CS 225: Pseudorandomness	**Prof. Salil Vadhan**
Problem Set 4	
Assigned: Thus. Apr. 2, 2009	Due: Wed. Apr. 15, 2009(1 PM)

- Recall that your problem set solutions must be typed. You can email your solutions to `cs225-hw@eecs.harvard.edu`, or turn in it to MD138. You may write formulas or diagrams by hand. Aim for clarity and conciseness in your solutions, emphasizing the main ideas over low-level details.
- If you use LaTeX, please submit both the source (`.tex`) and the compiled file (`.ps`). Name your files `PS4-yourlastname`.
- Starred problems are extra credit.

Problem 5.1. (Min-entropy and Statistical Difference)

1. Prove that for every two random variables X and Y,
$$\Delta(X,Y) = \max_f |\operatorname{E}[f(X)] - \operatorname{E}[f(Y)]| = \frac{1}{2}\cdot|X-Y|_1,$$
where the maximum is over all $[0,1]$-valued functions f. (Hint: first identify the functions f that maximize $|\operatorname{E}[f(X)] - \operatorname{E}[f(Y)]|$.)
2. Suppose that (W,X) are jointly distributed random variables where W takes values in $\{0,1\}^\ell$ and (W,X) is a k-source. Show that for every $\varepsilon > 0$, with probability at least $1-\varepsilon$ over $w \xleftarrow{R} W$, we have $X|_{W=w}$ is a $(k-\ell-\log(1/\varepsilon))$-source.
3. Suppose that X is an $(n-\Delta)$-source taking values in $\{0,1\}^n$, and we let X_1 consist of the first n_1 bits of X and X_2 the remaining $n_2 = n - n_1$ bits. Show that for every $\varepsilon > 0$, (X_1, X_2) is ε-close to some $(n_1-\Delta, n_2-\Delta-\log(1/\varepsilon))$ block source.

Problem 5.2. (Extractors vs. Samplers) One of the problems we have revisited several times is that of randomness-efficient sampling: Given oracle access to a function $f : \{0,1\}^m \to [0,1]$, approximate its average value $\mu(f)$ to within some small additive error. Most of the samplers we have seen work as follows: they choose some n random bits, use these to decide on some D samples $z_1,\ldots,z_D \in \{0,1\}^m$, and output the average of $f(z_1),\ldots,f(z_D)$. We call such a procedure a (δ,ε)-*averaging sampler* if, for any function f, the probability that the sampler's output differs from $\mu(f)$ by more than ε is at most δ. (An example of a non-averaging sampler is the median-of-averages sampler from Problem 1 of Problem Set 3.) In this problem, we will see that averaging samplers are essentially equivalent to extractors.

Given Ext: $\{0,1\}^n \times \{0,1\}^d \to \{0,1\}^m$, we obtain a sampler Smp which chooses $x \xleftarrow{R} \{0,1\}^n$, and uses $\{\mathrm{Ext}(x,y) : y \in \{0,1\}^d\}$ as its $D = 2^d$ samples. Conversely, every sampler Smp using n random bits to produce $D = 2^d$ samples in $\{0,1\}^m$ defines a function Ext: $\{0,1\}^n \times \{0,1\}^d \to \{0,1\}^m$.

1. Prove that if Ext is a $(k-1,\varepsilon)$-extractor, then Smp is a $(2^k/2^n,\varepsilon)$-averaging sampler.

2. Prove that if Smp is a $(2^k/2^n, \varepsilon)$-sampler, then Ext is a $(k + \log(1/\varepsilon), 2\varepsilon)$-extractor.

3. Suppose we are given a constant-error **BPP** algorithm which uses $r = r(n)$ random bits on inputs of length n. Show how, using Part 1 and the extractor of Theorem 8 from Lecture Notes 12, we can reduce its error probability to $2^{-\ell}$ using $O(r) + \ell$ random bits, for any polynomial $\ell = \ell(n)$. (Note that this improves the $r + O(\ell)$ given by expander walks for $\ell \gg r$.) Conclude that every problem in **BPP** has a randomized algorithm which only errs for $2^{q^{0.01}}$ choices of its q random bits!

