CS 700 Computational Mechanism Design
Homework 2: Mechanism Design

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Due: Tuesday October 7, 2008, at the beginning of class. You may use any sources that you want, but you must cite the sources that you use. You can also work in pairs– just list the person you work with. **If you already have advanced level economics please come chat with me about an alternative homework.** *Work hard on making the proofs clear, concise, and easy to read.*
Total points: 135

1. (15 pts) Consider the design of a mechanism for a simple double auction with one seller (agent 1), with a single item, and one buyer (agent 2). The outcome of the mechanism defines an *allocation*, (x_1, x_2), where $x_i \in \{0, 1\}$ and $x_i = 1$ if agent i receives the item in the allocation, and *payments* (p_1, p_2) by the agents. Let v_i denote the value of agent i for the item, and suppose quasilinear preferences, so that $u_i(x_i, p_i) = x_i v_i - p_i$ is the utility of agent i for outcome (x_1, x_2, p_1, p_2).

   (a) (10 pts) Specify the Vickrey-Clarke-Groves mechanism for the problem; i.e. define the strategy space, the rule to select the allocation based on agent strategies, and the rule to select the payments based on agent strategies. [Don't just give the general formula: write a succinct description for this application.]

   (b) (5 pts) Provide a simple example to show that the VCG mechanism for a double auction runs at a deficit.

2. (25 pts) Consider a problem with three advertising slots (slots 1, 2 and 3) and N bidders. The probability of a click is $p_1, \gamma p_1, \gamma^2 p_1$ in slots 1, 2 and 3 for $\gamma \in (0, 1)$ (for all advertisers). Bidders $i \in N$ have value $v_i > 0$ for a click, independent of the slot from which a click is received. Derive a description of the VCG mechanism for this problem (i.e., do not just the generic equations.) You can assume that $|N| > 3$. Specifically,

   (a) (5 pts) Describe a (simple!) algorithm for determining how to allocate adverts to slots.

   (b) (15 pts) Provide an equation for determining the per-click price for an advertiser in slot 1, slot 2 and slot 3.

   [*Hint: think about the problem as one of allocating a slot not a click, working with expected values. Finally, to determine the per-click price derive a price that is equal, in expectation, to the VCG payment for receiving a slot.*]

   (c) (5 pts) Compare the per-click price to that in a "Generalized second price auction" in which advertisers are allocated to slots in order of bid price and the price charged is the smallest bid amount for which an advertiser would retain the same slot. What do you notice?

3. (10 pts) Consider a SCF that is an affine maximizer, $f(v) \in \arg\max_{a \in A}(c_a + \sum_i w_i v_i(a))$ for some fixed agent weights $w_i, \ldots, w_n \geq 0$ and some outcome weights $c_a \in \mathbb{R}$ for every $a \in A$. Show that a Groves mechanism that picks the affine-maximizing alternative and collects payment

$$p_i(v) = h_i(v_{-i}) - \sum_{j \neq i} \frac{w_j}{w_i} v_j(a) - \frac{c_a}{w_i}$$

from every agent i, where h_i is an arbitrary function that does not depend on v_i is strategyproof.

4. (30 pts) Consider a second-price sealed-bid (Vickrey) auction of one item, with bidders, i, with values, $v_i \in [0, \overline{v}]$, and quasilinear preferences, i.e. with $u_i(v_i, p) = v_i - p$, given price p.

   (a) (10 pts) Show that bid $b_i(v_i) = v_i$ for all values, $v_i \in [0, \overline{v}]$, is a weakly dominant strategy for each bidder i. [Prove this from first principles, do not use the fact that the Vickrey auction is a special case of the Groves mechanism].

   (b) (5 pts) Let $b^{(k)}$ denote the kth highest bid. Suppose that the seller introduces a reservation price, $r \in [0, 1]$, such that the item is only sold if $b^{(1)} \geq r$, for price $p = \max[r, b^{(2)}]$. Show that truthful bidding remains a weakly dominant strategy for bidders.

