Combinatorial Agency of Threshold Functions

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Abstract. We study the combinatorial agency problem introduced by Babaioff, Feldman and Nisan [5] and resolve some open questions posed in their original paper. Our results include a characterization of the transition behavior for the class of *threshold* functions. This result confirms a conjecture of [5], and generalizes their results for the transition behavior for the OR technology and the AND technology. In addition to establishing a (tight) bound of 2 on the Price of Unaccountability (POU) for the OR technology for the general case of $n > 2$ agents (the initial paper established this for $n = 2$, an extended version establishes a bound of 2.5 for the general case), we establish that the POU is unbounded for all other threshold functions (the initial paper established this only for the case of the AND technology). We also obtain characterization results for certain compositions of anonymous technologies and establish an unbounded POU for these cases.

1 Introduction

The classic principal-agent model of microeconomics considers an agent with unobservable, costly actions, each with a corresponding distribution on outcomes, and a principal with preferences over outcomes [9, 15]. The principal cannot contract on the action directly (e.g. the amount of effort exerted), but only on the final outcome of the project. The main goal is to design contracts, with a payment from the principal to the agent conditioned upon the outcome, in order to maximize the payoff to the principal in equilibrium with a rational, self-interested agent.

The principal-agent model is a classic problem of moral hazard, with agents with potentially misaligned incentives and private actions. A related theory has considered the problem of moral hazard on teams of agents [4, 14, 13]. Much of this work involves a continuous action choice by the agent (e.g., effort) and a continuous outcome function, typically linear or concave in the effort of the agents. Moreover, rather than considering the design of an optimal contract that maximizes the welfare of a principal, considering the loss to the principal due to transfers to agents, it is more typical to design contracts that maximize the total value from the outcome net the cost of effort, and without consideration of the transfers other than requiring some form of budget balance.

Babaioff et al. [5] introduce the *combinatorial agency* problem. This a version of the moral hazard in teams problem in which the agents have binary actions and the outcome is binary, but where the outcome technology is a *complex combination* of the inputs of a team of agents. Each agent is able to exert high or low effort in its own hidden action, with the success or failure of an overall project depending on the specific technology function. In particular, these authors consider a number of natural technology functions such as the AND technology, the OR technology, the majority technology, and nested models such as AND-of-ORs and OR-of-ANDs. This can be conceptualized as a problem of moral hazard in teams where agents are situated on a graph, each controlling the effort at a particular vertex.

The combinatorial agency framework considers the social welfare, in terms of the cost to agents and the value to the principal, that can be achieved in equilibrium under an optimal contract where the principal seeks a contract that maximizes payoff, i.e. value net of transfers to the agents, in equilibrium. Thus the focus is on contracts that would be selected by a principal, not be a designer interested in finding an equilibrium that maximizes social welfare. In particular, Babaioff et al. consider the (social) *Price of Unaccountability* (POU), which is the worst case ratio between the *optimal social welfare* when actions are observable as compared to when they are not observable. The worst-case is taken over different probabilities of success for an individual agent's actions (and thus different, uncertain technology functions), and over the principal's value for a successful outcome. The optimal social welfare is obtained by requesting a particular set of agents to exert effort, in order to maximize the total expected value to the principal minus the cost incurred by these agents. In the agency case, the social welfare is again this value net cost, but optimized under the contract that maximizes the expected payoff of the principal.

The main contribution of this work is to characterize the *transition behavior* for the k-out-of-n (or *threshold*) technology, for n agents and $k \in \{1, \ldots, n\}$. The threshold technology is anonymous, meaning that the probability of a successful outcome only depends on the number of agents exerting high effort, not the specific set of agents. Because of this, the transition behavior — a characterization of the optimal contract, which specifies which agents to contract with, as a function of the principal's valuation — can be explained in terms of the number of agents with whom the principal contracts. We establish that the transition behavior (in both the non-strategic and agency cases) includes a transition from contracting between 0 and l agents for some $1 \leq l \leq n$, followed by all $n - l$ remaining transitions, for any $0 < \alpha < \beta < 1$, where α (resp. β) is the probability that the action of a low effort (resp. high effort) action by an agent results in a successful local outcome. This generalizes the prior result of Babaioff et al. [5] for the AND gate (a single transition from 0 agents to all n agents contracted) and the OR gate (all n transitions), and closes an important open question.

