A Regular Layout for Parallel Adders

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Abstract—With VLSI architecture, the chip area and design regularity represent a better measure of cost than the conventional gate count. We show that addition of $n$-bit binary numbers can be performed on a chip with a regular layout in time proportional to $\log n$ and with area proportional to $n$.

Index Terms—Addition, area-time complexity, carry lookahead, circuit design, combinational logic, models of computation, parallel addition, parallel polynomial evaluation, prefix computation, VLSI.

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I. INTRODUCTION

We are interested in the design of parallel "carry look-ahead" adders suitable for implementation in VLSI architecture. The addition problem has been considered by many other authors. See, for example, [1], [4], [6], [7], [11], [13], and [14]. Much attention has been paid to the tradeoff between time and the number of gates, but little attention has been paid to the problem of connecting the gates in an economical and regular way to minimize chip area and design costs. In this paper we show that a simple and regular design for a parallel adder is possible.

In Section II we briefly describe our computational model. Section III contains a description of the addition problem and shows how it reduces to a carry computation problem. The basis of our method, the reduction of carry computation to a "prefix" computation, is described in Section IV. Although the same idea was used by Ladner and Fischer [8], their results are not directly applicable because they ignored fan-out restrictions and used the gate count rather than area as a complexity measure.

In Section V we use the results of Section IV to give a simple and regular layout for carry computation. Our construction demonstrates that the addition of $n$-bit numbers can be performed in time $O(\log n)$, using area $O(n \log n)$. The implied constants are sufficiently small that the method is quite practical, and it is especially suitable for a pipelined adder. In Section VI we generalize the result of Section V, and show that $n$-bit numbers can be added in time $O(w \log w + w)$, where $w = n/\log n$ gives the result that $n$-bit addition can be performed in time $O(\log n)$ and area $O(n)$.

II. THE COMPUTATIONAL MODEL

Our model is intended to be general, but at the same time realistic enough to apply (at least approximately) to current VLSI technology. We assume the existence of circuit elements or "gates" which compute a logical function of two inputs in constant time. An output signal can be divided ("fanned out") into two signals in constant time. Gates have constant area, and the wires connecting them have constant minimum width (or, equivalently, must be separated by at least some minimal spacing). At most two wires can cross at any point.

We assume that a signal travels along a wire of any length in constant time. This is realistic as propagation delays are limited by line capacitances rather than the velocity of light. A longer wire will generally have a larger capacitance, and thus require a larger driver, but we can neglect the driver area as it typically need not exceed a fixed percentage of the wire area [10].

The computation is assumed to be performed in a convex planar region, with inputs and outputs available on the boundary of the region. Our measure of the cost of a design is the area rather than the number of gates required. This is an important difference between our model and earlier models of Brent [1], Winograd [14], and others. For further details of our model, see [3]; for motivation and discussion of models similar to ours, see [9] and [12]. A feature of our approach is that we strive for regular layouts in order to reduce design and implementation costs. For VLSI, regularity is one of the most important design criteria; so we shall not compromise the regularity of a design for the sake of efficiency. Since "regularity" is difficult to quantify, we have not included it in our theoretical cost measure, although this would be desirable.

III. OUTLINE OF THE GENERAL APPROACH

Let $a_0a_{n-1} \cdots a_1$ and $b_0b_{m-1} \cdots b_1$ be $n$-bit binary numbers with sum $s_{n+1} = s_{n+2} \cdots s_1$. The usual method for addition computes the $s_i$'s by

$c_0 = 0,$

$c_i = (a_i \land b_i) \lor (a_i \land c_{i-1}) \lor (b_i \land c_{i-1}),$

$s_i = a_i \lor b_i \lor c_{i-1}, \quad i = 1, \ldots, n,$

$s_{n+1} = c_n$

where $\land$ means the sum mod 2 and $c_i$ is the carry from bit position $i$.

It is well known that the $c_i$'s can be determined using the following scheme:

\[c_{i+1} = c_i \oplus a_i \oplus b_i,\]

\[s_{i+1} = c_i \lor a_i \lor b_i,\]

where $\oplus$ means the sum mod 2 and $c_0 = 0$.
\[ c_0 = 0, \]
\[ c_i = g_i \lor (p_i \land c_{i-1}) \quad (1) \]

where
\[ g_i = a_i \lor b_i \]
and
\[ p_i = a_i \land b_i \]
for \( i = 1, 2, \ldots, n \). One can view the \( g_i \) and \( p_i \) as the carry generate and carry propagate conditions at bit position \( i \). The relation (1) corresponds to the fact that the carry \( c_i \) is either generated by \( a_i \) and \( b_i \) or propagated from the previous carry \( c_{i-1} \). This is illustrated in Fig. 1.

