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A Bound on the Multiplicative Efficiency of Iteration*

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For a convergent sequence $\{x_i\}$ generated by $x_{i+1} = \varphi(x_i, x_{i-1}, ..., x_{i-d+1})$, define the multiplicative efficiency measure E to be $(\log_2 p)/M$, where p is the order of convergence and M is the number of multiplications or divisions needed to compute φ. Then, if φ is any multivariate rational function, $E \leqslant 1$. Since $E = 1$ for the sequence $\{x_i\}$ generated by $x_{i+1} = x_i^2 + x_i - \frac{1}{4}$ with the limit $-1/2$, the bound on E is sharp.

Let P_M denote the maximal order for a sequence generated by an iteration with M multiplications. Then $P_M \leqslant 2^M$ for all positive integers M. Moreover this bound is sharp.

1. INTRODUCTION

For a convergent sequence $\{x_i\}$ generated by $x_{i+1} = \varphi(x_i, x_{i-1}, ..., x_{i-d+1})$, define the multiplicative efficiency measure E to be $(\log_2 p)/M$, where p is the order of convergence and M is the number of multiplications or divisions needed to compute φ. In [1] Paterson showed that if

(i) φ is a rational function,

(ii) $d = 1$,

(iii) $\lim_{i\to\infty} x_i$ is an algebraic number, and

(iv) φ has rational coefficients,

then $E \leqslant 1$. In this note we show $E \leqslant 1$ removing all these restrictions except (i). Since condition (i) is not a restriction for a computer algorithm, this is a very general result. In particular, we shall show that $E = 1$ for the sequence $\{x_i\}$ defined by $x_{i+1} = x_i^2 + x_i - \frac{1}{4}$ with the limit $-1/2$. Hence our bound on E is sharp.

Let P_M denote the maximal order for a sequence generated by an iteration with M multiplications. Since $E \leqslant 1$, it follows that $P_M \leqslant 2^M$ for all positive integer M. Moreover, we shall show that this bound is sharp.

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Paterson used results from approximation by rational numbers to obtain his result, while we use a completely different approach here. With the technique we use here, the case $d = 1$ would be very easy to prove. We show that a rational iteration function which generates a pth order convergent sequence must have degree (degree will be defined below) $\geqslant p$, and therefore must employ at least $\lceil \log_2 p \rceil$ multiplications or divisions (except by constants). Hence, $E = (\log_2 p)/M \leqslant 1$.

The result belongs to analytic computational complexity which deals with optimality theory of analytic processes [2].

2. Notation

We work over the field of real numbers or the field of complex numbers. Let $\{x_i\}$ be any convergent sequence with limit α, and $x_i \neq \alpha$ for all i. Denote $e_i = | x_i - \alpha |$ for all i.

DEFINITION 1 (Order). The sequence $\{x_i\}$ has an order $p > 1$ (or $\{x_i\}$ is a pth order sequence) iff $\lim_{i\to\infty} e_{i+1}/e_i^{p-\epsilon} = 0$ and $\lim_{i\to\infty} e_{i+1}/e_i^{p+\epsilon} \neq 0$ for any $\epsilon > 0$.

From the above definition, it is easy to see that if $\{x_i\}$ has order p, then

(i) $p = \sup\{r \mid \lim_{i\to\infty} e_{i+1}/e_i^r = 0\}$, and

(ii) for any fixed positive integer n, $\{x_{in}\}_{i=0}^{\infty}$ has order p^n.

It should be noted that in our proofs the only properties of order needed are (i) and (ii), although (i) has been used as a definition of order by many people. Definition 1 is the weakest definition on order we have found which enjoys both properties (i) and (ii).

