Combinatorial Agency of Threshold Functions

Shaili Jain
Computer Science Department
Yale University
New Haven, CT 06520
shaili.jain@yale.edu

David C. Parkes
School of Engineering and Applied Sciences
Harvard University
Cambridge, MA 02138
parkes@eecs.harvard.edu

Abstract

In this paper, we study the combinatorial agency problem introduced by Babaioff, Feldman and Nisan [7] and resolve some open questions posed in their original paper. Our results include a characterization of the transition behavior for the class of *threshold* functions. This result confirms a conjecture of [7], and generalizes their results for the transition behavior for the OR technology and the AND technology. In addition to establishing a (tight) bound of 2 on the social Price of Unaccountability (POU) for the OR technology for the general case of $n > 2$ agents (the initial paper established this for $n = 2$, an extended version establishes a bound of 2.5 for the general case), we establish that the POU is unbounded for all other threshold functions (the initial paper established this only for the case of AND technology). We also obtain a characterization result for certain compositions of anonymous technologies and establish an unbounded POU for these cases.

1 Introduction

The classic principal-agent model of microeconomics considers an agent with unobservable, costly actions, each with a corresponding distribution on outcomes, and a principal with preferences over outcomes [11, 19]. The principal cannot contract on the action directly (e.g. the amount of effort exerted), but only on the final outcome of the project. The main goal is to design contracts, with a payment from the principal to the agent conditioned upon the outcome, in order to maximize the payoff to the principal in equilibrium with a rational, self-interested agent.

The principal-agent model is a classic problem of moral hazard, with agents with potentially misaligned incentives and private actions. A related theory has considered the problem of moral hazard on teams of agents [5, 18, 17]. Much of this work involves a continuous action choice by the agent (e.g., effort) and a continuous outcome function, typically linear or concave in the effort of the agents. Moreover, rather than considering the design of an optimal contract that maximizes the welfare of a principal, considering the loss to the principal due to transfers to agents, it is more typical to design contracts that maximize the total value from the outcome net the cost of effort, and without consideration of the transfers other than requiring some form of budget balance.

Babaioff et al. [7] introduce the *combinatorial agency* problem. This a very specific form of the moral hazard on team problem in which the agents have binary actions and the outcome is binary, but where the outcome technology is a *complex combination* of the inputs of a team of agents. Each agent is able to exert high or low effort in its own hidden action, with the success or failure of an overall project depending on the specific technology function. In particular, these authors consider the AND technology, in which all agents must exert effort in order for the global project to have some possibility of success. Other technologies include: the OR technology, the majority technology, and nested models such as AND-of-ORs and OR-of-ANDs. This can be conceptualized as a problem of moral hazard to teams where agents are situated on a graph, each controlling the effort at a particular vertex.

The combinatorial agency framework considers the social welfare, in terms of the cost to agents and the value to the principal, that can be achieved in equilibrium under an optimal contract where the principal seeks a contract that maximizes payoff, i.e. value net of transfers to the agents, in equilibrium. Thus the focus is on contracts that would be selected by a principal, not be a designer interested in finding an equilibrium that maximizes social welfare. In particular, Babaioff et al. suggest to consider the (social) *Price of Unaccountability* (POU), which is the worst case ratio between the *optimal social welfare* when actions are observable as compared to when they are not observable. The worst-case is taken over different probabilities of success for an individual agent's actions (and thus different, uncertain technology functions), and over the principal's value for a successful outcome. The optimal social welfare is obtained by requesting a particular set of agents to exert effort, in order to maximize the total expected value to the principal minus the cost incurred by these agents. In the agency case, the social welfare is again this value net cost, but optimized under the contract that maximizes the expected payoff of the principal.

The main contribution of this work is to characterize the *transition behavior* for the k-out-of-n (or *threshold*) technology, for n agents and $k \in \{1, \ldots, n\}$. The threshold technology is anonymous, meaning that the probability of a successful outcome only depends on the number of agents contracted to for high effort, not the specific set of agents. Because of this, the transition behavior — a characterization of the optimal contract, which specifies which agents to contract with, as a function of the principal's valuation — can be explained in terms of the number of agents with whom the principal contracts. We establish that the transition behavior (in both the non-strategic and agency cases) includes a transition from contracting between 0 and l agents for some $1 \leq l \leq n$, followed by all $n - l$ remaining transitions, for any $0 < \alpha < \beta < 1$, where α (resp. β) is the probability that the action of a low effort (resp. high effort) action by an agent results in a successful local outcome. This generalizes the prior result of Babaioff et al. [7] for the AND gate (a single transition from zero agents contracted to all agents contracted) and the OR gate (all n transitions), and closes an important open question. This result relies on the fact that a single function can exhibit increasing returns to scale, or IRS, followed by decreasing returns to scale, or DRS, whereas Babaioff et al. only considered the possibility that a function exhibits either IRS or DRS. In addition, we use properties of (log) convex functions to establish this result.

Considering the POU, we establish a tight bound of 2 for the OR technology, for all values of n, α and $\beta = 1 - \alpha$. The initial paper established this POU for the case of $n = 2$ agents only, while an extended version of the paper provides a bound of $n = 2.5$ for the general $n > 2$ case [8]. In addition, we establish that the POU is unbounded for the threshold technology for the general case of $k \geq 2, n \geq 2$, including Majority. The initial paper established this result only for AND technology, and so our result closes this for the more general threshold case for any $0 < \alpha < \beta < 1$. More specifically, we observe that as $\alpha \to 0$, the POU becomes unbounded.

In addition, we consider non-anonymous technology functions such as the Majority-of-AND, Majority-of-OR, and AND-of-Majority technologies, and study their transition behavior. Our result regarding the majority technology, and a technical lemma of Babaioff et al., give the transition behavior for the AND-of-majority technology. In particular, when a majority gate has its first transition to l agents, then the first transition as the principal's value increases under AND-of-Majority is to l agents on each Majority gate, and then follows the subsequent transitions, with an additional agent contracted with, in an increment deployed simultaneously on all Majority gates.

Our result for the Majority-of-OR technology is a bit surprising in that its transition behavior is similar to the case of the majority technology where there is a single transition from 0 to l followed by all remaining transitions. Thus far, we have only been able to characterize the transition behavior for the non-strategic version of Majority-of-OR, but we conjecture a similar transition behavior for the agency version. Though we have been unable to characterize the transition behavior in the agency case, we show that the POU for the majority-of-OR technology is unbounded. We also consider the majority-of-ANDs technology function introduced by [7] and prove the transition behavior in the non-strategic case consistent with a conjecture of Babaioff et al. [7] for the OR-of-ANDs. We are unable to prove the transition behavior for the agency case, but show that the POU is unbounded for the majority-of-ANDs.

We believe that this work is an interesting step in extending the combinatorial agency model in a direction of interest for crowd sourcing [23, 3, 1, 2]. In particular, it is relevant in applications where neither the effort nor the individual outcome of each worker is observable. All that is observable is the ultimate success or failure. One reason for this is that the boundaries between individual contributions are hard to define, or that the workers themselves preferred to anonymize or hide individual contributions in some way (e.g., to protect their privacy.) Another motivation is that it could be extremely costly, or even impossible, to determine the quality of the work performed by an individual worker when studied in isolation. One can know whether or not the overall project was a success or failure (lots of site traffic, or no site traffic, an overall artifact that passes required tests, or an artifact that crashes, etc.) but not know whether or not a counterfactual project outcome, where the work of any one worker was changed, would be different. For software engineering, the work of others to integrate individual components has already been done. For a web site, the opportunity to launch the site has already passed.

A threshold technology models a domain in which a project only succeeds when enough agents provide high effort (e.g., Wikipedia or the development of open-source software.) For Majority-of-OR, consider domains such as TopCoder [4], where mini competitions (e.g. OR gates) are used for each module and then ultimate success occurs if enough individual modules are judged to be successful. Many Games with a Purpose [23, 22, 24] can be modeled with a Majority-of-AND technology, since in an individual game, both agents involved must succeed at their task, however this task is given to more than one set of agents, and we need a majority of these games (or AND gates) to succeed in order to verify the quality of the output.

1.1 Related Work

A characterization of the transition behavior and the POU was first conjectured for Majority technology in Babaioff et al. [7], but almost all of the subsequent literature is restricted to *read-once* networks [9, 10, 15, 16].

A number of variations of the basic combinatorial agency model have been studied. Considering contracts that induce mixed Nash equilibria, this can sometimes improve the POU over insisting on a pure strategy NE, developing a number of upper and lower bound results on the relative gain from mixed strategies, and identifying a sufficient condition under which mixed strategies provide no advantage to the principal. These authors conjecture that for any technology function, the relative gain to the principal for inducing a mixed strategy Nash equilibrium is bounded above by a constant. Another variation considers the cost of "free labor", namely, if there are situations where the principal can benefit from having certain agents reduce their effort level, even when this effort is free [10]. The principal is hurt by free labor under the OR technology, because free labor can lead to free riding, while for the AND technology (and any technology with increasing returns to scale), the principal is not hurt by free labor. A third variation allows the principal to audit some fraction of the agents, and discover their individual private action [14]. Results provide the transition behavior for AND technology and also give some consideration to Majority and OR technologies.