In Section V we present a regular and area-efficient layout design for computing all the carries in parallel assuming that the \( g_i \)’s and \( p_i \)’s are given. The design of a parallel adder is then straightforward and is illustrated in Fig. 2. Notice that in Fig. 2(b) the bottom rectangle represents the combinational logic that transforms the \( a_i \)’s and \( b_i \)’s into the \( g_i \)’s and \( p_i \)’s. For computing the \( s_i \)’s we use the fact that \( s_i = p_i \land c_{i-1} \) for \( i = 1, \ldots, n \).

IV. REFORMULATION OF THE CARRY CHAIN COMPUTATION

We define an operator “\( \circ \)” as follows:
\[ (g, p) \circ (g', p') = (g \lor (p \land g'), p \land p') \]
for any Boolean variables \( g, p, g', \) and \( p' \).

Lemma 1: Let
\[ (G_i, P_i) = \begin{cases} (g_i, p_i) & \text{if } i = 1, \\ (g_i, p_i) \circ (G_{i-1}, P_{i-1}) & \text{if } 2 \leq i \leq n. \end{cases} \]

Then
\[ c_i = G_i \quad \text{for } i = 1, 2, \ldots, n. \]

Proof: We prove the lemma by induction on \( i \). Since \( c_0 = 0 \), (1) above gives
\[ c_1 = g_1 \lor (p_1 \land 0) = g_1 = G_1 \]
so the result holds for \( i = 1 \). If \( i > 1 \) and \( c_{i-1} = G_{i-1} \), then
\[ (G_i, P_i) = \begin{cases} (g_i, p_i) & \text{if } i = 1, \\ (g_i, p_i) \circ (G_{i-1}, P_{i-1}) & \text{if } 2 \leq i \leq n. \end{cases} \]

Thus
\[ G_i = g_i \lor (p_i \land c_{i-1}) \]
and from (1) we have
\[ G_i = c_i. \]
The result now follows by induction.

\[ \square \]

\[ \begin{array}{c}
\text{Fig. 1. Carry chain.}
\end{array} \]

\[ \begin{array}{c}
\text{Fig. 2. (a) Abstraction of a parallel carry chain computation, and (b) abstraction of a parallel adder based on the design for the carry chain computation.}
\end{array} \]

Lemma 2: The operator “\( \circ \)” is associative.

Proof: For any \( (g_3, p_3), (g_2, p_2), (g_1, p_1) \), we have
\[ ((g_3, p_3) \circ (g_2, p_2)) \circ (g_1, p_1) \]
\[ = (g_3 \lor (p_3 \land g_2), p_3 \land p_2) \lor (g_1, p_1) \]
\[ = (g_3 \lor (p_3 \land g_2) \lor (p_3 \land p_2 \land g_1), p_3 \land p_2 \land p_1) \]
and
\[ (g_3, p_3) \circ ((g_2, p_2) \circ (g_1, p_1)) \]
\[ = (g_3, p_3) \circ (g_2 \lor (p_2 \land g_1), p_2 \land p_1) \]
\[ = (g_3 \lor (p_3 \land g_2 \lor (p_2 \land g_1)), p_3 \land p_2 \land p_1). \]

One can check that the right-hand sides of the above two expressions are equal using the distributivity of “\( \land \)” over “\( \lor \)” (The dual distributive law is not required.)

To compute the \( c_i \)’s it suffices to compute all the \( (G_i, P_i)’s, \) but by Lemmas 1 and 2
\[ (G_i, P_i) = (g_i, p_i) \circ (g_{i-1}, p_{i-1}) \circ \cdots \circ (g_1, p_1) \]
can be evaluated in any order from the given \( g_i’s \) and \( p_i’s \). This is the motivation for the introduction of the operator “\( \circ \)” (Intuitively, \( G_i \) may be regarded as a “block carry generate” condition, and \( P_i \) as a “block carry propagate” condition.)

V. A LAYOUT FOR THE CARRY CHAIN COMPUTATION

Consider first the simpler problem of computing \( (G_i, P_i) \) for \( i = n \) only. Since the operator “\( \circ \)” is associative, \( (G_i, P_i) \) can be computed in the order defined by a binary tree. This is illustrated in Fig. 3 for the case \( n = 16 \). In the figure each black processor performs the function defined by the operator “\( \circ \)” and each white processor simply transmits data. The white and black processors are depicted in Fig. 4. Note that for Fig. 3 each processor is required to produce only one of its two identical outputs, and the units of time are such that one computation by a black processor and propagation of the results takes unit time.

Consider now the general problem of computing the \( (G_i, P_i) \) for all \( 1 \leq i \leq n \). This computation can be performed by using the tree structure of Fig. 3 once more, this time inverted (that is, the root is visited first). We illustrate the computation, for the case \( n = 16 \), in Fig. 5. It is easy to check that at time \( T = 7 \), all the \( (G_i, P_i) \) are computed along the top boundary of the network. As the final outputs, we only keep the \( G_i \) which are the carries \( c_i \). From the layout shown in Fig. 5, we have the following results.
Fig. 3. Computation of \((G_{16}, P_{16})\) using a tree structure.

