On the measurement error of the implicit volatility when \( r \) is misspecified.

We are going to consider a call. The reasoning for a put is similar but the result is different.

First recall that we have:

\[
C(t, S) = SN(d_1) - K \exp(-r\tau)N(d_2)
\]

\[
d_1 = \frac{\ln\left(\frac{S}{K}\right) + (r + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau}
\]

(\( N \) stands for the cdf of a standard normal variable).

Let’s compute some Greeks we will need:

\[
\partial_r C = SN'(d_1) \frac{dd_1}{dr} - K \exp(-r\tau)N'(d_2) \frac{dd_2}{dr} + K\tau \exp(-r\tau)N(d_2)
\]

\[
\partial_r C = (SN'(d_1) - K \exp(-r\tau)N'(d_2)) \frac{1}{\sigma\sqrt{\tau}} + K\tau \exp(-r\tau)N(d_2)
\]

But, and this is important in option pricing, \( SN'(d_1) - K \exp(-r\tau)N'(d_2) = 0 \) so that :

\[
\partial_r C = K\tau \exp(-r\tau)N(d_2) > 0
\]

Now,

\[
\partial_\sigma C = SN'(d_1) \frac{dd_1}{d\sigma} - K \exp(-r\tau)N'(d_2) \frac{dd_2}{d\sigma}
\]

\[
\partial_\sigma C = SN'(d_1) \frac{dd_1}{d\sigma} - K \exp(-r\tau)N'(d_2) \frac{d(d_1 - \sigma\sqrt{\tau})}{d\sigma}
\]

\[
\partial_\sigma C = \frac{dd_1}{d\sigma} \left( SN'(d_1) - K \exp(-r\tau)N'(d_2) \right) + K\sqrt{\tau} \exp(-r\tau)N'(d_2)
\]

\[
\partial_\sigma C = K\sqrt{\tau} \exp(-r\tau)N'(d_2) > 0
\]

The way we compute implicit volatility is by taking \( C \) fixed and calibrating \( \sigma \) so that the value of the BS formula is coherent with the market:

\[
C(r, \sigma_{imp}) = C^{market}
\]

Imagine now that the true value of the interest rate is \( r \) but that we make an error \( dr \) in the value of \( r \). The value we are going to get for \( \sigma_{imp} \) contains an error and is \( \sigma_{imp} + d\sigma_{imp} \) so that:

\[
C(r + dr, \sigma_{imp} + d\sigma_{imp}) = C^{market}
\]

\[
\Rightarrow C(r, \sigma_{imp}) + \partial_r Cdr + \partial_\sigma Cd\sigma_{imp} = C^{market}
\]

\[
\Rightarrow \partial_r Cdr + \partial_\sigma Cd\sigma_{imp} = 0
\]

\[1\text{Indeed, } \frac{N'(d_2)}{N'(d_1)} = \exp(-\frac{1}{2}(d_2^2 - d_1^2)) = \exp(\frac{(d_2 + d_1)(d_1 - d_2)}{2}) = \exp((d_1 - 0.5\sigma\sqrt{\tau})\sigma\sqrt{\tau})
\]

\[= \exp(\sigma\sqrt{\tau}d_1 - 0.5\sigma^2\tau) = \exp(ln\left(\frac{S}{K}\right) + rr) = \frac{S}{K \exp(-rr)}
\]
\[ \Rightarrow \frac{d\sigma_{\text{imp}}}{dr} = -\frac{\partial_C}{\partial \sigma_C} \]

\[ \Rightarrow \frac{d\sigma_{\text{imp}}}{dr} = -\frac{K\tau \exp(-r\tau)N(d_2)}{K\sqrt{\tau} \exp(-r\tau)N'(d_2)} \]

\[ \Rightarrow \frac{d\sigma_{\text{imp}}}{dr} = -\frac{\sqrt{\tau}N(d_2)}{N'(d_2)} \]

What we see here is that, for a call, \( \frac{d\sigma_{\text{imp}}}{dr} \) is negative: if \( r \) is downward biased, \( \sigma_{\text{imp}} \) will be overestimated.

Obviously, to answer the question, one could just say, that \( \frac{d\sigma_{\text{imp}}}{dr} = -\frac{\rho}{\nu} < 0 \) but we want to deal with the importance of the moneyness \( \left( \frac{S}{K} \right) \) in the size of the error.

Since \( d_2 \) is an increasing function of the moneyness,

\[ \text{sgn} \left[ \frac{d}{d \left( \frac{S}{K} \right)} \left( \frac{\sigma_{\text{imp}}}{d_2} \right) \right] = \text{sgn} \left[ \left( \frac{N'(x)}{N(x)} \right) \right] \]

This can be shown to be negative since \( \ln(N(x)) \) is a concave function (for example). Consequently, the error is more important for options with lower values of \( \frac{S}{K} \).

Turning now to the same problem for put options, we have that :

\[ \frac{d\sigma_{\text{imp}}}{dr} = -\frac{\partial_P}{\partial \sigma_P} \]

To know the Greeks for a Put, I usually consider the Call-Put parity which is :

\[ C - P = S - K \exp(-r\tau) \]

This gives \( \partial_C = \partial_P \) and \( \partial_P = \partial_C - K\tau \exp(-r\tau) = -K\tau \exp(-r\tau)N(-d_2) \).

Therefore,

\[ \frac{d\sigma_{\text{imp}}}{dr} = -\frac{K\tau \exp(-r\tau)N(-d_2)}{K\sqrt{\tau} \exp(-r\tau)N'(d_2)} \]

\[ \frac{d\sigma_{\text{imp}}}{dr} = \frac{\sqrt{\tau}N(-d_2)}{N'(d_2)} \]

We can know deduce that, for a put, \( \frac{d\sigma_{\text{imp}}}{dr} \) is positive: if \( r \) is downward biased, \( \sigma_{\text{imp}} \) will be underestimated.

Also, we can see that the result on moneyness is not changed as far as signed are concerned (easy to see because \( N'(d_2) = N'(-d_2) \)).