Consider \( N = \{1, \ldots, n\} \) agents with valuations \( V^n \in [0, h]^n \) and a seller with value \( v_0 \). There is a distribution function \( F_i(z) = \Pr[v_i \leq z] \) and a density function \( f_i(z) = \frac{d}{dz} F_i(z) \). Seek a direct-revelation mechanism (DRM) with allocation rule \( x : V^n \to [0, 1]^n \) and payment rule \( p : V^n \to \mathbb{R}^n \), where \( x_i(v) \) is the probability that the item is allocated to agent \( i \) and \( p_i(v) \) is the expected payment.

An optimal auction solves the following problem:

\[
\max_{x,p} \int_v \left[ v_0(1 - \sum_j x_j(v)) + \sum_j p_j(v) \right] f(v)dv \\
\text{s.t.} \quad \sum_j x_j(v) \leq 1, \quad x_i(v) \geq 0, \quad \forall i, \forall v \\
\quad \quad U_i(x, p, v_i) \geq 0, \quad \forall i, \forall v_i \\
\quad \quad U_i(x, p, v_i) \geq \int_{V_{-i}} [v_i x_i(v_i, v_{-i}) - p_i(v_i, v_{-i})] f_{-i}(v_{-i})dv_{-i}, \quad \forall i, \forall w_i, \forall v_i
\]

where

\[
U_i(x, p, v_i) = \int_{V_{-i}} (v_i x_i(v) - p_i(v))f_{-i}(v_{-i})dv_{-i}
\]

The road-map of Myerson’s proof is to show that the problem

\[
\max_{x,p} \text{ profit} \\
\text{s.t.} \text{ feasibility} \\
\text{IC} \\
\text{IR}
\]

is equivalent to the problem of

\[
\max_x \text{ expected virtual surplus} \\
\text{s.t.} \text{ feasibility} \\
\text{monotonicity}
\]

which can then be solved in term by an efficient auction designed with respect to virtual surplus under the technical condition of “monotone non-decreasing hazard rate” on the distribution function. Thus, Myerson converts the problem of optimal auction design into one of efficient auction design on a perturbed problem.

*These notes are based on discussions by Nisan and Hartline in “Algorithmic Game Theory” Cambridge University Press.
Lemma 1 Mechanism \( M = (x, p) \) is DSIC (in expectation) if and only if \( x_i(v) \) is monotone non-decreasing and \( p_i(v) = v_i x_i(v) - \int_{z=0}^{v_i} x_i(z, v_{-i}) dz \).

Proof 1 (\( \Leftarrow \)). Fix \( v_{-i} \), and drop this from the notation. For DSIC, we need \( vx(v) - p(v) \geq vx(v') - p(v') \) for every \( v' \). Substituting for \( p(v) \), this requires

\[
\begin{align*}
vx(v) - vx(v) + \int_0^v x(z) dz & \geq vx(v') - v'x(v') + \int_0^{v'} x(z) dz \\
\iff \int_0^v x(z) dz & \geq \int_0^{v'} x(z) dz - (v' - v)x(v')
\end{align*}
\]

If \( v' > v \) then we need \( (v' - v)x(v') \geq \int_0^v x(z) dz \). If \( v > v' \) then we need \( (v - v')x(v') \leq \int_0^v x(z) dz \). In both cases, the condition holds if \( x(v) \) is monotone non-decreasing.

(\( \Rightarrow \)). For monotonicity, we require \( vx(v) - p(v) \geq vx(v') - p(v') \) and \( v'x(v') - p(v') \geq v'x(v) - p(v) \), and adding that \( (v' - v)x(v') \geq (v' - v)x(v) \). If \( v' \geq v \) then we need \( x(v') \geq x(v) \). For the payments, fix \( v_i \) and define \( u(b) = vx(b) - p(b) \). For truthfulness, we need

\[
\frac{\partial u(b)}{\partial b} \bigg|_{b=v} = 0,
\]

so that whatever the value of agent \( i \) (and whatever \( v_i \)) the agent maximizes its utility by bidding \( b = v \). Expanding, we require:

\[
\iff v \frac{\partial x(b)}{\partial b} \bigg|_{b=v} - \frac{\partial p(b)}{\partial b} \bigg|_{b=v} = 0
\]

To emphasize that this must hold for all \( v_i \), substitute \( z = v_i \), for \( 0 = zx_i'(z) - p'_i(z) \), or \( p'_i(z) = zx_i'(z) \). Integrating both sides from \( 0 \) to some constant \( b_i \), we have

\[
\begin{align*}
\int_0^{b_i} p'_i(z) dz & = \int_0^{b_i} zx_i'(z) dz \\
\iff p_i(b_i) - p_i(0) & = zx_i(z)|_{0}^{b_i} - \int_0^{b_i} x_i(z) dz \\
& = b_i x_i(b) - \int_0^{b_i} x_i(z) dz
\end{align*}
\]

Finally, by voluntary participation \( p_i(v) \leq 0 \) and no positive transfers \( p_i(v) \geq 0 \), and we have the pricing rule.