For each number α, we define a class $F(\alpha)$ of convergent sequences with the same limit α as follows: $\{x_i\} \in F(\alpha)$ iff

(i) $x_i \neq \alpha$ for all but finitely many i,

(ii) $\{x_i\}$ has an order $p > 1$,

(iii) $x_{i+1} = \varphi(x_i, x_{i-1}, \ldots, x_{i-d+1})$ for all i, for some multivariate rational expression $\varphi(y_1, y_2, \ldots, y_d)$ of d variables, say,

$$\varphi(y_1, \ldots, y_d) = \frac{\varphi_1(y_1, y_2, \ldots, y_d)}{\varphi_2(y_1, y_2, \ldots, y_d)},$$

where $\varphi_1(y_1, y_2, \ldots, y_d)$ and $\varphi_2(y_1, y_2, \ldots, y_d)$ are two relatively prime multivariate polynomials of d variables $y_1, y_2, \ldots, y_d$. We say that $\{x_i\}$ is generated by the rational iteration φ. For examples of these φ's, see [3].

Consider a sequence in $F(\alpha)$ generated by φ. For the purpose of this note, we assume the cost in generating the sequence to be the number of multiplications or divisions needed to compute φ at each stage. Then it is natural to give the following definition about the measure of efficiency.

DEFINITION 2 (Multiplicative Efficiency). The multiplicative efficiency E of a sequence in $F(\alpha)$ generated by φ is defined to be $(\log_2 p)/M$, where p is the order of the sequence and M is the number of multiplications or divisions needed to compute φ, after doing any preconditioning of coefficients (i.e., preconditioning is not counted).

DEFINITION 3 (Optimality). A sequence in $F(\alpha)$ is called optimal if it has the largest multiplicative efficiency among all sequences in $F(\alpha)$.

From (ii) we can check that a very desirable property holds, namely, for any fixed positive integer n, $\{x_i\}$ and $\{x_{in}\}_{i=0}^{\infty}$ have the same multiplicative efficiency. In fact, this invariance under composition property implies that any efficiency measure must be a strictly increasing function of E [4]. Therefore, as far as optimality is concerned, it makes no difference if E or any other possible efficiency measure is used. For instance, the efficiency measure $p^{1/M}$ will give the same answer in optimality problems as E will since it is a strictly increasing function of E.

DEFINITION 4 (Degree). Let

$$\varphi(y_1, y_2, \dots, y_d) = \frac{\varphi_1(y_1, y_2, \dots, y_d)}{\varphi_2(y_1, y_2, \dots, y_d)}$$

be a multivariate rational expression, where $\varphi_1(y_1, y_2, \dots, y_d)$ and $\varphi_2(y_1, y_2, \dots, y_d)$ are two relatively prime multivariate polynomials. If $D(\varphi_i)$ is the degree of $\varphi_i(y_1, y_2, \dots, y_d)$ for $i = 1, 2$, then the degree $D(\varphi)$ of $\varphi(y_1, y_2, \dots, y_d)$ is defined to be $\max(D(\varphi_1), D(\varphi_2))$.

3. PRELIMINARY LEMMA

For each positive integer d, we define an order ($>$) on the set $I_d = \{(j_1, j_2, \dots, j_d) \mid j_i$ is a nonnegative integer for $i = 1, 2, \dots, d\}$ as follows: for $(j_1, j_2, \dots, j_d), (l_1, l_2, \dots, l_d) \in I_d$, $(j_1, j_2, \dots, j_d) > (l_1, l_2, \dots, l_d)$ iff there exists $k \in \{1, 2, \dots, d\}$ such that $j_k > l_k$ and $j_i = l_i$ for $i < k$.

LEMMA 1. *For any number α, let $\{x_i\}$ be any pth order sequence in $F(\alpha)$ generated by φ, and let $e_i = |x_i - \alpha|$ for all i. Suppose that φ has d variables. Then we have the following*:

(i) *if* $(j_1, j_2, ..., j_d) \in I_d$ *with* $\sum_{i=1}^{d} j_i < p$, *then*

$$\lim_{i\to\infty} \frac{e_i^{p-\epsilon}}{e_i^{j1} e_{i-1}^{j2} \cdots e_{i-d+1}^{jd}} = 0,$$

for $\epsilon > 0$ *and sufficiently small, and*

(ii) *if* $(j_1, j_2, ..., j_d)$, $(l_1, l_2, ..., l_d) \in I_d$ *with* $(j_1, j_2, ..., j_d) > (l_1, l_2, ..., l_d)$ *and* $\sum_{i=1}^{d} l_i < p$, *then*

$$\lim_{i\to\infty} \frac{e_i^{j_1} e_{i-1}^{j_2} \cdots e_{i-d+1}^{j_d}}{e_i^{l1} e_{i-1}^{l2} \cdots e_{i-d+1}^{ld}} = 0.$$