Some computational complexity results for identifying optimal contracts have also been developed. This problem is NP-hard for OR technology [15], and the difficulty is later shown to be a property of unobservable actions [16]. This is in contrast to the AND technology, which is shown to admit a polynomial time algorithm for computing the optimal contract [15]. An FPTAS is developed for OR technology, and extended to almost all "series-parallel" technologies [15].

A related topic in economic theory is that of *contest design* [20, 21, 6, 12]. Contests are situations in which multiple agents exert effort in order to win a prize. All agents bear the "cost" of the effort exerted regardless of whether they win a prize. Unlike the moral hazard on teams problem, or the combinatorial agency problem, the individual outcome from each worker is observable. One agent is unable to "hide behind" the success or failure of the overall project, since the result from its own effort are judged in isolation. Moreover, the contest design frameworks do not, to the best of our knowledge, consider combinations of inputs from workers. Rather, the outcome depends on the maximum quality outcome generated individually, by each agent. DiPalatino and Vojnovic [13] analyze a variation with multiple, simultaneous tasks and workers selecting the single task in which they will participate.

2 Model

In the combinatorial agency model, a principal employs a set of n self-interested agents. Each agent i has an action space A_i and a cost (of effort) associated with each action $c_i(a_i) \geq 0$ for every $a_i \in A_i$. We let $\vec{a}_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$ denote the action profile of all other agents besides agent i. Similar to Babaioff et al. [7], we focus on a binary-action model. That is, agents either exert effort ($a_i = 1$) or do not exert effort ($a_i = 0$), and the cost function becomes c_i if $a_i = 1$ and 0 if $a_i = 0$. If agent i exerts effort, she succeeds with probability β_i. If agent i does not exert effort, she succeeds with probability α_i, where $0 < \alpha_i < \beta_i < 1$. We deal with the case of homogenous agents (e.g. $\beta_i = \beta$, $\alpha_i = \alpha$ and $c_i = c$ for all i), though some of the prior work deals with the case of heterogenous agents. Sometimes we use the additional assumption of [7], that $\beta = 1 - \alpha$, where $0 < \alpha < \frac{1}{2}$.

Completing the description of the technology is the *outcome function* f, which determines the success or failure of the overall project as a function of the success or failure of each agent. Let $\vec{x} = (x_1, \ldots, x_n)$, with $x_i \in \{0, 1\}$ to denote the success or failure of the action of agent i given its selected effort level. Following Babaioff et al. [7] we focus on a binary outcome setting, so that the outcome is 1 (= success) or 0 (= failure.) Given this, we study the following outcome functions:

1. AND technology: $f(x_1, x_2, ..., x_n) = \wedge_{i \in N} x_i$. In other words, the project succeeds if and only if all agents succeed in their tasks.

2. OR technology: $f(x_1, x_2, ..., x_n) = \vee_{i \in N} x_i$. In other words, the project succeeds if and only if at least one agent succeeds in her task.

3. Majority technology: $f(x) = 1$ if a majority of the x_i are 1. In other words, the project succeeds if and only if a majority of the agents succeed at their tasks.[1]

[1] Note that this is different from the AND and OR technologies function since this is not a "read-once" network.

4. Threshold technology: We can generalize the majority technology into a threshold technology, where $f(x) = 1$ if and only if at least k of the x_i are 1, e.g. at least k of the n agents succeed in their tasks.

In fact, the threshold technology is a generalization of the OR, AND and majority technologies, since the $k = 1$ case is equivalent to the OR technology, the $k = n$ case is equivalent to the AND technology, and the $k = \lceil \frac{n}{2} \rceil$ case is equivalent to the majority technology. It should be noted that the set of threshold technologies is exactly the set of threshold functions. It is easy to see that each of these outcome functions is *anonymous*, meaning that the outcome is invariant to a permutation on the agent identities.

Given outcome function f, and success probabilities α and β, then action profile $\vec{a}$ induces a probability $p(\vec{a}) \in [0, 1]$ with which the project will succeed. This is just

$$p(\vec{a}) = E_{\vec{x}}[f(\vec{x}) \mid \vec{x} \sim \vec{a}] \quad (1)$$

where the local outcomes $\vec{x}$ are distributed according to α, β and as a result of the effort $\vec{a}$ by agents. Since p considers the combined effect of technology f, α and β, then we refer to p as the *technology function.*

The principal has a value v for a successful outcome and 0 for an unsuccessful outcome. Like [7], we assume that the principal is risk-neutral and seeks to maximize expected value minus expected payments to agents. The principal is unable to observe either the actions $\vec{a}$ or the (local) outcomes $\vec{x}$. The only thing the principal can observe is the success or failure of the overall project. Based on this, a *contract* specifies a payment $t_i \geq 0$ to each agent i when the project succeeds, with a payment of zero otherwise. The principal can pay the agents, but not fine them. It is convenient to include in a contract the set of agents that the principal intends to exert high effort; this is the set of agents that *will* exert high effort when the principal selects an appropriate payment function.

The utility to agent i under action profile $\vec{a}$ is $u_i(\vec{a}) = t_i \cdot p(\vec{a}) - c_i$ if the agent exerts effort, and $u_i(\vec{a}) = t_i \cdot p(\vec{a})$ otherwise. The principal's expected utility is $u(\vec{a}) = v \cdot p(\vec{a}) - \sum_{i \in N} t_i \cdot p(\vec{a})$. The principal's task is to design a contract so that its utility is maximized under an action profile $\vec{a}$ that is a Nash equilibrium. We make the same assumption as Babaioff et al. [7], that if there are multiple Nash equilibria (NE), the principal can contract for the best NE.[2] The *social welfare* for an action profile $\vec{a}$ is given by $u(\vec{a}) + \sum_{i \in N} u_i(\vec{a}) = v \cdot p(\vec{a}) - \sum_{i \in N} c_i \cdot a_i$, with payments from the principal to the agents canceling out.

Throughout, we focus on outcome functions that are *monotonic*, so that $f(\vec{x}) = 1 \Rightarrow f(x'_1, \vec{x}_{-1}) = 1$ for $x'_1 \geq x_1$. Based on this, then the technology function p is also *monotonic* in the amount of effort exerted, that is for all i and all $\vec{a}_{-i} \in \{0, 1\}^{n-1}$, $p(1, \vec{a}_{-i}) \geq p(0, \vec{a}_{-i})$. Similarly, a technology function p is *anonymous* if it symmetric with respect to the players. That is, it is anonymous if it only depends on the number of agents that exert effort and is indifferent to permutations of the joint action profile $\vec{a}$. This is true whenever the underlying outcome function is anonymous.

In the *non-strategic* variant of the problem, the principal can choose which agents exert effort and these agents need not be "motivated", the principal can simply bear their cost of exerting effort. Let S^*_a and S^*_{ns} denote the optimal set of agents to contract with in the agency case and the

A read-once network is a network that can be represented by a graph with a labeled source and sink, where there is a unique player corresponding to each edge. The project succeeds if and only if there exists a path, consisting of successful players, between the source and the sink [7]. Much of the previous work on the combinatorial agency problem applies to read-once networks and thus the understanding of the majority technology seems less well understood than the AND, OR, AND-of-ORs and OR-of-ANDs technologies.

[2] This is reasonable, since the principal can announce which set of agents should exert effort and also design the payment to provide strict incentive to exert effort for those contracted.

non-strategic case respectively. That is, these sets of agents are those that maximize the expected value to the principal net cost, first where the sets must be induced in a Nash equilibrium and second when they can be simply selected.

Definition 2.1. [7] The Price of Unaccountability (POU) for an outcome function f is defined as the worst case ratio (over v, α and β) of the social welfare in the non-strategic case and the social welfare of the agency case:

$$POU(f) = \sup_{v>0,\alpha,\beta} \frac{p(S^*_{\text{ns}}(v)) \cdot v - \sum_{i \in S^*_{\text{ns}}(v)} c_i}{p(S^*_a(v)) \cdot v - \sum_{i \in S^*_a(v)} c_i}, \tag{2}$$

where p is the technology function induced by f, α and β, with $0 < \alpha < \beta < 1$.

In studying the POU, it becomes useful to characterize the transition behavior for a technology. The transition behavior is, for a fixed technology function p, the optimal set of contracted agents as a function of the principal's value v. We know that when $v = 0$ it is optimal to contract with 0 agents and likewise, as $v \to \infty$, it is optimal to contract with all agents. However, we would like to understand what are the optimal sets of agents contracted between these two extreme cases. There are, in fact, two sets of transitions, for both the agency and the non-strategic case. For anonymous technologies, there can be at most n transitions in either case, since the number of agents in the optimal contract is (weakly) monotonically increasing in the principal's value. We seek to understand how many transitions occur, and the nature of each "jump" (i.e. the change in number of agents contracted with at a transition.)

