Deriving Performance Bounds for ISI Channels using Gallager Codes

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Abstract — **We develop density evolution methods for evaluating the performance of Gallager's low-density parity-check (LDPC) codes over binary intersymbol interference (ISI) channels. In contrast to previous work on memoryless channels, the noise tolerance thresholds for ISI channels depend on the transmitted sequences. The concentration statements are appropriately adjusted to reflect this difference. We compare the thresholds of regular Gallager codes over the $1-D$ partial response channel to the i.i.d. information rate, showing that at high code rates, regular Gallager codes are asymptotically optimal.**

I. Concentration and Density Evolution

We rely on the notation and graph-based framework presented in [1]. Define $p^{(\ell)}(\underline{s})$ as the probability that a branch from a variable node to a check node carries a wrong message when $\underline{s}$ is a the transmitted sequence. Similarly, define $p^{(\ell)}_{\text{i.i.d.}}$ as the probability that the branch carries a wrong message when the transmitted sequence consists of independent identically distributed (i.i.d.) binary symbols. We assume that the code graph is chosen uniformly at random from the ensemble of graphs with degree polynomials $\lambda(x)$ and $\rho(x)$ and that the coset is always chosen such that the transmitted sequence is a codeword. Let $Z^{(\ell)}$ denote the number of edges carrying wrong messages in the ℓ-th round of decoding, let n_e denote the total number of edges in the graph and let n denote the block length. The following theorems hold for an ISI channel and with additive white Gaussian noise (AWGN). The constant γ in these theorems is independent of n, see [1].

Theorem 1 *Let $\underline{s}$ be the transmitted codeword. For an arbitrarily small constant $\varepsilon > 0$, there exists a positive number β independent of ε, such that if $n > \frac{2\gamma}{\varepsilon}$, then*

$$\Pr\left(\left|\frac{Z^{(\ell)}}{n_e} - p^{(\ell)}(\underline{s})\right| \geq \varepsilon\right) \leq 2\mathrm{e}^{-\beta\varepsilon^2 n}. \tag{1}$$

Theorem 2 *Let a random sequence of i.i.d. equiprobable binary symbols be transmitted. For an arbitrary $\varepsilon > 0$, there exists $\beta' > 0$ independent of ε, such that if $n > \frac{2\gamma}{\varepsilon}$, then*

$$\Pr\left(\left|\frac{Z^{(\ell)}}{n_e} - p^{(\ell)}_{\text{i.i.d.}}\right| \geq \varepsilon\right) \leq 4\mathrm{e}^{-\beta'\varepsilon^2 n}. \tag{2}$$

Define the noise standard deviation threshold σ^* (dependent on the degree polynomials $\lambda(x)$ and $\rho(x)$) as the supremum of standard deviations σ for which $\lim_{\ell\to\infty} p^{(\ell)}_{\text{i.i.d.}} = 0$.

[1] Work supported by the NSF under Grant No. CCR-9904458 and by the National Storage Industry Consortium.

[2] Work supported by the NSF under Grant No. CCR-9983832.

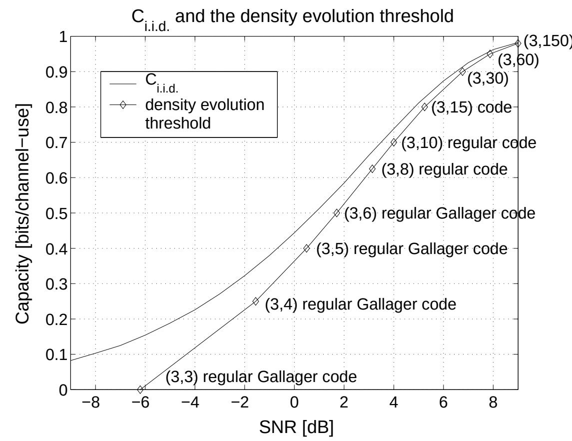

Figure 1: The i.i.d. capacity $C_{\text{i.i.d.}}$ and thresholds for regular Gallager codes with message node degree $L = 3$.

Corollary 2.1 *Let $\underline{m}$ be an information block chosen uniformly at random from $2^k = 2^{rn}$ binary sequences of length k. There exists a coset of a Gallager code with degree polynomials $\lambda(x)$ and $\rho(x)$, such that for any $\sigma < \sigma^*$*

$$\Pr\left(\frac{Z^{(\ell)}}{n_e} \geq 2\varepsilon\right) \leq 4\mathrm{e}^{-\beta'\varepsilon^2 n}. \tag{3}$$

The thresholds are computed by a combination of density evolution methods for variable and check nodes [1] and for trellis sections resembling the Monte Carlo methods for computing thresholds of turbo codes [2].

II. The Dicode Channel

We consider a dicode channel $(1-D)$ with AWGN and regular Gallager codes with variable node degree $L = 3$. Fig. 1 compares the thresholds for the regular Gallager codes to the i.i.d. information rate $C_{\text{i.i.d.}}$ (computed by the Arnold-Loeliger method [3]). Regular Gallager codes asymptotically approach the i.i.d rate at high signal-to-noise ratios (SNRs), or equivalently, high-rate regular Gallager codes achieve rates close to $C_{\text{i.i.d.}}$.

References

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