Proof. (i) Choose ϵ such that $0 < \epsilon < p - \sum_{i=1}^{d} j_i$ and $0 < \epsilon < p - 1$. Then

$$\lim_{i\to\infty} \frac{e_i}{e_{i-1}} = \lim_{i\to\infty} \frac{e_i}{e_{i-1}^{p-\epsilon}} \cdot e_{i-1}^{p-\epsilon-1} = 0,$$

and then

$$\lim_{i\to\infty} \frac{e_i}{e_{i-2}} = \lim_{i\to\infty} \frac{e_i}{e_{i-1}} \cdot \frac{e_{i-1}}{e_{i-2}} = 0.$$

In general, $\lim_{i\to\infty} e_i/e_{i-k} = 0$ for any positive integer k. Hence, if $d > 1$,

$$0 \leqslant \lim_{i\to\infty} \frac{e_i^{p-\epsilon}}{e_i^{j_1} \cdots e_{i-d+1}^{j_d}} \leqslant \lim_{i\to\infty} \frac{e_i^{\sum_{i=1}^{d} j_i}}{e_i^{j_1} \cdots e_{i-d+1}^{j_d}} = \lim_{i\to\infty} \left(\frac{e_i}{e_i}\right)^{j_1} \cdots \left(\frac{e_i}{e_{i-d+1}}\right)^{j_d} = 0.$$

(ii) Choose ϵ such that $0 < \epsilon < p - \sum_{i=1}^{d} l_i$. Let

$$Q_i = \frac{e_i^{j_1} e_{i-1}^{j_2} \cdots e_{i-d+1}^{j_d}}{e_i^{l_1} e_{i-1}^{l_2} \cdots e_{i-d+1}^{l_d}}.$$

Suppose that $j_k > l_k$ and $j_i = l_i$ for $i < k$. Then when i is so large that $e_i < 1$, we have

$$\begin{aligned} Q_i &= e_{i-k+1}^{j_k - l_k} \cdot \frac{e_{i-k}^{j_{k+1}} \cdots e_{i-d+1}^{j_d}}{e_{i-k}^{l_{k+1}} \cdots e_{i-d+1}^{l_d}} \\ &\leqslant e_{i-k+1} \cdot \frac{e_{i-k}^{j_{k+1}} \cdots e_{i-d+1}^{j_d}}{e_{i-k}^{l_{k+1}} \cdots e_{i-d+1}^{l_k}} \\ &= \frac{e_{i-k+1}}{e_{i-k}^{p-\epsilon}} \cdot e_{i-k}^{(p-\epsilon+j_{k+1}-l_{k+1})} \cdot \frac{e_{i-k-1}^{j_{k+2}} \cdots e_{i-d+1}^{j_d}}{e_{i-k-1}^{l_{k+2}} \cdots e_{i-d+1}^{l_d}}. \end{aligned}$$

Case 1. $p-\epsilon+j_{k+i}-l_{k+i}\geqslant 1$ for $k+i=k+1,\ldots,d$. Repeating the above procedure, we get

$$Q_i \leqslant \frac{e_{i-k+1}}{e_{i-k}^{p-\epsilon}}\cdot e_{i-k}\cdot\frac{e_{i-k-1}^{j_{k+2}}\cdots e_{i-d+1}^{j_d}}{e_{i-k-1}^{l_{k+2}}\cdots e_{i-d+1}^{l_d}}$$

$$=\frac{e_{i-k+1}}{e_{i-k}^{p-\epsilon}}\cdot\frac{e_{i-k}}{e_{i-k-1}^{p-\epsilon}}\cdot e_{i-k-1}^{(p-\epsilon+j_{k+2}-l_{k+2})}\cdot\frac{e_{i-k-2}^{j_{k+3}}\cdots e_{i-d+1}^{j_d}}{e_{i-k-2}^{l_{k+3}}\cdots e_{i-d+1}^{l_d}}$$

$$\leqslant\cdots\leqslant\frac{e_{i-k+1}}{e_{i-k}^{p-\epsilon}}\cdot\frac{e_{i-k}}{e_{i-k-1}^{p-\epsilon}}\cdot\ldots\cdot\frac{e_{i-d+2}}{e_{i-d+1}^{p-\epsilon}}.$$