We also consider compositions of these technologies such as majority-of-AND, Majority-of-OR, and AND-of-Majority. These technologies are no longer anonymous. For example, in the AND-of-Majority case, one can imagine that the probability of success will be different when i agents are contracted on the same majority function and when they are contracted on i different majority functions. With non-anonymous technologies, one needs to specify the contracted set of agents, in addition to the number of agents contracted. In considering composition of anonymous technologies, we assume we are composing identical technology functions, e.g. each AND gate in the majority-of-AND technology consists of the same number of agents.

3 Transition Behavior of the Optimal Contract

Below we will characterize the transition behavior of the threshold technology, which gives us the transition behavior for the majority technology. We show that there exists an $l \in \{1, ..., n\}$ such that the first transition is from 0 to l agents followed by all remaining transitions. This result holds for any value of α, β such that $0 < \alpha < \beta < 1$.

Our proof builds on the framework of Babaioff et al. [7]. In Babaioff et al., it was shown that the AND technology always exhibits "increasing returns to scale" (IRS) and the OR technology always exhibits "decreasing returns to scale" (DRS). It was also shown that any anonymous technology that exhibits IRS has a single transition from 0 to n agents for the optimal contract in the non-strategic case and that any anonymous technology that exhibits DRS exhibits all n transitions in the non-strategic case. Similar to the non-strategic case, it was shown in Babaioff et al. that the AND technology always exhibits overpayment (OP), in the agency case, where the OP condition guarantees a single transition from 0 to n, and the OR technology always exhibits increasing relative marginal payment (IRMP), in the agency case, where the IRMP condition guarantees all n transitions.

We show that the threshold technology exhibits IRS up to a certain number of agents contracted and DRS thereafter, which gives the transition characterization for the non-strategic case. Likewise, we show that the threshold function exhibits OP to a point and IRMP in the agency case, which is sufficient to give the transition characterization for the agency case. Our analysis is new, in the sense that we consider the possibility that a single technology can exhibit IRS up to a certain number of agents contracted, followed by DRS and likewise, that it can exhibit OP up to a certain number of agents contracted, followed by IRMP. Babaioff et al. only considered the possibility a function exhibits either IRS or DRS, and likewise, either OP or IRMP. In addition to this insight, we use properties of (log) convex functions to establish this result. We state our main theorems that give a complete characterization of the transition behavior of the majority technology below:

Theorem 3.12 For any threshold technology (any k, n, c, α and β) in the non-strategic case, there exists an $1 \leq l_{ns} \leq n$ where, such that the first transition is from 0 to l_{ns} agents, followed by all remaining $n - l_{ns}$ transitions.

Theorem 3.17 For any threshold technology (any k, n, c, α and β) in the agency case, there exists an $1 \leq l_a \leq l_{ns}$ such that the first transition is from 0 to l_a agents, followed by all remaining $n - l_a$ transitions.

The following observations give us the optimal payment rule for any technology and establish a monotonic property for the optimal contract as a function of v.

Definition 3.1. [7] The marginal contribution of agent i for a given a_{-i} is denoted by $\Delta_i(\vec{a}_{-i}) = p(1, \vec{a}_{-i}) - p(0, \vec{a}_{-i})$, and is the difference in the probability of success of the technology function when agent i exerts effort and when she does not.

For anonymous technologies, if exactly j entries in $\vec{a}_{-i}$ are 1, then $\Delta_i = p_{j+1} - p_j$, where p_j is the probability of success when exactly j agents exert effort. Since p is strictly monotone, we have $\Delta_i > 0$ for all i.

Remark 3.2. *[7] The best contracts (from the principal's point of view) that induce the action profile $\vec{a} \in \{0,1\}^n$ as a Nash equilibrium are $t_i = 0$ when the project is unsuccessful and $t_i = \frac{c_i}{\Delta_i(\vec{a}_{-i})}$ when the project succeeds and the principal requests effort $a_i = 1$ from agent i.*

The following remark of Babaioff et al. [7] establishes that the optimal contract for an anonymous technology function is (weakly) monotonically increasing with the principal's value and is used in establishing Lemma 3.6.

Remark 3.3. *[7] For any anonymous technology function p, if contracting with k_1 agents is optimal for v_1, and contracting with k_2 agents is optimal for v_2, and $v_1 > v_2$, then $k_1 \geq k_2$.*

The following two lemmas are from [7] and are used in the proof of Lemma 3.6, which gives a sufficient condition for a technology function to have a first transition to l, followed by all remaining transitions. Let Q_i be the total expected payment when contracting with i agents, or in other words, $Q_i = \frac{p_i \cdot i \cdot c}{p_i - p_{i-1}} = p_i \cdot i \cdot t_i$. In the non-strategic case, let Q_i be the total sum of costs of the number of agents contracted, or in other words, $Q_i = i \cdot c$. Note that the following lemmas hold in both the non-strategic case and the agency case. Finally, let $v_{i,j}$ denote the specific principal's value at which he is indifferent between contracting with i agents or j agents in the agency case. For the non-strategic case, $v_{i,j}$ is the principal's value at which he is indifferent between i agents exerting effort and j agents exerting effort. More formally, the point $v_{i,j}$ at which the principal is indifferent between contracting between i agents and j agents can be expressed as $p_i \cdot v_{i,j} - Q_i = p_j \cdot v_{i,j} - Q_j$. Solving for $v_{i,j}$, we get that $v_{i,j} = \frac{Q_j - Q_i}{p_j - p_i}$. In what follows, we mainly consider the value of $v_{0,i}$ for all i and the value of $v_{i,i+1}$ for all i. We define $u(i, v)$ as the utility to the principal of contracting with i agents when his value is v. In other words, $u(i, v) = p_i \cdot v - Q_i$, where Q_i is the total expected payment, either in the non-strategic case or the strategic case.

Lemma 3.4. *[7] $u(l, v_{0,l}) > u(i, v_{0,l})$ for all $i \neq l$ if and only if $\frac{Q_i}{Q_l} > \frac{p_i - p_0}{p_l - p_0}$ for all $i \neq l$.*

Lemma 3.5. *[7] $u(i, v_{i-1,i}) > u(i+1, v_{i,i+1})$ for all $i > l$ if and only if $\frac{Q_{i+1} - Q_i}{t_{i+1} - t_i} > \frac{Q_i - Q_{i-1}}{t_i - t_{i-1}}$ for all $i > l$.*

The following lemma gives a set of sufficient conditions for an anonymous technology to have a first transition from 0 to l, for some $l \in \{1, ..., n\}$, followed by all remaining $n - l$ transitions. This lemma holds for both the non-strategic case (by setting $Q_i = i \cdot c$) and the agency case (by setting $Q_i = \frac{i \cdot c}{\Delta_i}$). We view this lemma as a generalization of Theorem 9 from [7] and it follows a similar proof structure. This lemma states that as long as a technology function exhibits OP up to a certain number of agents contracted followed by IRMP, then the transition behavior involves a first transition from 0 to l, for some $l \in \{1, ..., n\}$, followed by all remaining $n - l$ transitions.

Lemma 3.6. *Any anonymous technology function that satisfies:*

1. $\frac{Q_i}{Q_l} > \frac{p_i - p_0}{p_l - p_0}$ *for all* $i \neq l$
2. $\frac{Q_{l+1} - Q_l}{p_{l+1} - p_l} > \frac{Q_l}{p_l - p_0}$
3. $\frac{Q_{i+1} - Q_i}{p_{i+1} - p_i} > \frac{Q_i - Q_{i-1}}{p_i - p_{i-1}}$ *for all* $i > l$

for some $l \in \{1, ..., n\}$ has a first transition from 0 to l and then all $n - l$ subsequent transitions, where Q_i is defined appropriate for the non-strategic case or the agency case.

Now that we have established a set of sufficient conditions for an anonymous technology to exhibit a first transition from 0 to l, followed by all remaining transitions (for either the non-strategic case or the agency case), we interpret what these conditions are for the non-strategic case.

Lemma 3.7. *Any anonymous technology that has a probability of success function that satisfies:*

1. $\frac{p_i - p_0}{i} > \frac{p_{i-1} - p_0}{i-1}$ *for all* $2 \leq i \leq l$ *and* $\frac{p_i - p_0}{i} < \frac{p_{i-1} - p_0}{i-1}$ *for all* $i > l$
2. $\frac{1}{p_{i+1} - p_i} > \frac{1}{p_i - p_{i-1}}$ *for all* $i > l$

for some $l \in \{1, ..., n\}$ has a first transition from 0 to l and then all $n - l$ subsequent transitions for the nonstrategic version of the problem.

In establishing that the threshold technology satisfies the conditions outlined in Lemma 3.7, it becomes useful to define a property of the probability of success function.