Considering the POU, we establish a tight bound of 2 for the OR technology, for all values of n, α and $\beta = 1 - \alpha$. The initial paper established this POU for the case of $n = 2$ agents only, while an extended version of the paper provides a bound of $n = 2.5$ for the general $n > 2$ case [6]. In addition, we establish that the POU is unbounded for the threshold technology for the general case of $k \geq 2, n \geq 2$. The initial paper established this result only for AND technology, and so our result closes this for the more general threshold case for any $0 < \alpha < \beta < 1$. In addition, we consider non-anonymous technology functions such as Majority-of-AND, Majority-of-OR, and AND-of-Majority, and study their transition behavior.

We believe that this work is an interesting step in extending the combinatorial agency model in a direction of interest for crowdsourcing [16, 3, 1, 2]. Combinatorial agency is relevant to applications where neither the effort nor the individual outcome of each worker is observable. All that is observable is the ultimate success or failure. Perhaps the boundaries between individual contributions are hard to define, or workers prefer to hide individual contributions in some way (e.g., to protect their privacy.) Perhaps it is extremely costly, or even impossible, to determine the quality of the work performed by an individual worker when studied in isolation. The threshold technology seems natural in modeling crowdsourcing problems in which success requires getting enough suitable contributions.

Related Work. A characterization of the transition behavior was first conjectured for the Majority technology in Babaioff et al. [5], but almost all of the subsequent literature is restricted to read-once networks [7, 8, 11, 12]. The combinatorial agency problem has also been studied under mixed Nash equilibria [7]. Babaioff et al. [8] study "free labor" and whether the principal can benefit from having certain agents reduce their effort level, even when this effort is free. The principal is hurt by free labor under the OR technology, because free labor can lead to free riding. Another variation allows the principal to audit some fraction of the agents, and discover their individual private action [10]. Some computational complexity results for identifying optimal contracts have also been developed. This problem is NP-hard for OR technology [11], and the difficulty is later shown to be a property of unobservable actions [12]. This is in contrast to the AND technology, which admits a polynomial time algorithm for computing the optimal contract. An FPTAS is developed for OR technology, and extended to almost all series-parallel technologies [11].

2 Model

In the combinatorial agency model, a principal employs a set of n self-interested agents. Each agent i has an action space A_i and a cost (of effort) associated with each action $c_i(a_i) \geq 0$ for every $a_i \in A_i$. We let $\boldsymbol{a}_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$ denote the action profile of all other agents besides agent i. Similar to Babaioff et al. [5], we focus on a binary-action model. That is, agents either exert effort ($a_i = 1$) or do not exert effort ($a_i = 0$), and the cost function becomes c_i if $a_i = 1$ and 0 if $a_i = 0$. If agent i exerts effort, she succeeds with probability β_i. If agent i does not exert effort, she succeeds with probability α_i, where $0 < \alpha_i < \beta_i < 1$. We deal with the case of homogenous agents (e.g. $\beta_i = \beta$, $\alpha_i = \alpha$ and $c_i = c$

for all i), though some of the prior work deals with the case of heterogenous agents. Sometimes we use the additional assumption of [5], that $\beta = 1 - \alpha$, where $0 < \alpha < \frac{1}{2}$.