Case 2. $p-\epsilon+j_{k+n}-l_{k+n}<1$ and $p-\epsilon+j_{k+i}-l_{k+i}\geqslant 1$ for $k+i=k+1,\ldots,k+n-1$ for some n with $k+n-1<d$. Since $p-\epsilon-l_{k+n}>0$, $j_{k+n}<p-\epsilon+j_{k+n}-l_{k+n}<1$. Hence we must have $j_{k+n}=0$. Consequently, $1>p-\epsilon-l_{k+n}>\sum_{i=1}^{d}l_i-l_{k+n}$. This implies that $l_i=0$ for all i except $i=k+n$. Then

$$Q_i\leqslant\frac{e_{i-k+1}}{e_{i-k}^{p-\epsilon}}\cdot\ldots\cdot\frac{e_{i-k-n+2}}{e_{i-k-n+1}^{p-\epsilon}}\cdot e_{i-k-n+1}^{p-\epsilon+j_{k+n}-l_{k+n}}\cdot e_{i-k-n}^{j_{k+n+1}}\cdot\ldots\cdot e_{i-d+1}^{j_d}.$$

Note that $p-\epsilon+j_{k+n}-l_{k+n}>0$. Therefore, in both cases, $\lim_{i\to\infty}Q_i=0$. ∎

4. Main Result

THEOREM 1. *For any number* α, *let* $\{x_i\}$ *be any pth order sequence generated by* φ. *Then* $D(\varphi)\geqslant p$.

Proof. Write

$$\varphi_1(y_1,y_2,\ldots,y_d)-\alpha\varphi_2(y_1,y_2,\ldots,y_d)=\sum_{(j_1,\ldots,j_d)\in I_d}C(j_1,\ldots,j_d)(y_1-\alpha)^{j_1}\cdots(y_d-\alpha)^{j_d}\tag{1}$$

for constants $C(j_1,\ldots,j_d)$. Suppose that $D(\varphi)<p$. Then $C(j_1,\ldots,j_d)=0$ for all $(j_1,\ldots,j_d)\in I_d$ with $\sum_{i=1}^{d}j_i\geqslant p$. Moreover, we shall use induction to show that $C(j_1,\ldots,j_d)=0$ for all $(j_1,\ldots,j_d)$ with $\sum_{i=1}^{d}j_i<p$. Note that for $\epsilon>0$,

$$0=\lim_{i\to\infty}\frac{|x_{i+1}-\alpha|}{|x_i-\alpha|^{p-\epsilon}}=\lim_{i\to\infty}\frac{|\varphi(x_i,x_{i-1},\ldots,x_{i-d+1})-\alpha|}{|x_i-\alpha|^{p-\epsilon}}.$$

Then, by (1), we have

$$\lim_{i\to\infty}\frac{\left|\sum_{j_1+j_2+\cdots+j_d<p}C(j_1,\ldots,j_d)(x_i-\alpha)^{j_1}\cdots(x_{i-d+1}-\alpha)^{j_d}\right|}{e_i^{p-\epsilon}}=0.\tag{2}$$

Since $\lim_{i\to\infty} e_k = 0$ for $k = i,..., i - d + 1$, from (2) it follows that $C(0,..., 0) = 0$. Suppose that $C(j_1,..., j_d) = 0$ whenever $(j_1,..., j_d) < (l_1,..., l_d)$ for some $(l_1,..., l_d) \in I_d$ with $\sum_{i=1}^{d} l_i < p$. (2) may be written as

$$\lim_{i\to\infty} \frac{\left| \sum_{(j_1,...,j_d)\geqslant(l_1,...,l_d)} C(j_1,..., j_d) \frac{(x_i - \alpha)^{j_1} \cdots (x_{i-d+1} - \alpha)^{j_d}}{e_i^{l_1} \cdots e_{i-d+1}^{l_d}} \right|}{\dfrac{e_i^{p-\epsilon}}{e_i^{l_1} \cdots e_{i-d+1}^{l_d}}} = 0.$$

Using Lemma 1 for sufficiently small ϵ, we must have $C(l_1,..., l_d) = 0$. This completes the induction proof.