Definition 3.8. We say that a probability of success p for a particular technology is unimodal if it satisfies one of three alternatives:

1. $p_i - p_{i-1} > p_{i-1} - p_{i-2}$ for all $2 \leq i \leq j$ and $p_i - p_{i-1} < p_{i-1} - p_{i-2}$ for all $i > j$
2. $p_i - p_{i-1} > p_{i-1} - p_{i-2}$ for all $2 \leq i \leq n$
3. $p_i - p_{i-1} < p_{i-1} - p_{i-2}$ for all $2 \leq i \leq n$

Let $f(i) = \frac{p_i - p_0}{i}$. This function is useful to consider, because in order to establish the first condition of Lemma 3.7, we need to show that $f(i)$ is unimodal.

Lemma 3.9. *If the probability of success function is unimodal over the set $\{1, ..., n\}$, then we know that $f(i)$ is also unimodal.*

Corollary 3.10. *For any anonymous technology function (p, c) that has a unimodal probability of success, there exists an $1 \leq l \leq n$ such that the first transition in the non-strategic case is from 0 to l agents (where l is the smallest value that satisfies $\frac{p_l - p_0}{l} > \frac{p_{l+1} - p_0}{l+1}$) followed by all remaining $n - l$ transitions.*

Therefore, it suffices to show that p is unimodal in order to establish that the technology (p, c) exhibits a first transition from 0 to l, for some $l \in \{1, ..., n\}$, followed by all remaining $n - l$ transitions, in the non-strategic case.

Lemma 3.11. *The probability of success function for any threshold technology is unimodal.*

The characterization of the transition behavior of the threshold technology in the non-strategic case follows from Lemmas 3.7, 3.9, and 3.11.

Theorem 3.12. *For any threshold technology (any k, n, c, α and β) in the non-strategic case, there exists an $1 \leq l_{ns} \leq n$ where, such that the first transition is from 0 to l_{ns} agents, followed by all remaining $n - l_{ns}$ transitions.*

Now that we have characterized the transition behavior of the threshold technology, for any k, in the non-strategic case, we focus on establishing the conditions of Lemma 3.6, for the agency case. The following lemma is used to show that the first condition in Lemma 3.6 is satisfied by the threshold technology.

Lemma 3.13. *The discrete valued function, $\frac{Q_i}{p_i - p_0}$, is convex.*

Lemma 3.14. *There exists a value of $1 \leq l_a \leq n$ such that $\frac{Q_i}{Q_{l_a}} > \frac{p_i - p_0}{p_{l_a} - p_0}$ for all $i \neq l_a$.*

Since there exists an l_a such that $\frac{Q_i}{p_i - p_0} > \frac{Q_{i+1}}{p_{i+1} - p_0}$ for all $1 \leq i < l_a$ and $\frac{Q_i}{p_i - p_0} < \frac{Q_{i+1}}{p_{i+1} - p_0}$ for all $l_a \leq i < n$, we have the following corollary.

Corollary 3.15. *We have $\frac{Q_{l_a+1} - Q_{l_a}}{p_{l_a+1} - p_{l_a}} > \frac{Q_{l_a}}{p_{l_a} - p_0}$, where $1 \leq l_a \leq n$ satisfies $\frac{Q_i}{Q_{l_a}} > \frac{p_i - p_0}{p_{l_a} - p_0}$ for all $i \neq l_a$.*

Lemma 3.16. *We have $\frac{Q_{i+1} - Q_i}{p_{i+1} - p_i} > \frac{Q_i - Q_{i-1}}{p_i - p_{i-1}}$ for all $i > l_a$ where l_a is the smallest value such that $\frac{Q_{l_a}}{p_{l_a} - p_0} < \frac{Q_{l_a+1}}{p_{l_a+1}}$.*

Lemmas 3.6, 3.14, 3.16 and 3.18 and Corollary 3.15 establish the following result.

Theorem 3.17. *For any threshold technology (any k, n, c, α and β) in the agency case, there exists an $1 \leq l_a \leq l_{ns}$ such that the first transition is from 0 to l_a agents, followed by all remaining $n - l_a$ transitions.*

Finally we show that the first transition in the agency is at most the value of the first transition in the non-strategic.

Lemma 3.18. *For any threshold technology, we get $l_a \leq l_{ns}$.*

Below we give the trend in transition behavior as a function of β, when $\alpha = 0$.

Remark 3.19. *For any threshold technology with fixed $k \geq 2$, n, c and $\alpha = 0$, we have that $l = k$ for β close enough to 1 and $l = n$ for β close enough to 0.*

4 Price of Unaccountability

In this section, we provide results regarding the Price of Unaccountability for OR and threshold technologies.

Lemma 4.1. *[7] For any technology function, the price of unaccountability is obtained at some value v which is a transition point, of either the agency or the non-strategic cases.*

We are able to improve slightly upon this result, for the OR technology, which is needed to establish Theorem 4.5. We suspect that this result can be improved further, in that the POU occurs at the *first* transition in the agency case.

Lemma 4.2. *For the OR technology, the price of unaccountability occurs at a transition in the agency case, as opposed to a transition in the non-strategic case.*

The following theorem is a result of Babaioff et al. [7], where they derive the price of unaccountability for AND technology where $\beta = 1 - \alpha$. (In fact, they give the price of unaccountability for any anonymous technology with a single transition in both the agency and non-strategic cases.) It is easy to see from the closed form expression of the POU that $POU \to \infty$ as $\alpha \to 0$.

Theorem 4.3. *[7] For the AND technology with $\alpha = 1 - \beta$, the price of unaccountability occurs at the transition point of the agency case and is $POU = (\frac{1}{\alpha} - 1)^{n-1} + (1 - \frac{\alpha}{1-\alpha})$.*

Remark 4.4. *[7] The price of unaccountability for the AND technology is not bounded. More specifically, $POU \to \infty$ as $\alpha \to 0$ and $POU \to \infty$ as $\beta \to 0$.*

In their original paper, Babaioff et al. [7] show that the Price of Unaccountability for the OR technology is bounded by 2 for exactly 2 agents and give an upper bound of 2.5 for the general case [8], when $\beta = 1 - \alpha$. We extend these results for the $\beta = 1 - \alpha$ case and show that the Price of Unaccountability is bounded above by 2 for any OR technology (i.e. for all n). This result is tight, namely, as $\alpha \to 0$, $POU \to 2$. We suspect that these results hold for the more general $0 < \alpha < \beta < 1$ case, but we have been unable to prove it for all values of α, β.

Theorem 4.5. *The POU for the OR technology is bounded by 2 for all $\alpha, \beta = 1 - \alpha$ and n.*

The following remark follows from the proof of Theorem 4.5.

Remark 4.6. *For any n, as $\alpha \to 0$, $POU \to 2$ for the OR technology.*

In contrast to the OR technology, we show that the POU for the threshold technology with $k \geq 2$ is unbounded. This result holds for any $0 < \alpha < \beta < 1$.

Theorem 4.7. *The Price of Unaccountability for the threshold technology is not bounded for all values of $k \geq 2$ and n. More specifically, when $\alpha \to 0$, $POU \to \infty$.*

Lemma 4.8. *As $\alpha \to 0$, we know that $k \leq l_a \leq l_{ns}$, where l_a is the first transition in the agency case and l_{ns} is the first transition in the non-strategic case.*

It should be noted that there is interesting structure to the social welfare ratio as a function of the principal's value v. For a fixed number of agents contracted in the agency case, the social welfare ratio is increasing. However, at a transition in the agency case, the social welfare ratio drops significantly such that the maximum ratio for each successive agency contract never reaches the maximum ratio for the previous agency contract. Proving this behavior could be useful in studying the Price of Unaccountability for restricted of v and α.

5 Composition of Anonymous Technologies

In this section, we study the composition of various technology functions. For the majority-of-AND and majority-of-OR technology, we are unable to provide the characterization of transition behavior for the agency case, but we provide the characterization of transition behavior in the non-strategic case and we provide a result regarding the Price of Unaccountability.

5.1 Majority-of-ANDs

We prove the transition behavior for the majority-of-AND technology in the non-strategic case. These results for the more general threshold-of-ORs case. For the following assume that in the majority-of-AND technology, the majority gate contains q AND gates, each with m agents. This builds on a conjecture of Babaioff et al. who conjecture the following behavior for both the non-strategic and agency cases. We are unable to prove the transition behavior for the agency case.

Lemma 5.1. *If the principal decides to contract with $j \cdot m + a$ agents for some $j \in Z^+$ and some $0 \leq a < m$, the probability of success is maximized by fully contracting j AND gates and contracting with a remaining agents on the same AND gate.*

Lemma 5.2. *For any principal's value v, the optimal contract involves a set of fully contracted AND gate.*

Theorem 5.3. *The transition behavior for the majority-of-AND technology in the non-strategic case has a first transition to l fully contracted AND gates, where $1 \leq l \leq n$, followed by each subsequent transition of fully contracted AND gates.*

While we are unable to characterize the transition behavior for the majority-of-AND technology in the agency case, we know that the first transition in the agency case must involve contracting at most $l \cdot m$ agents (proof similar to that of Lemma 4.8). This allows us to prove that the Price of Unaccountability is unbounded. The proof of Theorem 5.4 is omitted but has a virtually identical proof as Theorem 4.7.