Completing the description of the technology is the *outcome function* f, which determines the success or failure of the overall project as a function of the success or failure of each agent. Let $\boldsymbol{x} = (x_1, \ldots, x_n)$, with $x_i \in \{0, 1\}$ to denote the success or failure of the action of agent i given its selected effort level. Following Babaioff et al. [5] we focus on a binary outcome setting, so that the outcome is 1 (= success) or 0 (= failure.) Given this, we study the following outcome functions:

1. AND technology: $f(x_1, x_2, ..., x_n) = \wedge_{i \in N} x_i$. In other words, the project succeeds if and only if all agents succeed in their tasks.
2. OR technology: $f(x_1, x_2, ..., x_n) = \vee_{i \in N} x_i$. In other words, the project succeeds if and only if at least one agent succeeds in her task.
3. Majority technology: $f(x) = 1$ if a majority of the x_i are 1. In other words, the project succeeds if and only if a majority of the agents succeed at their tasks.
4. Threshold technology: We can generalize the majority technology into a threshold technology, where $f(x) = 1$ if and only if at least k of the x_i are 1, e.g. at least k of the n agents succeed in their tasks.

In fact, the threshold technology is a generalization of the OR, AND and majority technologies, since the $k = 1$ case is equivalent to the OR technology, the $k = n$ case is equivalent to the AND technology, and the $k = \lceil \frac{n}{2} \rceil$ case is equivalent to the majority technology. It should be noted that the set of threshold technologies is exactly the set of threshold functions. It is easy to see that each of these outcome functions is *anonymous*, meaning that the outcome is invariant to a permutation on the agent identities.

Given outcome function f, and success probabilities α and β, then action profile $\boldsymbol{a}$ induces a probability $p(\boldsymbol{a}) \in [0, 1]$ with which the project will succeed. This is just

$$p(\boldsymbol{a}) = E_{\boldsymbol{x}}[f(\boldsymbol{x}) \mid \boldsymbol{x} \sim \boldsymbol{a}] \tag{1}$$

where the local outcomes $\boldsymbol{x}$ are distributed according to α, β and as a result of the effort $\boldsymbol{a}$ by agents. Since p considers the combined effect of technology f, α and β, then we refer to p as the *technology function.*

The principal has a value v for a successful outcome and 0 for an unsuccessful outcome. Like [5], we assume that the principal is risk-neutral and seeks to maximize expected value minus expected payments to agents. The principal is unable to observe either the actions $\boldsymbol{a}$ or the (local) outcomes $\boldsymbol{x}$. The only thing the principal can observe is the success or failure of the overall project. Based on this, a *contract* specifies a payment $t_i \geq 0$ to each agent i when the project succeeds, with a payment of zero otherwise. The principal can pay the agents, but not fine them. It is convenient to include in a contract the set of agents that the principal intends to exert high effort; this is the set of agents that *will* exert high effort when the principal selects an appropriate payment function.

The utility to agent i under action profile $\boldsymbol{a}$ is $u_i(\boldsymbol{a}) = t_i \cdot p(\boldsymbol{a}) - c_i$ if the agent exerts effort, and $u_i(\boldsymbol{a}) = t_i \cdot p(\boldsymbol{a})$ otherwise. The principal's expected utility is $u(\boldsymbol{a}) = v \cdot p(\boldsymbol{a}) - \sum_{i \in N} t_i \cdot p(\boldsymbol{a})$. The principal's task is to design a contract so that its utility is maximized under an action profile $\boldsymbol{a}$ that is a Nash equilibrium. We make the same assumption as Babaioff et al. [5], that if there are multiple Nash equilibria (NE), the principal can contract for the best NE. The *social welfare* for an action profile $\boldsymbol{a}$ is given by $u(\boldsymbol{a}) + \sum_{i \in N} u_i(\boldsymbol{a}) = v \cdot p(\boldsymbol{a}) - \sum_{i \in N} c_i \cdot a_i$, with payments from the principal to the agents canceling out.

Throughout, we focus on outcome functions that are *monotonic*, so that $f(\boldsymbol{x}) = 1 \ \Rightarrow f(x'_1, \boldsymbol{x}_{-1}) = 1$ for $x'_1 \geq x_1$. Based on this, then the technology function p is also *monotonic* in the amount of effort exerted, that is for all i and all $\boldsymbol{a}_{-i} \in \{0,1\}^{n-1}$, $p(1, \boldsymbol{a}_{-i}) \geq p(0, \boldsymbol{a}_{-i})$. Similarly, a technology function p is *anonymous* if it symmetric with respect to the players. That is, it is anonymous if it only depends on the number of agents that exert effort and is indifferent to permutations of the joint action profile $\boldsymbol{a}$. This is true whenever the underlying outcome function is anonymous.