Hence, $C(j_1,..., j_d) = 0$ for all $(j_1,..., j_d) \in I_d$. From (1),

$$\varphi_1(y_1,..., y_d) - \alpha\varphi_2(y_1,..., y_d) \equiv 0.$$

Hence, $\varphi(y_1,..., y_d) \equiv \alpha$. This is a contradiction. Hence, $D(\varphi) \geqslant p$. ∎

THEOREM 2. *If* $\varphi(y_1,..., y_d)$ *is a multivariate rational expression and* $\overline{M}$ *is the number of multiplications or divisions (except by constants) needed to compute* $\varphi(y_1,..., y_d)$, *then* $\overline{M} \geqslant \log_2 D(\varphi)$.

Proof. Observe that we compute $\varphi(y_1,..., y_d)$ through a sequence of arithmetic operations. Let $R_i(y_1,..., y_d)$ be the result immediately following the ith multiplication or division (except by constants) for $i = 1, 2,..., \overline{M}$. Let $R_0(y_1,..., y_d) = 1$. Then for $n = 0, 1,..., \overline{M} - 1$, we have either

$$R_{n+1}(y_1,..., y_d) = \left(\sum_{i=0}^{n} M_{i,n+1}R_i(y_1,..., y_d) + A_{n+1}\right) \times \left(\sum_{i=0}^{n} N_{i,n+1}R_i(y_1,..., y_d) + B_{n+1}\right) \tag{3}$$

or

$$R_{n+1}(y_1,..., y_d) = \left(\sum_{i=0}^{n} M_{i,n+1}R_i(y_1,..., y_d) + A_{n+1}\right) \Big/ \left(\sum_{i=0}^{n} N_{i,n+1}R_i(y_1,..., y_d) + B_{n+1}\right), \tag{4}$$

where $M_{i,n+1}$, $N_{i,n+1}$ are numbers and A_{n+1}, B_{n+1} are linear combinations of $y_1,..., y_d$.

We claim that, for $n = 1, 2, \ldots, \overline{M}$, the following is true. For any numbers $k_0, \ldots, k_n$, and any linear combination C of $y_1, \ldots, y_d$, we have

$$\sum_{i=0}^{n} k_i R_i(y_1, \ldots, y_d) + C = \frac{P_n(y_1, \ldots, y_d; k_0, \ldots, k_n, C)}{Q_n(y_1, \ldots, y_d)}, \tag{5}$$

where $P_n(y_1, \ldots, y_d; k_0, \ldots, k_n, C)$ is a multivariate polynomial depending on $k_0, k_1, \ldots, k_n, C$ and $Q_n(y_1, y_2, \ldots, y_d)$ is a multivariate polynomial independent of $k_0, k_1, \ldots, k_n, C$; moreover, both polynomials have degrees $\leqslant 2^n$. We prove it by induction. It is clear that (5) is true for $n = 1$. Suppose that (5) is true for all $n \leqslant N$ for some $N < \overline{M}$. Suppose that (3) is true for $n = N$. Then by (5) for $n = N$, we have