Theorem 5.4. *The Price of Unaccountability is unbounded for the majority-of-AND technology.*

5.2 Majority of ORs

We will characterize the transition behavior for the non-strategic case of the majority of ORs below. In what follows, we assume that each OR gate has j agents and there are m of them comprising a majority function (i.e. $n = j \cdot m$). We also assume that $k = \lceil \frac{m}{2} \rceil$.[3] Since the following lemma is a statement regarding the probability of success, it holds for both the non-strategic and agency cases, because the probability of success is the same in both. In considering the majority-of-OR case, we further assume that $\beta = 1 - \alpha$ and $0 < \alpha < \frac{1}{2}$.

Lemma 5.5. *Consider an integer i such that $i = a \cdot m + b$, where $0 \leq b < m$. Fixing i, the probability of success for a majority-of-ORs function is maximized when $a+1$ agents are contracted on each of b OR gates and a agents are contracted on each of $n - b$ OR gates.*

The following lemma gives the complete transition behavior in the majority-of-OR technology in the nonstrategic case.

[3] It should be noted that these results do not hold for the more general threshold-of-ORs case. In fact we can construct a setting where this transition behavior will not occur.

Lemma 5.6. *The first transition for the non-strategic case of the majority-of-OR technology jumps from contracting with* 0 *agents to* l *agents, where* $1 \leq l \leq k$*, followed by all remaining transitions, where the transitions proceed in such a way so that no OR gate has more than 1 more agent contracted as compared to any other OR gate.*

We conjecture that a similar transition behavior holds in the agency case, but we have thus far been unable to prove it. Although we have been unable to characterize the transition behavior in the agency case, we do know that as $\alpha \to 0$, the first transition jumps to k. While we omit the proof of this lemma, it is very similar to Lemma 4.8. This is enough to determine that the POU is unbounded.

Lemma 5.7. *In the agency case of the majority-of-OR technology, as* $\alpha \to 0$*, the first transition occurs to a value* k*.*

The following theorem has a similar proof to Theorem 4.7.

Theorem 5.8. *The Price of Unaccountability is unbounded for the majority-of-OR technology.*

5.3 AND of Majority

In what follows, we will also characterize the transition behavior of AND-of-majorities. Similar to the previous case, these results hold for the more general AND-of-threshold's. We give a result from [7] that allows for the characterization of the transition behavior of AND-of-majority. Let g and h be two Boolean functions on disjoint inputs with any cost vectors, and let $f = g \wedge h$. An optimal contract S for f for some v is composed of some agents from the g-part (denoted by the set R) and some agents from the h-part (denoted by the set T).

Lemma 5.9. *[7] Let* S *be an optimal contract for* $f = g \wedge h$ *on* v*. Then,* T *is an optimal contract for* h *on* $v \cdot t_g(R)$*, and* R *is an optimal contract for* g *on* $v \cdot t_h(T)$*.*

The previous lemma gives us a characterization of the transition behavior in the AND-of-majorities technology. The statement of this result is analogous to the result given in [7] for the AND-of-ORs technology. Since the previous lemma holds for both the non-strategic and agency variations of the problem, the following theorem holds for both the non-strategic and agency variations of the problem.

Theorem 5.10. *Let* h *be an anonymous majority technology and let* $f = \bigwedge_{j=1}^{n_c}$ *be the AND of majority technology that is obtained by a conjunction of* n_c *of these majority technology functions on disjoint inputs. Then for any value* v*, an optimal contract contracts with the same number of agents in each majority component.*

Theorem 5.10 gives us a complete characterization of the transition behavior in the AND-of-majorities technology for both the non-strategic and the agency cases. Since we know that the first transition in both the agency and non-strategic cases for the AND-of-majority technology occurs to a value greater than 1, we have the following result. The proof structure is similar to that of Theorem 4.7.

Theorem 5.11. *The Price of Unaccountability is unbounded for the AND-of-majority technology.*

6 Conclusions

In this work, we advance the understanding of the combinatorial agency model. We prove the transition behavior for the threshold technology for general α, β. We study the majority technology, the majority-of-OR technology, and the AND-of-majority technology and observe the connection between these technologies and crowdsourcing systems. Babaioff et al. [7] showed that the POU was not bounded for the AND technology, for any n. We strengthen this result, and prove that the POU is not bounded for the threshold technology for all $k \geq 2$, any $n \geq 2$, and any $0 < \alpha < \beta < 1$. More specifically, the POU for the threshold technology (with $k \geq 2$) approaches ∞ as $\alpha \to 0$. Babaioff et al. [7] showed that the POU was bounded by 2 for the OR technology with 2 agents and bounded by 2.5 in the general case [8]. We show that the POU is bounded by 2 for the OR technology for all values of $\alpha, \beta = 1 - \alpha$ and n and this bound is tight.

While we do not study the entire class of anonymous functions, we do study a natural class in the k-out-of-n (or threshold) technology. The entire class of anonymous functions is easy to characterize using the set of "exact-value" functions [25]. The "exact value" function E_k is 1 if and only if exactly k agents succeed and 0 otherwise. The set of exact-value functions form a basis for the class of anonymous functions, or in other words, any anonymous function f can be written as follows: $f(x) = \bigvee_{0 \leq k \leq n} E_k(x) \wedge v_k$, where $(v_0, ..., v_n) \in \{0, 1\}^{n+1}$ and x is the success vector of the agents [25]. We leave studying the entire class of anonymous functions as a direction for future work. In fact, we suspect that the threshold technology has a more well-behaved transition behavior than other anonymous functions. This would imply that the threshold function is a desirable technology for crowdsourcing work and is the most significant open direction.

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A Proofs from Section 3

Lemma 3.6 Any anonymous technology function that satisfies:

1. $\frac{Q_i}{Q_l} > \frac{p_i - p_0}{p_l - p_0}$ for all $i \neq l$
2. $\frac{Q_{l+1} - Q_l}{p_{l+1} - p_l} > \frac{Q_l}{p_l - p_0}$
3. $\frac{Q_{i+1} - Q_i}{p_{i+1} - p_i} > \frac{Q_i - Q_{i-1}}{p_i - p_{i-1}}$ for all $i > l$

for some $l \in \{1, ..., n\}$ has a first transition from 0 to l and then all $n - l$ subsequent transitions, where Q_i is defined appropriate for the non-strategic case or the agency case.

Proof. From Lemma 3.4, we know that if $\frac{Q_i}{Q_l} > \frac{p_i - p_0}{p_l - p_0}$ for all $i \neq l$, then $u(l, v_{0,l}) > u(i, v_{0,l})$ for all $i \neq l$. By Remark 3.3, since l is the optimal contract at $v_{0,l}$, for any $v > v_{0,l}$, it must be the case that the optimal contract involves contracting with at least l agents. Likewise, since 0 is optimal at $v_{0,l}$, by Remark 3.3, if i were optimal for any $v < v_{0,l}$, then 0 could not be optimal at $v_{0,l}$. Therefore we know that for all $v < v_{0,l}$, contracting with 0 agents is the only optimal contract. Since at $v_{0,l}$, the only optimal contracts are 0 and l, there is no value of v for which it is optimal to contract with $i \in 1, 2, ..., l-1$ agents. Thus the technology function exhibits a jump between contracting between 0 agents and contracting with l agents.

From Lemma 3.5, we know that if $\frac{Q_{i+1} - Q_i}{p_{i+1} - p_i} > \frac{Q_i - Q_{i-1}}{p_i - p_{i-1}}$ for all $i > l$, then $u(i, v_{i-1,i}) > u(i+1, v_{i,i+1})$ for all $i > l$. Also, we know that the statement $\frac{Q_{i+1} - Q_i}{p_{i+1} - p_i} > \frac{Q_i - Q_{i-1}}{p_i - p_{i-1}}$ for all $i > l$, is equivalent to $v_{i,i+1} > v_{i-1,i}$ for all $i > l$. Since $\frac{Q_{l+1} - Q_l}{p_{l+1} - p_l} > \frac{Q_l}{p_l - p_0}$, we also know that $v_{l,l+1} > v_{0,l}$. In what follows, we show that for any $v \in (v_{i-1,i}, v_{i,i+1})$ for some $i > l$, contracting with exactly i agents is the only optimal contract. If we combine this with the fact that 0 is optimal for all $v \leq v_{0,l}$ and the fact that n is optimal for all $v \geq v_{n-1,n}$ we get that the first transition occurs from 0 to l and all remaining $n - l$ transitions occur.