In the *non-strategic* variant of the problem, the principal can choose which agents exert effort and these agents need not be "motivated", the principal can simply bear their cost of exerting effort. Let S^*_a and S^*_{ns} denote the optimal set of agents to contract with in the agency case and the non-strategic case respectively. That is, these sets of agents are those that maximize the expected value to the principal net cost, first where the sets must be induced in a Nash equilibrium and second when they can be simply selected.

Definition 1. *[5] The Price of Unaccountability (POU) for an outcome function f is defined as the worst case ratio (over v, α and β) of the social welfare in the non-strategic case and the social welfare of the agency case:*

$$POU(f) = \sup_{v>0,\alpha,\beta} \frac{p(S^*_{\text{ns}}(v)) \cdot v - \sum_{i \in S^*_{\text{ns}}(v)} c_i}{p(S^*_a(v)) \cdot v - \sum_{i \in S^*_a(v)} c_i}, \tag{2}$$

where p is the technology function induced by f, α and β, with $0 < \alpha < \beta < 1$.

In studying the POU, it becomes useful to characterize the transition behavior for a technology. The transition behavior is, for a fixed technology function p, the optimal set of contracted agents as a function of the principal's value v. We know that when $v = 0$ it is optimal to contract with 0 agents and likewise, as $v \to \infty$, it is optimal to contract with all agents. However, we would like to understand what are the optimal sets of agents contracted between these two extreme cases. There are, in fact, two sets of transitions, for both the agency and the non-strategic case. For anonymous technologies, there can be at most n transitions in either case, since the number of agents in the optimal contract is (weakly) monotonically increasing in the principal's value. We seek to understand how many transitions occur, and the nature of each "jump" (i.e. the change in number of agents contracted with at a transition.)

We also consider compositions of these technologies such as majority-of-AND, Majority-of-OR, and AND-of-Majority. These technologies are no longer anonymous. With non-anonymous technologies, one needs to specify the contracted set of agents, in addition to the number of agents contracted. In considering composition of anonymous technologies, we assume we are composing identical technology functions, e.g. each AND gate in the majority-of-AND technology consists of the same number of agents.

3 Transition Behavior of the Optimal Contract

Below we will characterize the transition behavior of the threshold technology. We show that there exists an $l \in \{1, ..., n\}$ such that the first transition is from 0 to l agents followed by all remaining transitions. This result holds for any value of α, β such that $0 < \alpha < \beta < 1$.

Our proof builds on the framework of Babaioff et al. [5]. In Babaioff et al., it was shown that the AND technology always exhibits "increasing returns to scale" (IRS) and the OR technology always exhibits "decreasing returns to scale" (DRS). It was also shown that any anonymous technology that exhibits IRS has a single transition from 0 to n agents for the optimal contract in the non-strategic case and that any anonymous technology that exhibits DRS exhibits all n transitions in the non-strategic case. Similar to the non-strategic case, it was shown in Babaioff et al. that the AND technology always exhibits overpayment (OP), in the agency case, where the OP condition guarantees a single transition from 0 to n, and the OR technology always exhibits increasing relative marginal payment (IRMP), in the agency case, where the IRMP condition guarantees all n transitions.

We show that the threshold technology exhibits IRS up to a certain number of agents contracted and DRS thereafter, which gives the transition characterization for the non-strategic case. Likewise, we show that the threshold function exhibits OP to a point and IRMP in the agency case, which is sufficient to give the transition characterization for the agency case. Our analysis is new, in the sense that we consider the possibility that a single technology can exhibit IRS up to a certain number of agents contracted, followed by DRS and likewise, that it can exhibit OP up to a certain number of agents contracted, followed by IRMP. Babaioff et al. only considered the possibility a function exhibits either IRS or DRS, and likewise, either OP or IRMP. In addition to this insight, we use properties of (log) convex functions to establish this result. We state our main theorems below:

Theorem 1 For any threshold technology (any k, n, c, α and β) in the non-strategic case, there exists an $1 \leq l_{ns} \leq n$ where, such that the first transition is from 0 to l_{ns} agents, followed by all remaining $n - l_{ns}$ transitions.

