Review: Dynamic Games

Strategy in an EG is a complete description of play at all information sets.

Nash eq.: No agent can benefit from a unilateral deviation in her strategy given the strategies of others.

Subgame Perfect NE: No agent can benefit from a unilateral deviation from her strategy in any subgame given strategies of other agents.

E.g. $s^* = ((0,2), (nnyy))$ is credible.

"One deviation" only need to check single deviation (in initial action) to verify SPNE.

Subgames are only defined from singleton information sets.
Repeated Games

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>1,1</td>
<td>-1,2</td>
</tr>
<tr>
<td>D</td>
<td>2,-1</td>
<td>0,0</td>
</tr>
</tbody>
</table>

$\Gamma(G)$ w/ discount factor $\delta$

Play $G$ in stages $0, 1, \ldots, \infty$

Histories $H = \{ \emptyset \} \cup \bigcup_{t=0}^{\infty} A^t$

$h^t \in H$ defines joint states played in $0 \ldots t$

$P(h^t) = N \quad \forall h^t \in H$ Strategy $\sigma^t(h^t) \in A^t$

Normalized payoff:

$U_i(\sigma) = (1 - \delta) \mathbb{E}_\sigma \left[ \sum_{t=0}^{\infty} \delta^t \pi_i(\sigma^t(h^t)) \right]$

$\sum_{t=0}^{\infty} \delta^t \nu = \frac{1}{1-\delta}$ Normalization brings units into "per-period" payoff

Main question:

what equilibria exist in $\Gamma(G)$?

what payoff do they support?

are they credible?
Restricted Folk Theorem

\[ V = \text{CH} \{ v : \exists a \in A \text{ with } \pi_i(a) = v \} \]

\[
\begin{array}{c|cc}
1 & 1 & 1,2 \\
2 & 1 & 0,0
\end{array}
\]

Thm. (Friedman '71) Consider \( x^* \in \text{NE}(G) \), then for any \( v \in V \) with \( v_i > \pi_i(x^*) \) \( \forall i \), there is a SPNE of \( G(\delta) \) with payoff \( v \) for some \( \delta < 1 \).

Proof Let a denote stage game strat. s.t. \( \pi(a) = v \).

Strategy \( s^+ \): (c) Play a while history is a ,

(p) if any deviation play \( x^* \) for ever.

SPNE

(a) in (c), need

\[(1 - \delta^t)v_i + \delta^t(1 - \delta) \max_a \left[ \pi_i(a) \right] + s^t e_i < v_i \]

(\(s^t\) \(\delta^t\) \(e_i\))

\(\Rightarrow \) \((1 - \delta) \max_a \left[ \pi_i(a) \right] + \delta e_i < v_i \)

satisfied strictly for \( \delta = 1 \)

(b) in (P), play open-loop strategy

=) NE.
Remark: if no strategy a gives $\Pi(a) = U$
then:

(a) Implement a rational convex combination
\[ \sum \gamma_a \cdot a \quad \text{with} \quad \gamma_a = \frac{\beta_a}{\sum \beta_a} \]
by playing each $a$ for $\beta_a$ steps.

(b) Assume a correlating device $\omega^t$ s.t.
replacing $a$ with $a(\omega)$ gives expected value
Any $\in c$ of feasible $u = u_{SNFE}$ for some $c < 1$

**Strategy**

(c) Play $a$. If single deviation in history $\Rightarrow P^3$

(R') Play $m^3$ for L periods, then $R^3$  
Single deviate $\Rightarrow P^k$

(R') Play $r^3$. Single deviate $\Rightarrow (P^k)$

$\Pi(a) = u$  
$m^3$ minimaxes $j$

$\Pi(r^3) = (v_1 + \varepsilon, v_2 + \varepsilon, \ldots, v_j, \ldots, v_{n+\varepsilon})$

$\Pi(r^1) = (v_1', v_2' + \varepsilon, v_2' + \varepsilon)$  
for $v < v' < v$

$\Pi(r^2) = (v_1' + \varepsilon, v_2', v_3' + \varepsilon)$  
$v' \in int(v)$

$\Pi(r^3) = (v_1' + \varepsilon, v_2' + \varepsilon, v_3')$  
$v' \in int(v)$
Example \( \sigma^t(h) = (D, D) \) where \( h \) is a SPNE

\( \delta^t: \) play \((C, C)\) if history is \((C, C)\) for all \( t' < t\),
or \((D, D)\) otherwise.

\[
\begin{array}{c|ccc}
 & C & D \\
\hline
C & 1, 1 & -1, 2 \\
D & 2, -1 & 0, 0 \\
\end{array}
\]

**SPNE:**
(a) In subgames where history is \((C, C)\), ...
Consider period \( t \). Show

\[
\begin{align*}
1 \cdots 1 & 2 \ 0 \cdots 0 \ vs. \ 1 \cdots 1 \\
\left(\frac{1-\delta^t}{1-\delta}\right) & 1 + \delta^t \cdot 2 + \frac{\delta^t \cdot 0}{1-\delta} > \frac{1}{1-\delta} \cdot 1
\end{align*}
\]

\{in normalized units?\}

\[
1 - \delta^t + (1-\delta) \delta^t \cdot 2 \ & < \ 1 \\
(\Rightarrow \delta \ > \ 1/2)
\]

(b) In subgames where history \( \neq (C, C) \) always

deviate, payoffs \(-1, \ldots, 0\)'s

e) stick with \((D, D)\)

**Remark**

Let \( \{ \alpha^j \}_{j=1}^m \) denote the (Stack) NE of stage game \( G \)

\( \Rightarrow \) "play \( \alpha^j \) in period \( t \)" is a SPNE

for \( j(t) \in \{1, \ldots, m\} \)