Spectral Radius of the Infinite D-Regular Tree

Salil Vadhan, for CS225 lecture on Ramanujan Graphs

December 8, 2016

Definition 1 *Let $H = (V, E)$ be a (countably) infinite undirected graph (i.e. the vertex set V is countably infinite) of bounded degree. Let $\ell_2(V)$ be the set of $g : V \to \mathbb{R}$ such that $\sum_{i \in V} f(i)^2 < \infty$. The* adjacency operator *A_H is the map that takes a function $g \in \ell_2(V)$ to the function $gA_H \in \ell_2(V)$ given by $(gA_H)(j) = \sum_{(i,j) \in E} g(i)$. The* spectral radius *$\rho(H)$ is defined to be $\sup_{g \in \ell_2(V)} \|gA_H\|/\|g\|$.*

Lemma 2 *If λ is a root of the matching polynomial $\mu_G(x)$ of a D-regular graph G, then $|\lambda| \leq \rho(T_D)$, where T_D is the infinite D-regular tree.*

Proof: Pick a vertex u in G, and let $P = P(G, u)$ be the path tree of G starting from u (Definition 3.3 in MSS). Consider the function g that assigns a vertex $(u, v_1, \ldots, v_\ell)$ of P the value $\mu_{G \setminus \{u, v_1, \ldots, v_\ell\}}(\lambda)$. The recurrence relation for $\mu_G(x)$ (Lemma 4.1 in Branden's notes) tells us that:

$$\lambda g(u) = \mu_G(\lambda) + \sum_{v_1 : (u, v_1) \in E} g(u, v_1) = \sum_{v_1 : (u, v_1) \in E} g(u, v_1) = (gA_P)(u),$$

and for every vertex $(u, v_1, \ldots, v_\ell)$ of P with $\ell \geq 1$, we have

$$\lambda g(u, v_1, \ldots, v_\ell) = g(u, v_1, \ldots, v_{\ell-1}) + \sum_{v_{\ell+1} : (v_\ell, v_{\ell+1}) \in E, v_{\ell+1} \notin \{u, v_1, \ldots, v_\ell\}} g(u, v_1, \ldots, v_{\ell+1}) = (gA_P)(u, v_1, \ldots, v_\ell).$$

Thus g is an eigenvector A_P of eigenvalue λ.

Now, consider P as an induced subgraph of T_D, and let $\tilde{g}$ be the extension of g to the vertices of T_D (assigning 0 to all vertices not in P).

Then we have:

$$\begin{aligned} \lambda \|g\| &= \|gA_P\| \\ &\leq \|\tilde{g} A_{T_D}\| \\ &\leq \rho(T_D) \cdot \|g\|. \end{aligned}$$

where the first inequality follows by observing that $\tilde{g} A_{T_D}$ and gA_P are equal on the vertices of P. ■

Lemma 3 $\rho(T_D) \leq 2\sqrt{D-1}$.

Proof: If we think of $T_D = (V, E)$ as an infinite tree in both directions (i.e. not rooted) then for every vertex $i \in V$, we have one parent $p(i)$ and $D-1$ children $c_1(i), \ldots, c_{D-1}(i)$. Then given any $g \in \ell_2(V)$, we can write:

$$
\begin{aligned}
\|gT_D\|^2 &= \sum_{i\in V} (gT_D)(i)^2 \\
&= \sum_{i\in V} (g(p(i)) + g(c_1(i)) + \cdots g(c_{D-1}(i)))^2 \\
&= \sum_{i\in V} \left\langle (g(p(i))/\sqrt{D-1}, g(c_1(i)), \ldots, g(c_{D-1}(i))), (\sqrt{D-1}, 1, \ldots, 1) \right\rangle^2 \\
&\leq \sum_{i\in V} \left(\left\| (g(p(i))/\sqrt{D-1}, g(c_1(i)), \ldots, g(c_{D-1}(i))) \right\| \cdot \left\| (\sqrt{D-1}, 1, \ldots, 1) \right\| \right)^2 \\
&= \sum_{i\in V} \left(g(p(i))^2/(D-1) + g(c_1(i))^2 + \cdots + g(c_{D-1}(i))^2 \right) \cdot 2(D-1) \\
&= \sum_{j\in V} g(j)^2 \cdot ((D-1) \cdot 1/(D-1) + 1) \cdot 2(D-1) \\
&= 4(D-1) \cdot \|g\|^2
\end{aligned}
$$

