Lesson Plan

- VCG mechanism
- Impossibility Results
- Monotonicity, Price-Based Characterization
- Combinatorial auctions
- Single-minded CAs

Reading


Vickrey-Clarke-Groves Mechanism

[Vickrey61, Clarke71, Groves73]

Def. [VCG mechanism] Outcome function

$$f(v) \in \arg \max_{a \in A} \sum_i v_i(a)$$

and payment function

$$p_i(v) = \sum_{j \neq i} v_j(f(v_{-i})) - \sum_{j \neq i} v_j(f(v_i, v_{-i}))$$

This is the marginal externality imposed on the other agents. (In computing $f(v_{-i})$ adjust the feasible alternatives $A$ as necessary; e.g., if $i$ is selling a good then can no longer allocate.)

Thm. The VCG mechanism is strategyproof and efficient.
Each agent’s equilibrium utility is:
\[ \pi_{vick,i} = v_i(a) - [v_i(a) - V(N) + V(N \setminus i)] \]
\[ = V(N) - V(N \setminus i), \]
where \( a = f(v) \), and \( V(N) = \max_{a \in A} \sum_i v_i(a) \) and \( V(N \setminus i) = \max_{a \not\in A} \sum_{j \neq i} v_j(a) \).
The payoff to agent \( i \) is its marginal contribution to the welfare of the system. (⇒ ex post IR, or participation.)

Example: Vickrey auction

Consider the Vickrey auction for a single item. Let \( b_1 \) denote the highest bid, and \( b_2 \) the second-highest bid. Every agent pays marginal externality imposed on rest of system.

For agent 1 (with highest bid), this is:
\[ p_1 = b_2 - 0 = b_2 \]

For all other agents, this is 0 (because agent 1 wins with or without them in the system.) ⇒ VCG mechanism reduces to sealed-bid second price auction.

Aside: History

- Second price auction used by stamp collectors for mail-in auctions since the mid nineteenth century.
- German writer Goethe (1749-1832), wrote on Jan 16, 1797 to his publisher
  “I am inclined to offer Mr. Vieweg... an epic poem... Concerning the royalty we will proceed as follows: I will hand over to Mr. Counsel Bottiger a sealed note which contains my demand, and I wait for what Mr. Vieweg will suggest to offer for my work. If his offer is lower than my demand, then I take my note back, unopened and the negotiation is broken. If, however, his offer is higher, then I will not ask for more than what is written in the note to be opened by Mr. Bottiger.”
- In a letter to Boisseree, dated Jan 12, 1828
  “Let me... name the main evil. It is this: the publisher always knows the profit to himself and his family, whereas the author is totally in the dark.”

Example: Shortest Path.

Biconnected graph, \( G = (N,E) \), cost \( c_l \geq 0 \) per edge \( l \in E \), edges strategic. Assume large value \( V \) to send message.

Goal: route packets along the lowest-cost path from \( S \) to \( T \).

VCG Payment edge \( e \):
\[ p_{vick,e} = -c_l - [(V - d_G) - (V - d_{G/l})] \]
\[ = -c_l - (d_{G/l} - d_G) \]
Example: Multi-unit Auction

$m$ units of a homogeneous item. First, consider the special case in which each bidder demands a single unit. Let $v_i \geq 0$ denote the value of bidder $i$.

**Def.** The VCG auction for this special case sells the items to the $m$ highest bidders, each pays the $m+1$st highest bid price.

$$p_i = b_i - \left( \sum_{j \leq m} b_j - \sum_{j \leq m+1, j \neq i} b_j \right) = b_{m+1}$$

Applications of VCG

- Double auction (one buyer with value $v_b$, one seller with value $v_s$) [Exercise]
- Multi-unit auction with $m$ units to sell of an identical good and $n$ buyers, each with private value $v_i$ for exactly $q_i$ units
- Reverse, single item auction. (One buyer that must buy the item, and $n > 1$ sellers each with private value $v_i$ for the item.)
- Public project of cost $C > 0$ and $n$ agents, each with private value $v_i$ for the project. Want to build the project if and only if in the public interest.

From positive to...

- Median rule: Single-peaked preferences, Pareto optimal, Strategyproof, No payments
- Groves and VCG mechanism: Quasi-linear preferences, Efficient, Strategyproof, payments

... lots of impossibility results as well!

