Class Problems, Lecture 2

Problem 1

You need a new staff assistant, and you have n people to interview. You want to hire the best candidate for the position. When you interview a candidate, you can give them a score, with the highest score being the best and no ties being possible. You interview the candidates one by one. Because of your company's hiring practices, after you interview the k-th candidate, you either offer the candidate the job before the next interview, or you lose the chance to ever hire that candidate. We suppose the candidates are interviewed in a random order, chosen uniformly at random from all $n!$ possible orderings.

We consider the following strategy: first, interview m candidates, but reject them all. These candidates give you an idea of how strong the candidates are. After the m-th candidate, hire the first candidate you interview who is better than all of the previous candidates you have interviewed.

1. Let E be the even that we hire the best available assistant, and let E_i be the event that i-th candidate is the best and is hired. Determine $\Pr(E_i)$, and show that

$$\Pr(E) = \frac{m}{n} \sum_{j=m+1}^{n} \frac{1}{j-1}.$$

2. Bound $\sum_{j=m+1}^{n} \frac{1}{j-1}$ to obtain

$$\frac{m}{n}(\ln n - \ln m) \leq \Pr(E) \leq \frac{m}{n}(\ln(n-1) - \ln(m-1)).$$

3. Show that $m(\ln n - \ln m)/n$ is maximized when $m = n/\mathrm{e}$, and explain why this means $\Pr(E) \geq 1/\mathrm{e}$ if we choose the optimal strategy.