The penultimate equality follows from observing that each vertex j occurs as the parent of $D-1$ vertices and the child of 1 vertex. This also motivates the choice to scale $g(p(i))$ by a factor of $\sqrt{D-1}$ in the second equality before applying Cauchy-Schwartz, to balance out the contributions coming from parenthood and childhood. ■

For the material on interlacing families, I recommend starting with Sections 4 and 5 in MSS. I prefer to think of the property that two real-rooted polynomials $f_1(x)$ and $f_2(x)$ of degree n have a common interlacing as follows: if we sort the roots $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$ and $\beta_1 \leq \beta_2 \leq \cdots \beta_n$, then both α_1 and β_1 are less than or equal to both α_2 and β_2, which are both less than or equal to α_3 and β_3. (However the α_i and β_i can have any ordering between them.) It is not hard to see that, when both f_1 and f_2 have leading coefficients of the same sign, this is equivalent to every convex combination $pf_1(x) + (1-p)f_2(x)$ being real-rooted (see Lemma 5.5 in Branden's lecture notes). Thus it is natural that the proofs that the family $\{f_s\}$ is an interlacing family involve working with generalizations of real-rootedness ("real-stable polynomials" in MSS' proof, "hyperbolic polynomials" in Branden's proof) and applying various closure properties of these generalized properties. (That's all I can tell you about the proofs!)

Notice the similarity between the proof of MSS' Theorem 4.4 and the Method of Conditional Expectations. Similarly to the Method of Conditional Expectations, it could be implemented in polynomial time and give a mildly explicit construction, if we could compute the univariate polynomials $f_{s_1,\ldots,s_k,1}$ and $f_{s_1,\ldots,s_k,-1}$ at each step, because then we could find their roots in polynomial time and pick the one with the smaller max-root. However, in the construction of MSS2013, these polynomials are NP-hard (even #P-hard) to compute. For example, $f_\emptyset$ is the matching polynomial. Computing this polynomial requires counting the number of matchings of all sizes in G, which is a classic #P-hard problem. The recent mildly explicit construction of Cohen2016 is obtained by showing that for the MSS2015 (Bipartite Ramanujan Graphs of all Sizes) existence proof (which is obtained by taking the union of D perfect matchings on N vertices, rather than a sequence of 2-lifts), the polynomials in the interlacing family can be computed in polynomial time.

More notes and cautions about T_D and the spectra of infinite graphs:

- T_D is the "universal cover" for every D-regular graph (under a topology-like definition of covering maps between graphs). Most of what we have said generalizes if we replace G with any graph and T_D with its universal cover T: If we start with a constant-sized graph G_0 whose nontrivial eigenvalues are bounded by the spectral radius of its universal cover T, by repeatedly taking 2-lifts we obtain an infinite family of graphs whose non-trivial eigenvalues are bounded by the spectral radius of T (and all of whom are covered by T).

- T_D does not have any eigenvectors or eigenvalues in the usual sense. You can find functions $f : V \to \mathbb{R}$ such that $fT_D = \lambda f$ (such as a constant function, with $\lambda = D$), but none of these are in $\ell_2(V)$.

- The supremum in the definition of the $\rho(T_D)$ is not achieved by any function g (if it were, g would be an eigenvector of eigenvalue $2\sqrt{D-1}$). This is possible because the unit ball in $\ell_2(V)$ is not compact when V is infinite.

- In general, the *spectrum* of an infinite graph H is defined as $\{\lambda : \lambda I - A_H \text{ is not invertible}\}$. When H is finite, then this is exactly the set of eigenvalues of A_H. When H is infinite, λ can be in the spectrum because $\lambda I - A_H$ has a nontrivial kernel (so λ is an eigenvalue) *or* because $\lambda I - A_H$ is not surjective. As mentioned earlier, in the case of $H = T_D$, all the eigenvalues come from the latter case.

- It turns out that the spectral radius $\rho(H)$ equals the largest absolute value of elements of the spectrum of H. (In the case of directed graphs, i.e. non-symmetric operators, a slightly more involved definition of spectral radius is needed.) In fact, in the case of T_D the spectral radius *equals* $[-2\sqrt{D-1}, 2\sqrt{D-1}]$.

For more on this, see the survey "Expander Graphs and their Applications" by Hoory, Linial, and Wigderson.