Gibbard-Satterthwaite Impossibility

[Arrow 51, Gibbard & Satterthwaite 73, 75]

Consider SCF, $f(\theta)$, an outcome space $O$, and a type space $\Theta_i$ that allows all possible strict preference orderings over $O$, for every $i$.

**Def. [Dictatorial]** Agent $i$ is a dictator in SCF $f$ if for all $\theta_1, \ldots, \theta_n$, $f(\theta)$ is always agent $i$’s most preferred outcome. SCF $f$ is dictorial if some $i$ is a dictator.

**[Gibbard-Satterthwaite Impossibility]** A SCF $f(\theta)$ onto $O$ where $|O| \geq 3$ is implementable in dominant strategies (strategyproof) if and only if it is dictatorial.

Proof: as a corollary Arrow’s theorem (see p.214-215 in Nisan IMD chapter.)
Implications

- Voting systems: If at least 3 outcomes, then all interesting voting protocols will be manipulable. – the majority vote rule is an obvious counterexample for 2 outcomes

⇒ Consequences for design of computational systems to promote group decision making

What to do?

Two circumventions

- Median-voting rule
- VCG mechanism

Both results circumvent GS by providing additional structure on the type space.

Roberts’ Theorem

- **Def.** A SCF $f_{w,c}$ is an affine maximizer if for some agent weights $w_1, \ldots, w_n \in \mathbb{R}_{\geq 0}$ and some outcome weights $c_a \in \mathbb{R}$ for every $a \in A$, that $f(v) \in \arg \max_{a \in A} (c_a + \sum_i w_i v_i(a))$.

- **Def.** A weighted VCG mechanism selects outcome $f_{w,c}(v)$ and collects payment $p_i(v) = h_i(v-i) - \sum_{j \neq i} (w_j/w_i) v_j(a) - c_a/w_i$ from every agent $i$, where $h_i$ is an arbitrary function that does not depend on $v_i$.

**Claim:** Weighted-VCG mechanism is strategyproof.

- **Thm.** If $|A| \geq 3$, $f$ is onto, $V_i = \mathbb{R}^A$ for every $i$, and $(f, p_1, \ldots, p_n)$ is strategyproof then $f$ is an affine maximizer.

[Proof in Chapter 12 of AGT book]

Centrality of Groves mechanisms

- **Thm.** Roberts’79: if unconstrained valuation domain $V_i = \mathbb{R}^A$ and SP, must be Groves.

- **Thm.** Holmstrom’80: if smoothly connected preferences, efficient and SP, must be Groves mechanism.

- **Thm.** Krishna and Perry’98. VCG maximizes expected revenue across all efficient and interim IR mechanisms, even amongst BNIC mechanisms.
What about budget balance?

[Hurwicz 75; Green & Laffont 79]

**Def.** A mechanism is budget balanced if the total payment collected is exactly equal to the total payments made.

**Thm.** If $|A| \geq 3$, $f$ is onto, $V_i = \mathbb{R}^A$, then no mechanism can be efficient, strategyproof and budget-balanced.

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Does Bayes-Nash help?

A mech. satisfies *interim* individual rationality (IR) if $E_{v_{-i}}[v_i(f(v_i, v_{-i})) - p_i(v_i, v_{-i})] \geq 0$. A mechanism satisfies *ex ante* IR if $E_v[v_i(f(v)) - p_i(v)] \geq 0$.

- **Thm.** (Myerson-Satterthwaite 83) In the bilateral trading problem, no mechanism can be efficient, BNIC, budget-balanced, and satisfy interim IR.
  - Proof via the VCG mechanism
- **Thm.** (Arrow’79, d’Aspremont and Gerard-Varet’79)
  The expected-externality mechanism (VCG but with payments averaged across all possible types of other agents), is EFF, BB, BNIC and satisfies ex ante IR.