Theorem 2 For any threshold technology (any k, n, c, α and β) in the agency case, there exists an $1 \leq l_a \leq l_{ns}$ such that the first transition is from 0 to l_a agents, followed by all remaining $n - l_a$ transitions.

The following observations give us the optimal payment rule for any technology and establish a monotonic property for the optimal contract as a function of v.

Definition 2. *[5] The marginal contribution of agent i for a given* a_{-i} *is denoted by* $\Delta_i(\boldsymbol{a}_{-i}) = p(1, \boldsymbol{a}_{-i}) - p(0, \boldsymbol{a}_{-i})$*, and is the difference in the probability of success of the technology function when agent i exerts effort and when she does not.*

For anonymous technologies, if exactly j entries in $\boldsymbol{a}_{-i}$ are 1, then $\Delta_i = p_{j+1} - p_j$, where p_j is the probability of success when exactly j agents exert effort. Since p is strictly monotone, we have $\Delta_i > 0$ for all i.

Remark 1. [5] The best contracts (from the principal's point of view) that induce the action profile $\boldsymbol{a} \in \{0, 1\}^n$ as a Nash equilibrium are $t_i = 0$ when the project is unsuccessful and $t_i = \frac{c_i}{\Delta_i(\boldsymbol{a}_{-i})}$ when the project succeeds and the principal requests effort $a_i = 1$ from agent i.

The following lemma gives a set of sufficient conditions for an anonymous technology to have a first transition from 0 to l, for some $l \in \{1, ..., n\}$, followed by all remaining $n - l$ transitions. This lemma holds for both the non-strategic case (where $Q_i = i \cdot c$) and the agency case (where $Q_i = \frac{i \cdot c}{\Delta_i}$). We view this lemma as a generalization of Theorem 9 from [5] and it follows a similar proof structure in that it uses Lemmas 12 and 13 from [5] that relate the principal's utility of contracting with a fixed number of agents to the Q_i values. This lemma states that as long as a technology function exhibits OP up to a certain number of agents contracted followed by IRMP, then the transition behavior involves a first transition from 0 to l, for some $l \in \{1, ..., n\}$, followed by all remaining $n - l$ transitions.

Lemma 1. *Any anonymous technology function that satisfies:*

1. $\frac{Q_i}{Q_l} > \frac{p_i - p_0}{p_l - p_0}$ *for all* $i \neq l$
2. $\frac{Q_{l+1} - Q_l}{p_{l+1} - p_l} > \frac{Q_l}{p_l - p_0}$
3. $\frac{Q_{i+1} - Q_i}{p_{i+1} - p_i} > \frac{Q_i - Q_{i-1}}{p_i - p_{i-1}}$ *for all* $i > l$

for some $l \in \{1, ..., n\}$ *has a first transition from* 0 *to* l *and then all* $n - l$ *subsequent transitions, where* Q_i *is defined appropriate for the non-strategic case or the agency case.*

Now that we have established a set of sufficient conditions for an anonymous technology to exhibit a first transition from 0 to l, followed by all remaining transitions (for either the non-strategic case or the agency case), we interpret what these conditions are for the non-strategic case.

Lemma 2. *Any anonymous technology that has a probability of success function that satisfies:*

1. $\frac{p_i - p_0}{i} > \frac{p_{i-1} - p_0}{i-1}$ *for all* $2 \leq i \leq l$ *and* $\frac{p_i - p_0}{i} < \frac{p_{i-1} - p_0}{i-1}$ *for all* $i > l$
2. $\frac{1}{p_{i+1} - p_i} > \frac{1}{p_i - p_{i-1}}$ *for all* $i > l$

for some $l \in \{1, ..., n\}$ *has a first transition from* 0 *to* l *and then all* $n - l$ *subsequent transitions for the nonstrategic version of the problem.*

In establishing that the threshold technology satisfies the conditions outlined in Lemma 2, it becomes useful to define a property of the probability of success function.