   (c) (5 pts) Consider the special case of an auction with a single bidder, with a Uniformly distributed value $v_1 \sim U(0, \overline{v})$. In addition, suppose that the seller has value, v_0, for the item. Verify that strategies, $r^*(v_0) = (v_0 + \overline{v})/2$, $b_1^*(v_1) = v_1$ form a Bayesian-Nash eq. of this reserve-price Vickrey auction. [Hint: this uses Bayesian-Nash analysis because the auction is analyzed from the seller's perspective as well as the buyer.]

   (d) (5 pts) In fact, $((v_0 + \overline{v})/2, v_1, \ldots, v_n)$, is also the Bayes-Nash eq. of the auction with n bidders, each with value v_i. Assuming, $\overline{v} = 1$ and $v_0 = 0$, determine the seller's expected revenue for the special case of **two** bidders. [**Hint:** construct an expression, by case analysis of the bids received, for the expected revenue to the seller. The fact, $E[v^{(2)} | v^{(2)} \geq 1/2] = 2/3$, where $v^{(2)}$ is the second-highest value across two bidders, will be useful.]

   (e) (5 pts) For this two-bidder case, compare the expected revenue in the reserve-price Vickrey auction to that in the Vickrey auction with no reserve price, and provide an intuitive argument about the effect on allocative-efficiency. [**Hint:** The following fact is helpful: the expected k^{th} highest value among n values independently drawn from the uniform distribution on $[\underline{v}, \overline{v}]$ is $\underline{v} + \left(\frac{n+1-k}{n+1}\right)(\overline{v} - \underline{v})$.]

5. (20 pts) Consider a sealed-bid auction that is defined in terms of:

   (A1) an agent-independent price-function so that for every v_{-i}, every allocation $x_i \in X$ to agent i, and for all v_i, the payment $p_i(v_i, v_{-i}) = p(x_i, v_{-i})$ and depends only on the allocation and the valuations of other agents.

   (A2) an allocation rule $x_i(v) \in X$ that selects an allocation $x_i(v) \in \arg\max_{x \in X}\{v_i(x) - p(x_i, v_{-i})\}$, for every agent i.

   Assume that X contains a "null" allocation, for which $v_i(x) = 0$ for all possible valuations v_i.

   (a) (10 pts) Prove formally that an auction that satisfies A1 and A2 is truthful in a dominant-strategy equilibrium.

   (b) (10 pts) Define the second-price Vickrey auction in these terms (i.e. exhibit an agent-independent price function (A1) and demonstrate that the winner determination rule satisfies (A2).)

   (c) (**extra credit**) Define the Vickrey-Clarke-Groves mechanism for a combinatorial allocation problem in these terms.

6. (25 pts) Consider a VCG mechanism applied to a combinatorial auction with two goods $\{A, B\}$ and bids $(AB, \$2), (A, \$2), (B, \$2)$ from three different bidders.

   (a) (5 pts) What is the outcome of the VCG mechanism in this example?

(b) (5 pts) Use a variation on the example to show that the revenue of the VCG mechanism is not monotonic-increasing in the number of bidders.

(c) (5 pts) Use a variation on the example to show why the VCG mechanism is susceptible to collusion by losers.

(d) (5 pts) Use a variation on the example to show why the VCG mechanism is susceptible to manipulation by false-name bidders (or "sybil-attack"), where one bidder bids under multiple identities.

(e) (5 pts) Do any of these problems occur in the single-item Vickrey auction?

7. (10 pts) Consider a single-parameter domain with known interesting set $W_i \subseteq A$ for each agent i and private value $v_i \in \mathbb{R}$. Fixing v_{-i}, show that the WMON condition implies that if the alternative $f(v_i, v_{-i}) \in W_i$ then $f(v_i', v_{-i}) \in W_i$ for $v_i' \geq v_i$. (See lecture notes slides 23–25).