A nice pictorial proof of sufficiency is available by drawing a plot of \( x(v) \) vs. \( v \) (fixing \( v_{-i} \)) and noting that the payment is the area above the curve, profit the area below the curve.

Remark. In the special case of a deterministic allocation rule, the payment is precisely that of the critical value (i.e., the minimal value at which the agent is allocated.)

Definition 1 The virtual valuation of agent \( i \) with valuation \( v_i \) is

\[
\phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}
\]

Definition 2 The virtual surplus of allocation \( x(v) \) is

\[
\sum_i \phi_i(v_i)x_i(v) + v_0(1 - \sum_j x_j(v))
\]
Theorem 1 The expected profit of any truthful mechanism is equal to its expected virtual surplus.

Proof Since the mechanism is truthful, can adopt the lemma and have $x_i(v) \leq v_i x_i(v) - \int_{z=0}^{v_i} x_i(z, v_{-i})dz$. Now fix $v_{-i}$ and consider bidder $i$. Neglecting $i$ from the notation, we now have:

$$E_v[p(v)] = \int_{v=0}^{h} p(v) f(v) dv$$

$$= \int_{v=0}^{h} v x(v) f(v) dv - \int_{v=0}^{h} \int_{z=0}^{v} x(z) f(v) dz dv$$

$$= \int_{v=0}^{h} v x(v) f(v) dv - \int_{z=0}^{h} x(z) \int_{v=z}^{h} f(v) dv dz$$

$$= \int_{v=0}^{h} v x(v) f(v) dv - \int_{z=0}^{h} x(z)(1 - F(z)) dz$$

$$= \int_{v=0}^{h} \left( v - \frac{1 - F(v)}{f(v)} \right) x(v) f(v) dv$$

$$= E_v[\phi(v) x(v)],$$

and thus the agent’s expected payment is exactly its expected virtual valuation. This completes the proof, since virtual surplus for an allocation is the sum of the virtual valuations of each agent and the valuation of the seller when the item is not sold.

Thus, to maximize expected profit the task is to choose an allocation that maximizes expected virtual surplus across all monotonic and feasible allocation rules:

$$\max_x E_v \left[ \sum_i \phi_i(v_i) x_i(v) + v_0(1 - \sum_j x_j(v)) \right]$$

s.t. feasibility

monotonicity

Consider the allocation rule $x$ that maximizes virtual surplus; i.e., $x(v)$ sets $x_i^*(v) = 1$ for the bidder $i^*$ that solves $\arg\max \phi_i(v_i)$ when $\max_i \phi_i(v_i) \geq v_0$ (breaking ties at random), or $x_i(v) = 0$ for all $i$ otherwise. This is the

Lemma 2 Virtual surplus maximization is monotone when the virtual valuations $\phi_i(v_i)$ are monotone non-decreasing in $v_i$ for all $i$.

A sufficient condition for $\phi_i(v_{-i})$ to be monotone non-decreasing is the monotone hazard rate condition.

Definition 3 The hazard rate for distribution function $F_i(v)$ is

$$\frac{f(v)}{1 - F(v)}$$

If $\frac{f(v)}{1 - F(v)}$ is monotone non-decreasing then $v - \frac{1 - F(v)}{f(v)}$ is monotone non-decreasing.

Theorem 2 Allocating the item to maximize virtual surplus is optimal and truthful when the distribution has a monotone non-decreasing hazard rate. Collect payment equal to the “critical value,” i.e. the smallest amount the winner could have bid and still been allocated.
Remark. In the symmetric case with $F_i(v) = F(v)$ for all $i$, this is equivalent to Vickrey auction with reserve price $\phi^{-1}(v_0)$, i.e. with a reserve price $r^*$ that solves

$$r^* - \frac{1 - F(r^*)}{f(r^*)} = v_0$$

Remark. To gain some intuition for why this reserve price maximizes revenue, consider selling an item to a single buyer. The seller fixes a price $r$, and the buyer either accepts (and pays) the price or walks away. The optimal price solves

$$\max_r r (1 - F(r)),$$

and by the first-order conditions this is solved by $r^*$ satisfying

$$1 - (r^* f(r^*) + F(r^*)) = 0 \iff r^* - \frac{1 - F(r^*)}{f(r^*)} = 0$$

Remark. Myerson also discusses the idea of “input ironing” in the case in which the allocation rule is non-monotonic. The idea is to construct the “nearest monotonic transformation” of the virtual valuations (in a well-defined sense) and then use those in place of virtual valuations.

Example. Consider $v_i \sim U(0, 100)$. In this case, $\phi_i(v_i) = v_i - \frac{1 - v_i/100}{1/100} = v_i - (100 - v_i) = 2v_i - 100$. Thus, the optimal auction is a Vickrey auction with reserve price $\phi^{-1}(0) = 50$. [The exponential distribution $F(v) = 1 - e^{-\lambda v}$ is another one with monotone virtual valuations, here the hazard rate is constant, $\frac{f(v)}{1 - F(v)} = \frac{\lambda e^{-\lambda v}}{e^{-\lambda v}} = \lambda$.]

Example. A two-step uniform distribution with a higher probability density in the initial section has a decreasing hazard rate because $f(v)$ falls from step 1 to step 2.