Definition 3. *We say that a probability of success p for a particular technology is unimodal if it satisfies one of three alternatives:*

1. $p_i - p_{i-1} > p_{i-1} - p_{i-2}$ *for all* $2 \leq i \leq j$ *and* $p_i - p_{i-1} < p_{i-1} - p_{i-2}$ *for all* $i > j$
2. $p_i - p_{i-1} > p_{i-1} - p_{i-2}$ *for all* $2 \leq i \leq n$
3. $p_i - p_{i-1} < p_{i-1} - p_{i-2}$ *for all* $2 \leq i \leq n$

Let $f(i) = \frac{p_i - p_0}{i}$. This function is useful to consider, because in order to establish the first condition of Lemma 2, we need to show that $f(i)$ is unimodal.

Lemma 3. *If the probability of success function is unimodal over the set* $\{1, ..., n\}$, *then we know that* $f(i)$ *is also unimodal.*

Corollary 1. *For any anonymous technology function* (p, c) *that has a unimodal probability of success, there exists an* $1 \leq l \leq n$ *such that the first transition in the non-strategic case is from* 0 *to* l *agents (where* l *is the smallest value that satisfies* $\frac{p_l - p_0}{l} > \frac{p_{l+1} - p_0}{l+1}$*) followed by all remaining* $n - l$ *transitions.*

Therefore, it suffices to show that p is unimodal in order to establish that the technology (p, c) exhibits a first transition from 0 to l, for some $l \in \{1, ..., n\}$, followed by all remaining $n - l$ transitions, in the non-strategic case.

Lemma 4. *The probability of success function for any threshold technology is unimodal.*

The characterization of the transition behavior of the threshold technology in the non-strategic case follows from Lemmas 2, 3, and 4.

Theorem 1. *For any threshold technology (any* k*,* n*,* c*,* α *and* β*) in the non-strategic case, there exists an* $1 \leq l_{ns} \leq n$ *where, such that the first transition is from* 0 *to* l_{ns} *agents, followed by all remaining* $n - l_{ns}$ *transitions.*

Now that we have characterized the transition behavior of the threshold technology, for any k, in the non-strategic case, we focus on establishing the conditions of Lemma 1, for the agency case. The following lemma is used to show that the first condition in Lemma 1 is satisfied by the threshold technology.

Lemma 5. *The discrete valued function,* $\frac{Q_i}{p_i - p_0}$*, is convex.*

Lemma 6. *There exists a value of* $1 \leq l_a \leq n$ *such that* $\frac{Q_i}{Q_{l_a}} > \frac{p_i - p_0}{p_{l_a} - p_0}$ *for all* $i \neq l_a$.

Since there exists an l_a such that $\frac{Q_i}{p_i - p_0} > \frac{Q_{i+1}}{p_{i+1} - p_0}$ for all $1 \leq i < l_a$ and $\frac{Q_i}{p_i - p_0} < \frac{Q_{i+1}}{p_{i+1} - p_0}$ for all $l_a \leq i < n$, we have the following corollary.

Corollary 2. *We have* $\frac{Q_{l_a+1}-Q_{l_a}}{p_{l_a+1}-p_{l_a}} > \frac{Q_{l_a}}{p_{l_a}-p_0}$, *where* $1 \le l_a \le n$ *satisfies* $\frac{Q_i}{Q_{l_a}} > \frac{p_i-p_0}{p_{l_a}-p_0}$ *for all* $i \neq l_a$.