Now consider any value $v \in (v_{0,l}, v_{l,l+1})$, we know that contracting with l agents yields higher utility to the principal than contracting with $j < l$ agents from above. Likewise, consider any value $v \in (v_{l,l+1}, v_{l+1,l+2})$, we know know by the definition of $v_{l,l+1}$, that contracting with $l+1$ agents is strictly better than contracting with l agents for all $v > v_{l,l+1}$ and we know that contracting with $l+1$ agents is strictly better than contracting with j agents, where $j < l$, since contracting with l is strictly better than contracting with $j < l$ agents for all $v > v_{0,l}$. Now we will proceed inductively (much like the proof of Theorem 9 in [7]), as follows: consider any value $v \in (v_{i,i+1}, v_{i+1,i+2})$ for any $i > l$, we know that contracting with $i+1$ is strictly better than contracting with i for all $v > v_{i,i+1}$. We know that contracting with $i+1$ agents is strictly better than contracting with j agents, where $j < i$, because the induction hypothesis gives us that contracting with i agents is strictly better than contracting with $j < i$ agents.

Now we induct backwards as in [7]. Consider the $v > v_{n-1,n}$, we know that contracting with n agents has strictly greater utility than contracting with $j > n$ agents (trivially true). Now consider $v \in (v_{i-1,i}, v_{i,i+1})$ for all $l < i < n$, contracting with i agents is strictly better than contracting with $i+1$ agents by the definition of $v_{i,i+1}$ and by the induction hypothesis, we know that contracting with i agents is strictly better than contracting with $j > i+1$ agents. Now consider $v \in (v_{0,l}, v_{l,l+1})$, we know that contracting with l agents is strictly better than contracting with $l+1$ agents and all $j > l+1$ agents, by the induction hypothesis. Finally consider $v \in (0, v_{0,l})$, we know that contracting with 0 agents is strictly better than contracting with $j \in \{1, 2, ..., l\}$ agents from above. The induction hypothesis gives us that contracting with 0 agents is strictly better than contracting with $j > l$ agents.

Combining the two inductive arguments gives us that contracting with 0 agents is optimal for $v \in (0, v_{0,l})$, contracting with l agents is optimal for $v \in (v_{0,l}, v_{l,l+1})$ and contracting with $i+1$ agents is optimal for $v \in (v_{i,i+1}, v_{i+1,i+2})$ for all $i \geq l$. □

Lemma 3.7 Any anonymous technology that has a probability of success function that satisfies:

1. $\frac{p_i - p_0}{i} > \frac{p_{i-1} - p_0}{i-1}$ for all $2 \leq i \leq l$ and $\frac{p_i - p_0}{i} < \frac{p_{i-1} - p_0}{i-1}$ for all $i > l$
2. $\frac{1}{p_{i+1} - p_i} > \frac{1}{p_i - p_{i-1}}$ for all $i > l$

for some $l \in \{1, ..., n\}$ has a first transition from 0 to l and then all $n - l$ subsequent transitions for the nonstrategic version of the problem.

Proof. We show that the conditions of Lemma 3.6 are satisfied. Since $Q_i = i \cdot c$ for the nonstrategic case, the condition that $\frac{Q_i}{Q_l} > \frac{p_i - p_0}{p_l - p_0}$ for all $i \neq l$ is equivalent to $\frac{i}{l} > \frac{p_i - p_0}{p_l - p_0}$ for all $i \neq l$ or $\frac{p_l - p_0}{l} > \frac{p_i - p_0}{i}$, for all $i \neq l$. The latter is clearly satisfied by condition 1 of this Lemma. The condition $\frac{Q_{i+1} - Q_i}{p_{i+1} - p_i} > \frac{Q_i - Q_{i-1}}{p_i - p_{i-1}}$ for all $i > l$ is equivalent to $\frac{1}{p_{i+1} - p_i} > \frac{1}{p_i - p_{i-1}}$ for all $i > l$, which is equivalent to condition 3 of this Lemma. The condition $\frac{Q_{l+1} - Q_l}{p_{l+1} - p_l} > \frac{Q_l}{p_l - p_0}$ is equivalent to $\frac{1}{p_{l+1} - p_l} > \frac{l}{p_l - p_0}$. Since condition 1 of this Lemma gives us $\frac{p_l - p_0}{l} > \frac{p_{l+1} - p_0}{l+1}$, we know $\frac{p_l - p_0}{l} > p_{l+1} - p_l$, which gives us the desired result. □

Lemma 3.9 If the probability of success function is unimodal over the set $\{1, ..., n\}$, then we know that $f(i)$ is also unimodal.

Proof. If $p_i - p_{i-1} > p_{i-1} - p_{i-2}$ for all $2 \leq i \leq n$, then $\frac{p_i - p_0}{i} > \frac{p_{i-1} - p_0}{i-1}$ for all $2 \leq i \leq n$ as well. Likewise, if $p_i - p_{i-1} < p_{i-1} - p_{i-2}$ for all $2 \leq i \leq n$, then $\frac{p_i - p_0}{i} < \frac{p_{i-1} - p_0}{i-1}$ for all $2 \leq i \leq n$ as well. Finally, consider the case that $p_i - p_{i-1} > p_{i-1} - p_{i-2}$ for all $2 \leq i \leq j$ and $p_i - p_{i-1} < p_{i-1} - p_{i-2}$ for all $i > j$. Since $p_i - p_{i-1} > p_{i-1} - p_{i-2}$ for all $2 \leq i \leq j$, we know that $\frac{p_i - p_0}{i} > \frac{p_{i-1} - p_0}{i-1}$ for all $2 \leq i \leq j$. Now consider the smallest value of l for which $\frac{p_l - p_0}{l} < \frac{p_{l-1} - p_0}{l-1}$. Note that if $p_l - p_{l-1} > p_{l-1} - p_{l-2} > ... > p_1 - p_0$, it must be the case that $\frac{p_l - p_0}{l} > \frac{p_{l-1} - p_0}{l-1}$, so therefore we know that $p_l - p_{l-1} < p_{l-1} - p_{l-2}$. We also know that $p_l - p_{l-1} < \frac{p_{l-1} - p_0}{l-1}$. Since $p_{l+1} - p_l < p_l - p_{l-1} < \frac{p_{l-1} - p_0}{l-1}$, we know that $p_{l+1} - p_l < \frac{p_l - p_0}{l}$, and therefore $\frac{p_{l+1} - p_0}{l+1} < \frac{p_l - p_0}{l}$. Applying this reasoning inductively, we get the desired result. □

Corollary 3.10 For any anonymous technology function (p, c) that has a unimodal probability of success, there exists an $1 \leq l \leq n$ such that the first transition in the non-strategic case is from 0 to l agents (where l is the smallest value that satisfies $\frac{p_l - p_0}{l} > \frac{p_{l+1} - p_0}{l+1}$) followed by all remaining $n - l$ transitions.

Proof. It suffices to show that the conditions of Lemma 3.7 are met. We know from Lemma 3.9, that $f(i)$ is unimodal, so condition 1 is satisfied. We also know from the proof of Lemma 3.9, that if $\frac{p_i - p_0}{i} < \frac{p_{i-1} - p_0}{i-1}$, then $p_i - p_{i-1} < p_{i-1} - p_{i-2}$. Since $\frac{p_i - p_0}{i} < \frac{p_{i-1} - p_0}{i-1}$ for all $i > l$, $p_i - p_{i-1} < p_{i-1} - p_{i-2}$ for all $i > l$ and condition 2 is satisfied. □

Lemma 3.11 The probability of success function for any threshold technology is unimodal.

Proof. Denote the probability of success when contracting with j agents as $P(n, j, \geq k)$. More specifically, let $P(n, j, \geq k)$ denote the probability of success when you contract with j agents out n and at least k succeed. Note that, $\Delta_{j+1} = P(n, j+1, \geq k) - P(n, j, \geq k) = (\beta - \alpha) \cdot (P(n-1, j, \geq k-1) - P(n-1, j, \geq k)) = (\beta - \alpha) \cdot P(n-1, j, = k-1)$, where $P(n-1, j, = k-1)$ is the probability that exactly $k-1$ agents succeed when j agents are contracted out of the $n-1$. Note that: $P(n-1, j+1, = k-1) - P(n-1, j, = k-1) = (\beta - \alpha) \cdot (P(n-2, j, = k-2) - P(n-2, j, = k-1))$. Note that the discrete distribution: $P(n-2, j, 0), P(n-2, j, 1), ..., P(n-2, j, n-2)$ is the convolution of two binomial random variables. Since binomial random variables are strongly unimodal and the convolution of any two strongly unimodal function is also strongly unimodal, we know that the distribution: $P(n-2, j, 0), P(n-2, j, 1), ..., P(n-2, j, n-2)$ is strongly unimodal. Note that if $(P(n-2, j, = k-2) < P(n-2, j, = k-1))$, this means the mode of this distribution is greater than $k-1$. Therefore the mode of the distribution, $P(n-2, j+1, 0), P(n-2, j+1, 1), ..., P(n-2, j+1, n-2)$, is also greater than $k-1$, so we know $(P(n-2, j+1, = k-2) < P(n-2, j+1, = k-1))$. Hence we know if $P(n-1, j+1, = k-1) < P(n-1, j, = k-1)$, $P(n-1, j+2, = k-1) < P(n-1, j+1, = k-1)$, which gives us that Δ_j is a unimodal function. □

Lemma 3.13 The discrete valued function, $\frac{Q_i}{p_i - p_0}$, is convex.