Fig. 4. (a) White processor, and (b) black processor.

Theorem 3: All the carries in an \(n\)-bit addition can be computed in time proportional to \(\log n\) and in area proportional to \(n \log n, n \geq 2\), and so can the addition.

VI. A PIPELINE SCHEME FOR ADDITION OF LONG INTEGERS

We define the width \(w\) of a parallel adder to be the number of bits it accepts at one time from each operand. For the parallel adder corresponding to the network in Fig. 5, \(w = 16\). We have hitherto assumed that the width of a network is equal to the number \(n\) of bits in each operand. Here we consider the case \(w < n\). We show that this case can be handled efficiently using a pipeline scheme on a network which is a modification of the one depicted in Fig. 5.

For simplicity, assume that \(n\) is divisible by \(w\). One can partition an \(n\)-bit integer into \(n/w\) segments, each consisting of \(w\)-consecutive bits. To illustrate the idea, suppose that \(w = 16\). Then the carry chain computation corresponding to each segment can be done on the network in Fig. 5, and the computations for all the segments can be pipelined, starting from the least significant segment. The results coming out from the top of the network are not the final solutions, though. Results corresponding to the \(i\)th least significant segment \((i > 1)\) have to be modified by applying \((G_{16} - 1, P_{16} - 1)\) on the right using the operator \(\times\). To facilitate this modification, we superimpose another tree structure on the top half of the network, as shown in Fig. 6. Using this additional tree, the contents of the "square" processor (denoted by \(G")\) are sent to all the leaves, which are black processors. The square processor, shown in Fig. 7, is an accumulator which initially has value \((g, p) = (0, 1), \) and successively has values \((g, p) = (G_{16} - 1, P_{16} - 1)\) for \(i = 2, 3, \ldots\). At the time when a particular \((G_{16} - 1, P_{16} - 1)\) reaches the leaves, it is combined with the results just coming out from the old network there. By this pipeline scheme, we have the following result.

**Theorem 4:** Let \(1 \leq w \leq n\). Then all the carries in an \(n\)-bit addition can be computed in time proportional to \((n/w) + \log w\) and in area proportional to \(w \log w + 1\), and so can the addition. When \(w = 1\), the method outlined in this section is essentially the usual serial carry-chain computation. From Theorem 4 we have the following.

**Corollary 1:** The area-time product for \(n\)-bit addition is \(O(n \log w + w \log^2 w + 1)\), which is \(O(n \log^2 n)\) when \(w = n\), and \(O(n \log n)\) when \(w = n \log n\), and \(O(n)\) when \(w\) is a constant.

One can similarly obtain an upper bound on \(AT^\alpha\) (where \(A\) and \(T\) stand for area and time, respectively) for any \(\alpha \geq 0\), and for each \(\alpha\) one can choose \(w\) to minimize the upper bound [2].

VII. SUMMARY AND CONCLUSIONS

The preliminary and final stages of binary addition with our scheme (generation of \((g_i, p_i)\) and computation of \(s_i = p_i \oplus c_{i-1}\) respectively) are straightforward. Figs. 4 and 5 illustrate that the intermediate phase (fast carry computation) is conceptually simple, and the layout illustrated in Fig. 5 is regular. The design of the white processor is trivial, and the black processor is about as complex as a one-bit adder. After these two basic processors are designed, we can simply replicate them and connect their copies in the regular way illustrated in Fig. 5. We conclude that using the approach of this paper, parallel adders with carry lookahead are well-suited for VLSI implementation.

Mead and Conway [10] considered several lookahead schemes, but concluded that "they added a great deal of complexity to the system without much gain in performance." To show that this comment does not apply to our scheme, suppose that the operations \(\times\), \(\times\), \(\times\) and \(\oplus\) take unit time. Table 1 gives the computation time for our scheme and for a straightforward serial scheme, where the \(c_i\) are computed from (1) for various \(n\) (\(n\) is the number of bits in each operand.) For \(n = 2^4\) the general formulas are \(4k\) and \(2n - 1\), respectively.

**Table 1**

<table>
<thead>
<tr>
<th>(n)</th>
<th>(\text{Time (parallel)})</th>
<th>(\text{Time (serial)})</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>12</td>
<td>13</td>
</tr>
<tr>
<td>16</td>
<td>16</td>
<td>21</td>
</tr>
<tr>
<td>32</td>
<td>20</td>
<td>63</td>
</tr>
<tr>
<td>64</td>
<td>24</td>
<td>127</td>
</tr>
</tbody>
</table>
where \( g_i, p_i \) are numbers and the black processor in Fig. 4(b) now computes \( g_{\text{out}} = g_i + p_1 g_j + p_2 g_k \) and \( p_\text{out} = p_1 p_j + p_2 p_k \). Note that when \( p_1 = \ldots = p_n = 1 \) expression (2) corresponds to the polynomial

\[
g_n + g_{n-1}x + \cdots + g_1x^{n-1}.
\]

REFERENCES