Lemma 7. *We have* $\frac{Q_{i+1}-Q_i}{p_{i+1}-p_i} > \frac{Q_i-Q_{i-1}}{p_i-p_{i-1}}$ *for all* $i > l_a$ *where* l_a *is the smallest value such that* $\frac{Q_{l_a}}{p_{l_a}-p_0} < \frac{Q_{l_a+1}}{p_{l_a+1}}$.

Lemmas 1, 6, and 7 and Corollary 2 establish the following result.

Theorem 2. *For any threshold technology (any* k, n, c, α *and* β*) in the agency case, there exists an* $1 \le l_a \le l_{ns}$ *such that the first transition is from* 0 *to* l_a *agents, followed by all remaining* $n - l_a$ *transitions.*

These results beg the question, how do the values of l_a and l_{ns} relate to k? Below we give the trend in transition behavior as a function of β, when $\alpha = 0$. This remark holds for both the non-strategic and agency cases. We also provide a technical lemma regarding the value of l_a and l_{ns} as $\alpha \to 0$. This lemma is used in the next section to establish an unbounded POU for the threshold function.

Remark 2. For any threshold technology with fixed $k \ge 2$, n, c and $\alpha = 0$, we have that $l = k$ for β close enough to 1 and $l = n$ for β close enough to 0.

Lemma 8. *As* $\alpha \to 0$, *we know that* $k \le l_a \le l_{ns}$, *where* l_a *is the first transition in the agency case and* l_{ns} *is the first transition in the non-strategic case.*

We note that it is not always the case that $l_a \ge k$. For example, when $\alpha = \frac{1}{2} - \epsilon$, $\beta = \frac{1}{2} + \epsilon$ and ϵ is sufficiently small, we have all n transitions, regardless of the value of k.

4 Price of Unaccountability

Lemma 9. *[5] For any technology function, the price of unaccountability is obtained at some value* v *which is a transition point, of either the agency or the non-strategic cases.*

We are able to improve upon this result, for the OR technology, which is needed to establish Theorem 4.

Lemma 10. *For the OR technology, the price of unaccountability occurs at a transition in the agency case, as opposed to a transition in the non-strategic case.*

The following theorem is a result of Babaioff et al. [5], where they derive the price of unaccountability for AND technology where $\beta = 1 - \alpha$.

Theorem 3. *[5] For the AND technology with* $\alpha = 1 - \beta$, *the price of unaccountability occurs at the transition point of the agency case and is* $POU = (\frac{1}{\alpha} - 1)^{n-1} + (1 - \frac{\alpha}{1-\alpha})$.

Remark 3. [5] The price of unaccountability for the AND technology is not bounded. More specifically, $POU \to \infty$ as $\alpha \to 0$ and $POU \to \infty$ as $\beta \to 0$.

Babaioff et al. [5] show that the Price of Unaccountability for the OR technology is bounded by 2 for exactly 2 agents and give an upper bound of 2.5 for the general case [6], when $\beta = 1 - \alpha$. We extend these results for the $\beta = 1 - \alpha$ case and show that the Price of Unaccountability is bounded above by 2 for any OR technology (i.e. for all n). This result is tight, namely, as $\alpha \to 0$, $POU \to 2$. We suspect that these results hold for the more general $0 < \alpha < \beta < 1$ case, but we have been unable to prove it for all values of α, β.

Theorem 4. *The POU for the OR technology is bounded by 2 for all $\alpha, \beta = 1-\alpha$ and n.*

In contrast to the OR technology, we show that the POU for the threshold technology with $k \geq 2$ is unbounded. This result holds for any $0 < \alpha < \beta < 1$.

Theorem 5. *The Price of Unaccountability for the threshold technology is not bounded for all values of $k \geq 2$ and n. More specifically, when $\alpha \to 0$, $POU \to \infty$.*

5 Composition of Anonymous Technologies

5.1 Majority-of-ANDs

We prove the transition behavior for the majority-of-AND technology in the non-strategic case. These results hold for the more general threshold-of-ORs case. For the following assume that in the majority-of-AND technology, the majority gate contains q AND gates, each with m agents. This builds on a conjecture of Babaioff et al. who conjecture the following behavior for both the non-strategic and agency cases. We are unable to prove the transition behavior for the agency case.