$$\begin{aligned}
&\sum_{i=0}^{N+1} k_i R_i(y_1, \ldots, y_d) + C \\
&\quad = k_{N+1} R_{N+1}(y_1, \ldots, y_d) + \sum_{i=0}^{N} k_i R_i(y_1, \ldots, y_d) + C \\
&\quad = k_{N+1} \left(\sum_{i=0}^{N} M_{i,N+1} R_i(y_1, \ldots, y_d) + A_{N+1} \right) \\
&\qquad \times \left(\sum_{i=0}^{N} N_{i,N+1} R_i(y_1, \ldots, y_d) + B_{N+1} \right) + \sum_{i=0}^{N} k_i R_i(y_1, \ldots, y_d) + C \\
&\quad = \frac{P_{N+1}(y_1, \ldots, y_d; k_0, \ldots, k_N, C)}{Q_{N+1}(y_1, \ldots, y_d)},
\end{aligned}$$

where

$$\begin{aligned}
P_{N+1}(y_1, \ldots, y_d; k_0, \ldots, k_N, C) = {} & k_{N+1} P_N(y_1, \ldots, y_d; M_{0,N+1}, \ldots, M_{N,N+1}, A_{N+1}) \\
& \times P_N(y_1, \ldots, y_d; N_{0,N+1}, \ldots, N_{N,N+1}, B_{N+1}) \\
& + P_N(y_1, \ldots, y_d; k_0, \ldots, k_N, C) Q_N(y_1, \ldots, y_d),
\end{aligned}$$

and

$$Q_{N+1}(y_1, \ldots, y_d) = Q_N(y_1, \ldots, y_d)^2.$$

Then by the induction hypothesis, we have that $\sum_{i=0}^{N+1} k_i R_i(y_1, \ldots, y_d) + C$ has degree $\leqslant 2^{N+1}$.

Similarly, from (4) we also have that

$$\sum_{i=0}^{N+1} k_i R_i(y_1, \ldots, y_d) + C$$

has the form

$$\frac{P_{N+1}(y_1, \ldots, y_d; k_0, \ldots, k_N, C)}{Q_{N+1}(y_1, \ldots, y_d)}$$

with degree $\leqslant 2^{N+1}$ for some $P_{N+1}(y_1, \ldots, y_d; k_0, \ldots, k_N, C)$ and $Q_{N+1}(y_1, \ldots, y_d)$.

Hence, both cases imply that (5) is true for $n = N + 1$. This completes the induction. Therefore, for any $k_0, \ldots, k_n$, and any C, the degree of

$$\sum_{i=0}^{n} k_i R_i + C$$

will not reach $D(\varphi)$ until $n \geqslant \log_2 D(\varphi)$. This implies that $\bar{M} \geqslant \log_2 D(\varphi)$. This completes the proof. ▮

Note that $M \geqslant \bar{M}$, since preconditioning is only performed on constant coefficients. Thus, by Theorems 1 and 2, we have

$$M \geqslant \bar{M} \geqslant \log_2 D(\varphi) \geqslant \log_2 p \tag{6}$$

Therefore, we have the following

MAIN RESULT: $E = (\log_2 p)/M \leqslant 1$

Now consider the sequence generated by $\psi(x) = x^2 + x - \frac{1}{4}$ with the limit $-1/2$. Since $\psi'(-1/2) = 0$ and $\psi''(-1/2) \neq 0$, we can easily show that this sequence has order 2. Obviously $M = 1$ for this sequence. Thus $E = (\log_2 2)/1 = 1$. Similarly, $E = 1$ for the second order sequence generated by $\Gamma(x) = 1/x + x - 1$ with the limit 1. Either example shows that our bound on E is sharp. Moreover, we have the following interesting result.

Let P_M denote the maximal order for a sequence generated by an iteration with M multiplications or divisions. From our main result, we have the following

COROLLARY. *$P_M \leqslant 2^M$ for all positive integer M. Moreover this bound is sharp.*

Proof. Let ψ_M be the composition of ψ with itself M times, where $\psi(x) = x^2 + x - \frac{1}{4}$ as before. Then the sequence generated by ψ_M has order 2^M and ψ_M employs M multiplications. Hence for each M the maximal order is achieved by the sequence generated by ψ_M. ▮

In [1], Paterson defined his efficiency measure $\bar{E}$ as $\bar{E} = (\log_2 p)/\bar{M}$, and showed that $\bar{E} \leqslant 1$ under some restrictions (see Sec. 1 of this note). We note here that (6) implies his result.

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