Proof. Since $\frac{1}{\Delta_i}$ is log-convex (Lemma A.1) and $\frac{p_i}{p_i - p_0}$ is log-convex (Lemma A.3), we know that $\frac{p_i}{\Delta_i(p_i - p_0)}$ is also log-convex. Since a log convex function is also convex, we know that $\frac{p_{i+1}}{\Delta_{i+1}(p_{i+1}-p_0)} - \frac{p_i}{\Delta_i(p_i - p_0)} > \frac{p_i}{\Delta_i(p_i - p_0)} - \frac{p_{i-1}}{\Delta_{i-1}(p_{i-1}-p_0)}$ for all i. Therefore we know that $\frac{p_{i+1}\cdot(i+1)}{\Delta_{i+1}(p_{i+1}-p_0)} - \frac{p_i\cdot(i+1)}{\Delta_i(p_i - p_0)} > \frac{p_i\cdot(i-1)}{\Delta_i(p_i - p_0)} - \frac{p_{i-1}}{\Delta_{i-1}(p_{i-1}-p_0)}$. Adding $\frac{p_i}{\Delta_i(p_i - p_0)}$ to both sides, $\frac{p_{i+1}\cdot(i+1)}{\Delta_{i+1}(p_{i+1}-p_0)} - \frac{p_i \cdot i}{\Delta_i(p_i - p_0)} > \frac{p_i \cdot i}{\Delta_i(p_i - p_0)} - \frac{p_{i-1}}{\Delta_{i-1}(p_{i-1}-p_0)}$, as desired. □

Lemma 3.14 There exists a value of $1 \leq l_a \leq n$ such that $\frac{Q_i}{Q_{l_a}} > \frac{p_i - p_0}{p_{l_a} - p_0}$ for all $i \neq l_a$.

Proof. Since $\frac{Q_i}{p_i - p_0}$ is convex, $\frac{Q_{i+1}}{p_{i+1}-p_0} - \frac{Q_i}{p_i - p_0} > \frac{Q_i}{p_i - p_0} - \frac{Q_{i-1}}{p_{i-1}-p_0}$ for all i. Therefore if $\frac{Q_i}{p_i - p_0} - \frac{Q_{i-1}}{p_{i-1}-p_0} > 0$, then $\frac{Q_{i+1}}{p_{i+1}-p_0} - \frac{Q_i}{p_i - p_0} > 0$. Let l_a be the smallest l such that $\frac{Q_{l+1}}{p_{l+1}-p_0} - \frac{Q_l}{p_l - p_0}$, therefore we know that $\frac{Q_i}{p_i - p_0} > \frac{Q_{i+1}}{p_{i+1}-p_0}$ for all $1 \leq i < l_a$ and $\frac{Q_i}{p_i - p_0} < \frac{Q_{i+1}}{p_{i+1}-p_0}$ for all $l_a \leq i < n$, so $\frac{Q_i}{Q_{l_a}} > \frac{p_i - p_0}{p_{l_a} - p_0}$ for all $i \neq l_a$. If $\frac{Q_i}{p_i - p_0} - \frac{Q_{i-1}}{p_{i-1}-p_0} < 0$ for all i, then $\frac{Q_1}{p_1 - p_0} > \frac{Q_2}{p_2 - p_0} > ... > \frac{Q_n}{p_n - p_0}$, so $\frac{Q_i}{Q_1} > \frac{p_i - p_0}{p_1 - p_0}$ for all $i \neq 1$. □

Lemma 3.16 We have $\frac{Q_{i+1} - Q_i}{p_{i+1} - p_i} > \frac{Q_i - Q_{i-1}}{p_i - p_{i-1}}$ for all $i > l_a$ where l_a is the smallest value such that $\frac{Q_{l_a}}{p_{l_a} - p_0} < \frac{Q_{l_a+1}}{p_{l_a+1}}$.

Proof. We know for all $i \geq l_a$, $\frac{Q_i}{p_i - p_0} < \frac{Q_{i+1}}{p_{i+1}-p_0}$ or in other words $\frac{p_i \cdot i}{\Delta_i(p_i - p_0)} < \frac{p_{i+1}\cdot(i+1)}{\Delta_{i+1}(p_{i+1}-p_0)}$. Since $\frac{p_i}{p_i - p_0} > \frac{p_{i+1}}{p_{i+1}-p_0}$ for any value of i, we know that $\frac{i}{\Delta_i} < \frac{i+1}{\Delta_{i+1}}$. Note that if $\frac{p_i - p_0}{i} > \frac{p_{i+1}-p_0}{i+1}$, it must be that $\Delta_{i+1} < \Delta_i$, since p is unimodal, so $\frac{p_{i+1}}{\Delta_{i+1}} > \frac{p_i}{\Delta_i}$. If $\frac{p_i - p_0}{i} \leq \frac{p_{i+1}-p_0}{i+1}$, then it must be that $\frac{\Delta_i}{p_i} > \frac{\Delta_{i+1}}{p_{i+1}}$ or in other words, $\frac{p_{i+1}}{\Delta_{i+1}} > \frac{p_i}{\Delta_i}$. Note that if $\frac{p_{i+1}}{\Delta_{i+1}} > \frac{p_i}{\Delta_i}$, then $\frac{p_{i+1}}{\Delta_{i+1}} \cdot \frac{p_i}{p_{i+1}} > \frac{p_i}{\Delta_i} \cdot \frac{p_{i-1}}{p_i}$ or in other words $\frac{p_i}{\Delta_{i+1}} > \frac{p_{i-1}}{\Delta_i}$, since p is log-concave.

Since $\frac{p_i}{p_i - p_0} > \frac{p_{i+1}}{p_{i+1}-p_0}$, it must be that $\frac{i+1}{\Delta_{i+1}} > \frac{i}{\Delta_i}$ if $\frac{p_i \cdot i}{\Delta_i(p_i - p_0)} < \frac{p_{i+1}\cdot(i+1)}{\Delta_{i+1}(p_{i+1}-p_0)}$. Thus $\frac{i+1}{\Delta_{i+1}} > \frac{i}{\Delta_i}$ for all $i \geq l_a$.

We know from Lemma A.1 that $\frac{1}{\Delta_i}$ is a log-convex function so therefore $\frac{1}{\Delta_{i+1}} - \frac{1}{\Delta_i} > \frac{1}{\Delta_i} - \frac{1}{\Delta_{i-1}}$ for all i. Therefore we know that $\frac{i+1}{\Delta_{i+1}} - \frac{i+1}{\Delta_i} > \frac{i-1}{\Delta_i} - \frac{i-1}{\Delta_{i-1}}$ for all i. Adding $\frac{1}{\Delta_i}$ to both sides we get that $\frac{i+1}{\Delta_{i+1}} - \frac{i}{\Delta_i} > \frac{i}{\Delta_i} - \frac{i-1}{\Delta_{i-1}}$ for all i. Combining this with the fact that $\frac{p_{i+1}}{\Delta_{i+1}} > \frac{p_i}{\Delta_i}$ for all $i \geq l*$ and $\frac{i+1}{\Delta_{i+1}} > \frac{i}{\Delta_i}$ for all $i \geq l_a$, we get that $\frac{p_i}{\Delta_{i+1}}(\frac{i+1}{\Delta_{i+1}} - \frac{i}{\Delta_i}) + \frac{i+1}{\Delta_{i+1}} > \frac{p_{i-1}}{\Delta_i}(\frac{i}{\Delta_i} - \frac{i-1}{\Delta_{i-1}}) + \frac{i}{\Delta_i}$ for all $i \geq l_a$ as desired. □

Lemma A.1. Δ_i *is log-concave.*

Proof. $\Delta_i = p_i - p_{i-1} = P(n, i, \geq k) - P(n, i-1, \geq k)$, where $P(n, i, \geq k)$ is the probability that at least k agents succeed when i succeed with probability β and $n - i$ succeed with probability of α.

Note that $P(n, i, \geq k) - P(n, i-1, \geq k) = (\beta - \alpha) \cdot (P(n-1, i-1, \geq k-1) - P(n-1, i-1, \geq k)) = (\beta - \alpha) \cdot P(n-1, i-1, = k-1)$, where $P(n-1, i-1, = k-1)$ is the probability that exactly $k-1$ agents succeed when $i-1$ agents succeed with probability β and $n-i$ succeed with probability α.

We abbreviate the following: $f_{i+1} = P(n-1, i+1, = k-1)$, $f_i = P(n-1, i, = k-1)$ and $f_{i-1} = P(n-1, i-1, = k-1)$. It suffices to show that: $f_i^2 \geq f_{i+1} f_{i-1}$.