Problem 5.3. (Encryption and Deterministic Extraction) A (one-time) *encryption scheme* with key length n and message length m consists of an encryption function $\mathrm{Enc}\colon \{0,1\}^n \times \{0,1\}^m \to \{0,1\}^\ell$ and a decryption function $\mathrm{Dec}\colon \{0,1\}^n \times \{0,1\}^\ell \to \{0,1\}^m$ such that $\mathrm{Dec}(k, \mathrm{Enc}(k,u)) = u$ for every $k \in \{0,1\}^n$ and $u \in \{0,1\}^m$. Let K be a random variable taking values in $\{0,1\}^n$. We say that (Enc, Dec) is *(statistically) ε-secure with respect to K* if for every two messages $u, v \in \{0,1\}^m$, we have $\Delta(\mathrm{Enc}(K,u), \mathrm{Enc}(K,v)) \le \varepsilon$. For example, the *one-time pad*, where $n = m = \ell$ and $\mathrm{Enc}(k,u) = k \oplus u = \mathrm{Dec}(k,u)$ is 0-secure (aka perfectly secure) with respect to the uniform distribution $K = U_m$. For a class $\mathcal{C}$ of sources on $\{0,1\}^n$, we say that the encryption scheme (Enc, Dec) is *ε-secure with respect to $\mathcal{C}$* if Enc is ε-secure with respect to every $K \in \mathcal{C}$.

1. Show that if there exists a deterministic ε-extractor $\mathrm{Ext}\colon \{0,1\}^n \to \{0,1\}^m$ for $\mathcal{C}$, then there exists an 2ε-secure encryption scheme with respect to $\mathcal{C}$.

2. Conversely, use the following steps to show that if there exists an ε-secure encryption scheme (Enc, Dec) with respect to $\mathcal{C}$, where $\mathrm{Enc}\colon \{0,1\}^n \times \{0,1\}^m \to \{0,1\}^\ell$, then there exists a deterministic 2ε-extractor $\mathrm{Ext}\colon \{0,1\}^n \to \{0,1\}^{m - 2\log(1/\varepsilon) - O(1)}$ for $\mathcal{C}$, provided $m \ge \log n + 2\log(1/\varepsilon) + O(1)$.

   (a) For each fixed key $k \in \{0,1\}^n$, define a source X_k on $\{0,1\}^\ell$ by $X_k = \mathrm{Enc}(k, U_m)$, and let $\mathcal{C}'$ be the class of all these sources (i.e., $\mathcal{C}' = \{X_k : k \in \{0,1\}^n\}$). Show that there exists a deterministic ε-extractor $\mathrm{Ext}'\colon \{0,1\}^\ell \to \{0,1\}^{m-2\log(1/\varepsilon)-O(1)}$ for $\mathcal{C}'$, provided $m \ge \log n + 2\log(1/\varepsilon) + O(1)$.

   (b) Show that if Ext' is a deterministic ε-extractor for $\mathcal{C}'$ and Enc is ε-secure with respect to $\mathcal{C}$, then $\mathrm{Ext}(k) = \mathrm{Ext}'(\mathrm{Enc}(k, 0^m))$ is a deterministic 2ε-extractor for $\mathcal{C}$.

Thus, a class of sources can be used for secure encryption iff it is deterministically extractable.

Problem 5.5. (The Building-Block Extractor) Assume the condenser stated in Theorem 5.30. Show that for every *constant* $t > 0$ and all positive integers $n \ge k$ and all $\varepsilon > 0$, there is an *explicit* (k, ε)-extractor $\mathrm{Ext}\colon \{0,1\}^n \times \{0,1\}^d \to \{0,1\}^m$ with $m = k/2$ and $d = k/t + O(\log(n/\varepsilon))$. (Hint: convert the source into a block source with blocks of length $k/O(t) + O(\log(n/\varepsilon))$.)

Problem 5.6. (Extracting from Symbol-Fixing Sources*) A generalization of a bit-fixing source is a *symbol-fixing source* X taking values in Σ^n for some alphabet Σ, where a subset of the coordinates of X are fixed and the rest are uniformly distributed and independent elements of Σ. For $\Sigma = \{0,1,2\}$ and $k \in [0,n]$, give an explicit ε-extractor $\mathrm{Ext} : \Sigma^n \to \{0,1\}^m$ for the class of of symbol-fixing sources on Σ^n with min-entropy at least k, with $m = \Omega(k)$ and $\varepsilon = 2^{-\Omega(k)}$. (Hint: use a random walk on a consistently labelled 3-regular expander graph.)