1 Overview

The authors study the convergence of a balanced-bidding (BB) strategy in a GSP auction on a single keyword for bidders without budget constraints and with complete knowledge of the bids of other bidders. The main result is that BB is shown to converge under asynchronous updates with probability 1.

The model has \( k \) slots with CTR \( \theta_1, \ldots, \theta_k \), \( n \) players with value per click, and defines \( \gamma_j = \frac{\theta_j}{\theta_{j-1}} \) and \( \gamma^* = \max_i \gamma_i \). As \( \gamma^* \to 1 \), adjacent slots have closer CTR.

The purpose of this note is to give a complete proof of Lemma 2, one of the early results for the restricted balanced bidding strategy and also to include the remark about where the \( \log k \) term comes from in the asynchronous analysis.

2 An early Lemma

One lemma they establish is for a restricted balanced bidding (RBB) strategy.

**Lemma 2** At every round \( t \) s.t. \( t > t_1 = 2 \log_{\gamma^*}(1 - \gamma^*)(v_k - v_{k+1})/v_{k+1} \), where \( \gamma^* = \max_{i>0} \theta_i/\theta_{i-1} \), we have

\[
\begin{align*}
b_i &> v_{k+1}, \text{ if } i \leq k \\
b_i & = v_i, \text{ if } i \geq k+1
\end{align*}
\]

**Proof 1** Let \( b \) denote the \( k+1 \)st bid and assume that \( b < v_{k+1} \). Consider any player \( i \in \{1, 2, \ldots, k+1\} \) (a bidder with one of the top \( k+1 \) values). If this player bids \( b'_i \) its value then

\[
(v_{k+1} - b'_i) \leq \gamma^*(v_{k+1} - b)
\]

Otherwise, if in RBB it targets slot \( j \in \{1, \ldots, k\} \) then its bid is

\[
\begin{align*}
b'_i & = (1 - \gamma_j)v_i + \gamma_j p_j \\
& = v_i - \gamma_j(v_j - p_j) \\
& \geq v_{k+1} - \gamma^*(v_{k+1} - b)
\end{align*}
\]

since \( p_j \geq b \) because \( b \) is the \( k+1 \)st highest bid.
Now, we initially have
\[ v_{k+1} - b \leq v_{k+1} \]

Since \( b < v_{k+1} \). And for all \( i \in \{1, \ldots, k+1\} \) we have that Eq. (1) holds irrespective of how such an agent \( i \) bids. From this, we see that the \((k+1)st\) highest bid must satisfy
\[ (v_{k+1} - b') \leq \gamma^*(v_{k+1} - b) \]  \hspace{1cm} (2)
where \( b' \) is this bid in the next round after an update by every agent \( i \) in the top \( k + 1 \). To reach
\[ v_{k+1} - b < (1 - \gamma^*)(v_k - v_{k+1}) \]  \hspace{1cm} (3)
we need at most \( r \) rounds such that
\[ (\gamma^*)^r v_{k+1} \leq (1 - \gamma^*)(v_k - v_{k+1}) \]

and thus
\[ r \geq \log_{\gamma^*} \frac{(v_k - v_{k+1})}{v_{k+1}} \]

Now in round \( r + 1 \), every agent \( i \in \{1, \ldots, k\} \) bids \( v_i > v_{k+1} \) or at least
\[ b'_i = (1 - \gamma_j)v_i + \gamma_jp_j \geq (1 - \gamma^*)v_k + \gamma^*b \\
= b + (1 - \gamma^*)(v_k - b) \]
\[ \geq b + (1 - \gamma^*)(v_k - v_{k+1}) \]  \hspace{1cm} (4)
\[ > v_{k+1} \]  \hspace{1cm} (5)
where Eq. (4) follows since \( b \leq v_{k+1} \) and Eq. (5) follows by the property (Eq. 3) established.

Thus, in round \((r + 2)\), player \( k + 1 \) bids \( v_{k+1} \) and players \( 1 \ldots k \) bid above \( v_{k+1} \). This completes the proof, with \( b_i > v_{k+1} \) for players \( i \leq k \) and \( b_i = v_i \) otherwise.

3 The asynchronous analysis

I had wondered why it takes \( O(n \log k) \) random attempts to activate \( k \) players from \( n \) players. The argument is that one of \( k \) players is picked with probability \( k/n \), then one of the remaining \( k - 1 \) players is picked with probability \( (k - 1)/n \) and so on, so that the expected number of activations required to activate each of \( k \) players is
\[ T(n, k) = \frac{n}{k} + \frac{n}{k-1} + \frac{n}{k-2} + \ldots + \frac{n}{1} \]
\[ = n \left( \frac{1}{k} + \frac{1}{k-1} + \ldots + \frac{1}{1} \right) \]
\[ = O(n \log(k)) \]