Lemma 11. *If the principal decides to contract with $j \cdot m + a$ agents for some $j \in Z^+$ and some $0 \leq a < m$, the probability of success is maximized by fully contracting j AND gates and contracting with a remaining agents on the same AND gate.*

Lemma 12. *For any principal's value v, the optimal contract involves a set of fully contracted AND gate.*

Theorem 6. *The transition behavior for the majority-of-AND technology in the non-strategic case has a first transition to l fully contracted AND gates, where $1 \leq l \leq n$, followed by each subsequent transition of fully contracted AND gates.*

While we are unable to characterize the transition behavior for the majority-of-AND technology in the agency case, we know that the first transition in the agency case must involve contracting at most $l \cdot m$ agents. This allows us to prove that the Price of Unaccountability is unbounded.

Theorem 7. *The Price of Unaccountability is unbounded for the majority-of-AND technology.*

5.2 Majority of ORs

We will characterize the transition behavior for the non-strategic case of the majority of ORs below. In what follows, we assume that each OR gate has j agents and there are m of them comprising a majority function (i.e. $n = j \cdot m$). We also assume that $k = \lceil \frac{m}{2} \rceil$. In considering the majority-of-OR case, we further assume that $\beta = 1 - \alpha$ and $0 < \alpha < \frac{1}{2}$.

Lemma 13. *Consider an integer i such that $i = a \cdot m + b$, where $0 \leq b < m$. Fixing i, the probability of success for a majority-of-ORs function is maximized when $a + 1$ agents are contracted on each of b OR gates and a agents are contracted on each of $n - b$ OR gates.*

The following lemma gives the complete transition behavior in the majority-of-OR technology in the nonstrategic case.

Lemma 14. *The first transition for the non-strategic case of the majority-of-OR technology jumps from contracting with 0 agents to l agents, where $1 \leq l \leq k$, followed by all remaining transitions, where the transitions proceed in such a way so that no OR gate has more than 1 more agent contracted as compared to any other OR gate.*

We conjecture that a similar transition behavior holds in the agency case, but we have thus far been unable to prove it. However, we do know that as $\alpha \to 0$, the first transition jumps to k. This is enough to determine that the POU is unbounded.

Theorem 8. *The Price of Unaccountability is unbounded for the majority-of-OR technology.*

5.3 AND of Majority

In what follows, we will also characterize the transition behavior of AND-of-majorities. These results hold for the more general AND-of-threshold's. We give a result from [5] that allows for the characterization of the transition behavior of AND-of-majority. Let g and h be two Boolean functions on disjoint inputs with any cost vectors, and let $f = g \wedge h$. An optimal contract S for f for some v is composed of some agents from the g-part (denoted by the set R) and some agents from the h-part (denoted by the set T).

Lemma 15. *[5] Let S be an optimal contract for $f = g \wedge h$ on v. Then, T is an optimal contract for h on $v \cdot t_g(R)$, and R is an optimal contract for g on $v \cdot t_h(T)$.*

The previous lemma gives us a characterization of the transition behavior in the AND-of-majorities technology. The statement of this result is analogous to the result given in [5] for the AND-of-ORs technology. Since the previous lemma holds for both the non-strategic and agency variations of the problem, the following theorem holds for both the non-strategic and agency variations of the problem.

Theorem 9. *Let h be an anonymous majority technology and let $f = \bigwedge_{j=1}^{n_c}$ be the AND of majority technology that is obtained by a conjunction of n_c of these majority technology functions on disjoint inputs. Then for any value v, an optimal contract contracts with the same number of agents in each majority component.*

Theorem 9 gives us a complete characterization of the transition behavior in the AND-of-majorities technology for both the non-strategic and the agency cases. Since we know that the first transition in both the agency and non-strategic cases for the AND-of-majority technology occurs to a value greater than 1, we have the following result.

Theorem 10. *The Price of Unaccountability is unbounded for the AND-of-majority technology.*

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