**Demonstrates:**
- (a) *ex ante* IR helps (MS vs. d’AGVA);
- (b) BNIC helps (GL vs. d’AGVA).

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Aside: Additional Properties

- **Group strategyproofness.** In any coalition, a manipulation that strictly benefits some member will also strictly hurt another member.
  - assumes no payoffs between agents
- **t-collusion resistant.** No group of agents of size $\leq t$ can useful deviate to improve the total utility to the group.
  - (Goldberg & Hartline ’04; Schummer’01) for $t \geq 2$, the $t$-truthful mechanisms are “posted price” mechanisms
- **Core.** An allocation mechanism implements an outcome in the core if $\sum_{a \in T} v_i(a) - p_i(v) \geq V(T)$ for all $T \subseteq N$, where $a = f(v)$. i.e., no coalition of agents wants to “break away” and not participate.

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The Big Question

- What social choice functions other than linear affine functions can be implemented in dominant strategy in domains with structure on agent preferences?

Why? Computational tractability, and just to have more flexibility in what can be implemented.

Note: Robert’s requires $V_i = \mathbb{R}^A$; and this assumption of unrestricted preferences is not without limitation.
Warm-up: Understanding IC

Consider again a setting without money and agents with preference orderings $\prec$, so that $a \prec b$ if $b$ is preferred to $a$. Let $\prec_a = (\prec_{a_1}, \ldots, \prec_{a_n})$.

**Def.** A SCF $f$ is monotone if $f(\prec) = a \neq a'$ implies $f(\prec_a', \prec_{-i})$ implies that $a' \prec_a a$ and $a \prec_a' a'$.

**Thm.** A SCF is strategyproof if and only if it is monotone.

**Proof.** ($\Rightarrow$) For any change in alternative, the new alternative $a'$ is less preferred than the current alternative $a$. ($\Rightarrow$) Fix some $\prec_{-i}$ and $\prec_i$ and $\prec_i'$ for which $f(\prec) = a \neq a' = f(\prec_i', \prec_{-i})$. For strategyproofness, must have (i) type $\prec_i$ pretending to be $\prec_i'$; need $a' \prec_i a$ (ii) type $\prec_i'$ pretending to be $\prec_i$; need $a \prec_i' a'$

Price-Based Characterization

**Thm.** A mechanism is strategyproof if and only if:

(i) [agent-independent prices] for every $v_{-i}$, there exist prices $p_a \in \mathbb{R}$ for every $a \in A$ s.t. for all $v_i$ with $f(v_i, v_{-i}) = a$, we have $p(v_i, v_{-i}) = p_a$

(ii) [maximizes] for every $v_i$, $f(v_i, v_{-i}) \in \arg \max_a (v_i(a) - p_a)$, where quantification on alternatives in range of $f(v_i, v_{-i})$.

**Proof.** ($\Rightarrow$) By construction: mechanism maximizes agent’s utility against agent-independent prices when report true valuation.

($\Rightarrow$) (i) If $f(v_i, v_{-i}) = f(v_i', v_{-i})$ but $p_i(v_i, v_{-i}) > p_i(v_i', v_{-i})$ then $v_i$ should report $v_i'$.

(ii) If $f(v_i, v_{-i}) \neq \arg \max_a (v_i(a) - p_a)$ then should report $v_i'$ for which $a' = f(v_i', v_{-i}) \in \arg \max_a (v_i(a) - p_a)$.

Weak-Monotonicity

Rochet’87, Saks & Yu’05

Previous characterization is on both SCF and payment rule. What about a condition just on SCF?

**Def.** A SCF $f$ satisfies weak monotonicity (WMON) if for all $i$, all $v_{-i}$, we have $f(v_i, v_{-i}) = a \neq b = f(v_i', v_{-i})$ implies that $v_i(a) - v_i(b) \geq v_i'(a) - v_i'(b)$.

**Thm.** If mechanism $(f, p)$ is strategyproof then $f$ satisfies WMON. If domains $V_i$ are convex sets then every $f$ that satisfies WMON is implementable in a strategyproof mechanism.

