CS 700 Computational Mechanism Design
Homework 1: Game Theory and Intro Mechanism Design

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Due: Tuesday September 30, 2008, at the beginning of class. You may use any sources that you want, but you must cite the sources that you use. You can also work in pairs– just list the person you work with. **If you already have advanced level economics please come chat with me about an alternative homework.** *Work hard on making the proofs clear, concise, and easy to read.*
Total points: 140

1. (10 pts) (a) In the following strategic-form game, what strategies survive iterated elimination of strictly-dominated strategies?
(b) What are the pure strategy Nash eq.?
(c) Find a non-trivial (support > 1) mixed-strategy NE.

	L	C	R
T	2,0	1,1	4,2
M	3,4	1,2	2,3
B	1,3	0,2	3,0

2. (10 pts) Two agents are bargaining over how to split a dollar. Each simultaneously names the share it would like to have, s_1 and s_2, where $0 \leq s_1, s_2 \leq 1$. If $s_1 + s_2 \leq 1$, then the agents receive the shares they named; if $s_1 + s_2 > 1$, then the agents receive zero. What are the pure strategy Nash eq.?

3. (5 pts) Show that there are no (non-trivial) mixed-strategy Nash eq. (i.e. with support greater than one) in the Prisoners' Dilemma game.

	C	D
C	1,1	-1,2
D	2,-1	0,0

4. (15 pts) *Battle of the Sexes.* Pat and Chris must choose to go for dinner or go to the movies. Both players would rather spend the evening together than apart, but Pat would rather the go for dinner, and Chris would rather they go to the movies.

		Chris	
		Dinner	Movie
Pat	Dinner	2,1	0,0
	Movie	0,0	1,2

(a) Find three Nash equilibria of this game.
(b) Let $(q, 1-q)$ be the mixed strategy in which Pat plays Dinner with prob. q, and let $(r, 1-r)$ be the mixed strategy in which Chris plays Dinner with prob. r. By first determining the best-response correspondences $q^*(r)$ and $r^*(q)$, or otherwise, find the mixed-strategy Nash equilibrium.

5. (20 pts) Iterated elimination of strictly dominated strategies.

   (a) (10 pts) Prove that if strategies, $s^* = (s_1^*, \dots, s_n^*)$, are a Nash eq. in a strategic-form game $G =< N, (S_i), (u_i) >$, then they survive iterated elimination of strictly dominated strategies. **(hint)** *By contradiction, assume that one of the strategies in the Nash eq. is eliminated by iterated elimination of strictly dominated strategies.*

   (b) (10 pts) Prove that if the process of iterated elimination of strictly dominated strategies in game $G =< N, (S_i), (u_i) >$ results in a *unique* strategy profile, $s^* = (s_1^*, \dots, s_n^*)$, that this is a Nash eq. of the game. **(hint)** By contradiction, suppose there exists some agent i for which $s_i \neq s_i^*$ is preferred over s_i^*, and show a contradiction with the fact that s_i was eliminated.

6. (20 pts) Consider a problem in which the outcome space, $\mathcal{O} \subset \mathbb{R}$, and each agent i, with type θ_i, has *single-peaked* preferences, $u_i(\theta_i, o)$ over outcomes. In particular, each agent, i, with type θ_i, has a *peak*, $p_i(\theta_i) \in \mathcal{O}$, such that for any $d', d \in \mathbb{R}$ for which $d' < d \leq p(\theta_i)$ or $p(\theta_i) \leq d < d'$, then $u_i(\theta_i, d) > u_i(\theta_i, d')$ (see p.10–11, M.Jackson "Mechanism Theory" reading).

   (a) (10 pts) Show that the "median selection" mechanism, in which each agent declares its peak and the mechanism selects the median (with a tie break in the case of an even number of agents) is *strategyproof*, and implements a *Pareto Optimal* outcome.

   (b) (5 pts) Let n denote the number of agents. Suppose, in addition, that the mechanism designer can position $n - 1$ "phantom peaks" before the peaks from the agents are received. Why does the median selection mechanism combined with $2n - 1$ peaks remain strategyproof?

   (c) (5 pts) In combination with the phantom peaks, the median selection mechanism can implement a rich variety of outcomes. Describe a method to position the peaks to implement the kth order statistic *of the peaks announced by agents*, for any $1 \leq k \leq n$. (i.e. implement the outcome at the kth largest peak)

7. (10 pts) [Quite easy.] Show that if $f \ : V \rightarrow \mathcal{O}$ is truthfully implementable in dominant strategies when the set of possible types is V_i for $i = 1, \dots, N$ [i.e. the direct revelation mechanism, $\mathcal{M} = (V, f)$, is strategyproof], then when each agent i's set of possible types is $\hat{V}_i \subset V_i$ (for $i = 1, \dots, N$) the social choice function $\hat{f} \ : \hat{V} \rightarrow \mathcal{O}$ satisfying $\hat{f}(v) = f(v)$ for all $v \in \hat{V}$ is truthfully implementable in dominant strategies.

8. (15 pts) Consider the design of a mechanism for a simple double auction with one seller (agent 1), with a single item, and one buyer (agent 2). The outcome of the mechanism defines an *allocation*, (x_1, x_2), where $x_i \in \{0, 1\}$ and $x_i = 1$ if agent i receives the item in the allocation, and *payments* (p_1, p_2) by the agents. Let v_i denote the value of agent i for the item, and suppose quasilinear preferences, so that $u_i(x_i, p_i) = x_i v_i - p_i$ is the utility of agent i for outcome (x_1, x_2, p_1, p_2).

   (a) (10 pts) Specify the Vickrey-Clarke-Groves mechanism for the problem; i.e. define the strategy space, the rule to select the allocation based on agent strategies, and the rule to select the payments based on agent strategies. [Don't just give the general formula: write a succinct description for this application.]

   (b) (5 pts) Provide a simple example to show that the VCG mechanism for a double auction runs at a deficit.

9. (25 pts) Consider a problem with three advertising slots (slots 1, 2 and 3) and N bidders. The probability of a click is $p_1, \gamma p_1, \gamma^2 p_1$ in slots 1, 2 and 3 for $\gamma \in (0, 1)$ (for all advertisers). Bidders $i \in N$ have value $v_i > 0$ for a click, independent of the slot from which a click is received. Derive a description of the VCG mechanism for this problem (i.e., do not just the generic equations.) You can assume that $|N| > 3$. Specifically,

   (a) Describe a (simple!) algorithm for determining how to allocate adverts to slots.

   (b) Provide an equation for determining the per-click price for an advertiser in slot 1, slot 2 and slot 3.

[*Hint: think about the problem as one of allocating a slot not a click, working with expected values. Finally, to determine the per-click price derive a price that is equal, in expectation, to the VCG payment for receiving a slot.*]

(c) Compare the per-click price to that in a "Generalized second price auction" in which advertisers are allocated to slots in order of bid price and the price charged is the smallest bid amount for which an advertiser would retain the same slot. What do you notice?

10. (10 pts) Consider a problem with alternatives A and $|A| \geq 3$. Let $a \succ_i b$, for $a, b \in A$ denote a preference type in which agent i prefers a to b. Suppose $\succ_i \in L_i$ where preference domain L_i allows all possible preference orderings. Explain (from first principles) why the following social-choice function cannot be implemented in a dominant-strategy equilibrium by any mechanism:

$$f(\theta) = \begin{cases} a & \text{, if for all } i \text{ we have } a \succ_i b \text{ for all } b \neq a \\ a^* & \text{, otherwise.} \end{cases}$$

where θ denotes the preferences of agents and a^* is an arbitrary member of A.