101). So i cannot gain by participating for any $\underline{v} < v_0$.

Now consider the case where i's value is $\overline{v} > v_0$. Let $p(y)$ equal y if $y \geq v_0$ and 0 otherwise. If i participates then his expected utility is:

$$u_i(\overline{v}) = \int_0^{\overline{v}} [\overline{v} - p(y)]g(y)\,dy - c(\overline{v}) \tag{105}$$
$$= \int_0^{v_0} [\overline{v} - p(y)]g(y)\,dy - c(\overline{v}) + \int_{v_0}^{\overline{v}} [\overline{v} - p(y)]g(y)\,dy \tag{106}$$
$$= \overline{v}G(v_0) - c(\overline{v}) + \int_{v_0}^{\overline{v}} [\overline{v} - p(y)]g(y)\,dy \tag{107}$$
$$\geq \overline{v}G(v_0) - c(\overline{v}) \tag{108}$$
$$= \frac{\overline{v}}{v_0}c(v_0) - c(\overline{v}) \geq 0 \tag{109}$$

Again the inequality holds by Eq. (101). So i cannot gain from *not* participating when his value is greater than v_0,

There always exists a v_0 such that $v_0G(v_0) = c(v_0)$ (by regularity of c and support $[0,1]$ of f), and there exists exactly one such v_0 if $c(v)$ and $vG(v)$ are distinct and each is either concave or convex, which implies a unique symmetric equilibrium. □