**Proof.** ($\Rightarrow$) Fix $v_{-i}$. Consider $f(v_i, v_{-i}) = a \neq b = f(v_i', v_{-i})$. For SP, need

1. $v_i(a) - p_i(v_i, v_{-i}) \geq v_i(b) - p_i(v_i', v_{-i})$
2. $v_i'(b) - p_i(v_i', v_{-i}) \geq v_i'(a) - p_i(v_i, v_{-i})$
3. $v_i(a) + v_i'(b) \geq v_i(b) + v_i'(a)$

$\Rightarrow v_i(a) - v_i(b) \geq v_i'(a) - v_i'(b)$

Implications of WMON result

- WMON is a local condition for each player separately: not immediately clear which functions (or algorithms) satisfy this
- WMON $\Rightarrow$ affine maximizer for unrestricted preference domains
- Many interesting preference domains are not Convex (see Constantin & Parkes, 2005)
- Payments that make a WMON function strategyproof can be computed using graph-theoretic methods (see Gui et al., 2004)
**Single-Parameter Domains**

**Examples:** job scheduling (cost per unit delay); path on network with unknown value.

Known interesting set $W_i \subseteq A$, unknown value $v_i$ for $a \in W_i$. WMON here implies that if $f(v_i, v_{-i}) \in W_i$, then $f(v_i', v_{-i}) \in W_i$ for all $v_i' \geq v_i$. [Exercise]

Define **critical value** $c_i(v_{-i})$ as minimal $v_i$ for which $f_i(v_i, v_{-i}) \in W_i$.

**Thm.** A normalized mechanism $(f, p)$ on a single-parameter domain is strategyproof if and only if

1. $f$ is monotone in every $v_i$
2. every winning bid pays the critical value

Note: allows for more than affine maximizers; e.g.,

$$\arg \max_a \sum_i v_i(a)^2 \text{ or } \arg \max_a \min_i v_i(a).$$

**Example: Combinatorial Auction**

$G$ items, values $v_i(L)$ for $L \subseteq G$. Allocation $L = (L_1, \ldots, L_n)$ is feasible if $L_i \cap L_j = \emptyset$ for all $i, j$.

Efficient: maximize $\sum_i v_i(S_i)$.

**Examples:** course registration; take-off/landing; logistics; bus routes, etc.

- VCG mechanism:
  - collect reported valuations
  - solve winner determination for main economy and each marginal economy
  - collect marginal externality as payment

**Bidding Languages**

Don’t want a representation in which values for all $2^m$ bundles are transmitted.

Should be expressive (“can express succinctly naturally occurring valuations”) and simple.

E.g., XOR language: $(\{a, b\}, 3) XOR(\{c, d\}, 5)$

OR language: $(\{a, b\}, 3) OR(\{c, d\}, 5)$

OR* language: $(\{a, b, c_1\}, 3) OR(\{c, d, c_2\}, 5)$

(see p.279-283 of CAS reading)

In OR* language, winner-determination problem is maximal weighted set-packing and NP-hard. (Also inapproximable to better than $O(m^{1/2})$ for $m$ goods.)

**Single-Minded CAs**

- Single-minded bidders: each bidder has a single interesting bundle $L_i \subseteq G$, and a value $v_i$. (Still NP-hard).

- An algorithm is a $c$-approximation algorithm if $\sum_i v_i(x^*) / \sum_i v_i(x^{alg}) \leq c$ for all inputs, where $x^*$ is the optimal solution.

- Consider a greedy approximation algorithm: sort bids by some criterion, then take bids in order if not in conflict.
  - e.g. scheme with norm $v_i / |L|^{1/2}$ approximates within factor of $m^{1/2}$ for $m$ goods.

- But VCG-based mechanism is not truthful!
  - Consider an example with bidders Red (10, A); Green (19, AB); Blue (8, B).
All is not lost!

(Lehmann et al. 1999)

**Thm.** A normalized mechanism for single-minded bidders is strategyproof if and only if it satisfies

(i) **Monotonicity:** if win with bid \((L_i, v_i)\) then win for any bid \(v'_i > v_i\) and any \(L'_i \subset L_i\)

(ii) **Critical payment:** pay the minimum value needed to win, i.e. smallest \(v'_i\) such that bid \((L_i, v'_i)\) still wins.

Consider greedy algorithm that sorts by \(v_i/|L_i|\).
Example: Red \((10, A)\); Green \((19, AB)\); Blue \((8, B)\).

*shows that by moving from affine maximization can circumvent computational intractability*

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**Next class**

- Auction valuation models
- Optimal auctions
- Dynamic auctions
- Revenue equivalence
- Proxy agents, closing rules, trust and collusion