We can write:

$f_{i+1} = \beta \cdot P(n-2, i, = k-2) + (1-\beta) \cdot P(n-2, i, = k-1)$

$f_i = \alpha \cdot P(n-2, i, = k-2) + (1-\alpha) \cdot P(n-2, i, = k-1)$
$f_i = \beta \cdot P(n-2, i-1, = k-2) + (1-\beta) \cdot P(n-2, i-1, = k-1)$
$f_{i-1} = \alpha \cdot P(n-2, i-1, = k-2) + (1-\alpha) \cdot P(n-2, i-1, = k-1)$

Note that:

$f_{i+1}f_{i-1} = \alpha\beta P(n-2, i, = k-2)P(n-2, i-1, = k-2) + \alpha(1-\beta)P(n-2, i, = k-1)P(n-2, i-1, = k-2) + \beta(1-\alpha)P(n-2, i, = k-2)P(n-2, i-1, = k-1) + (1-\alpha)(1-\beta)P(n-2, i, = k-1)P(n-2, i-1, = k-1)$

Also note that:

$f_i f_i = \alpha\beta P(n-2, i, = k-2)P(n-2, i-1, = k-2) + \alpha(1-\beta)P(n-2, i, = k-2)P(n-2, i-1, = k-1) + \beta(1-\alpha)P(n-2, i, = k-1)P(n-2, i-1, = k-2) + (1-\alpha)(1-\beta)P(n-2, i, = k-1)P(n-2, i-1, = k-1)$

$f_i^2 - f_{i-1}f_{i+1} = (\beta-\alpha)(P(n-2, i, = k-1)P(n-2, i-1, = k-2) - P(n-2, i-1, = k-1)P(n-2, i, = k-2))$.

Note that we can then write:

$P(n-2, i, = k-1) = \beta P(n-3, i-1, = k-2) + (1-\beta)P(n-3, i-1, = k-1)$
$P(n-2, i-1, = k-1) = \alpha P(n-3, i-1, = k-2) + (1-\alpha)P(n-3, i-1, = k-1)$
$P(n-2, i, = k-2) = \beta P(n-3, i-1, = k-3) + (1-\beta)P(n-3, i-1, = k-2)$
$P(n-2, i-1, = k-2) = \alpha P(n-3, i-1, = k-3) + (1-\alpha)P(n-3, i-1, = k-2)$

So we can write:
$P(n-2, i, = k-1)P(n-2, i-1, = k-2) = (\beta P(n-3, i-1, = k-2) + (1-\beta)P(n-3, i-1, = k-1))(\alpha P(n-3, i-1, = k-3) + (1-\alpha)P(n-3, i-1, = k-2)) = \alpha\beta P(n-3, i-1, = k-2)P(n-3, i-1, = k-3) + \beta(1-\alpha)P(n-3, i-1, = k-2)P(n-3, i-1, = k-2) + \alpha(1-\beta)P(n-3, i-1, = k-1)P(n-3, i-1, = k-3) + (1-\alpha)(1-\beta)P(n-3, i-1, = k-1)P(n-3, i-1, = k-2)$

And:
$P(n-2, i-1, = k-1)P(n-2, i, = k-2) = (\alpha P(n-3, i-1, = k-2) + (1-\alpha)P(n-3, i-1, = k-1))(\beta P(n-3, i-1, = k-3) + (1-\beta)P(n-3, i-1, = k-2)) = \alpha\beta P(n-3, i-1, = k-2)P(n-3, i-1, = k-3) + \beta(1-\alpha)P(n-3, i-1, = k-3)P(n-3, i-1, = k-1) + \alpha(1-\beta)P(n-3, i-1, = k-2)P(n-3, i-1, = k-2) + (1-\alpha)(1-\beta)P(n-3, i-1, = k-1)P(n-3, i-1, = k-2)$

So $(P(n-2, i, = k-1)P(n-2, i-1, = k-2) - P(n-2, i-1, = k-1)P(n-2, i, = k-2)) = (\beta-\alpha)(P(n-3, i-1, = k-2)P(n-3, i-1, = k-2) - P(n-3, i-1, = k-1)P(n-3, i-1, = k-3))$

Note that $P(n, j, k)$ for fixed n, k is strongly unimodal since it is the convolution of two binomial random variables, which are also strongly unimodal. Therefore we know that $P(n-3, i-1, = k-2)P(n-3, i-1, = k-2) - P(n-3, i-1, = k-1)P(n-3, i-1, = k-3) > 0$, so $f_j^2 - f_{j-1}f_{j+1} > 0$ for all $n > 3$ and all $n > k > 2$.

Now we address the $k = 2$ case. When $k = 2$:

$f_{i+1} = \beta \cdot (1-\beta)^i(1-\alpha)^{n-i-2} + (1-\beta) \cdot P(n-2, i, = k-1)$
$f_i = \alpha \cdot (1-\beta)^i(1-\alpha)^{n-i-2} + (1-\alpha) \cdot P(n-2, i, = k-1)$
$f_i = \beta \cdot (1-\beta)^{i-1}(1-\alpha)^{n-i-1} + (1-\beta) \cdot P(n-2, i-1, = k-1)$
$f_{i-1} = \alpha \cdot (1-\beta)^{i-1}(1-\alpha)^{n-i-1} + (1-\alpha) \cdot P(n-2, i-1, = k-1)$

Note that:

$f_{i+1}f_{i-1} = \beta \cdot (1-\beta)^i(1-\alpha)^{n-i-2}\alpha \cdot (1-\beta)^{i-1}(1-\alpha)^{n-i-1} + \alpha(1-\beta)P(n-2, i, = k-1)(1-\beta)^{i-1}(1-\alpha)^{n-i-1} + \beta(1-\alpha)(1-\beta)^i(1-\alpha)^{n-i-2}P(n-2, i-1, = k-1) + (1-\alpha)(1-\beta)P(n-2, i, = k-1)P(n-2, i-1, = k-1)$

Also note that:

$f_i f_i = \alpha \cdot (1-\beta)^i(1-\alpha)^{n-i-2}\beta \cdot (1-\beta)^{i-1}(1-\alpha)^{n-i-1} + \alpha(1-\beta)(1-\beta)^i(1-\alpha)^{n-i-2}P(n-2, i-1, = k-1) + \beta(1-\alpha)P(n-2, i, = k-1)(1-\beta)^{i-1}(1-\alpha)^{n-i-1} + (1-\alpha)(1-\beta)P(n-2, i, = k-1)P(n-2, i-1, = k-1)$

$f_i^2 - f_{i-1}f_{i+1} = (\beta-\alpha)(P(n-2,i,=k-1)(1-\beta)^{i-1}(1-\alpha)^{n-i-1} - P(n-2,i-1,=k-1)(1-\beta)^i(1-\alpha)^{n-i-2}) = (\beta-\alpha)^2(1-\beta)^{i-1}(1-\alpha)^{n-i-2}P(n-2,i,=k-1) > 0$.

Finally we consider the case that $n=3$ (and $k=2$ necessarily). For the $n=3$ case:

$P(3,1,\geq 2) - P(3,0,\geq 2) = (\beta-\alpha)P(2,0,=1) = 2\alpha(1-\alpha)$
$P(3,2,\geq 2) - P(3,1,\geq 2) = (\beta-\alpha)P(2,1,=1) = \beta(1-\alpha)+\alpha(1-\beta)$
$P(3,3,\geq 2) - P(3,2,\geq 2) = (\beta-\alpha)P(2,2,=1) = 2\beta(1-\beta)$

We know that: $(\beta(1-\alpha)-\alpha(1-\beta)) > 0$
$\beta^2(1-\alpha)^2 + 2\alpha\beta(1-\alpha)(1-\beta) + \alpha^2(1-\beta)^2 > 4\alpha\beta(1-\alpha)(1-\beta)$
$(P(3,2,\geq 2) - P(3,1,\geq 2))^2 > (P(3,1,\geq 2) - P(3,0,\geq 2))(P(3,3,\geq 2) - P(3,2,\geq 2))$

Now consider the case that $k=1$. When $k=1$, $p_i - p_{i-1} = (1-(1-\beta)^i(1-\alpha)^{n-i}) - (1-(1-\beta)^{i-1}(1-\alpha)^{n-i+1}) = (1-\beta)^{i-1}(1-\alpha)^{n-i}(\beta-\alpha$. Therefore $\Delta_i/\Delta_{i-1} = \frac{1-\beta}{1-\alpha}$ for any i and Δ_i is a log-concave function.

Now consider the case that $k=n$. When $k=n$, $p_i - p_{i-1} = \beta^i\alpha^{n-i} - \beta^{i-1}\alpha n - i + 1 = \beta^{i-1}\alpha^{n-i}(\beta-\alpha)$. Therefore $\Delta_i/\Delta_{i-1} = \frac{\beta}{\alpha}$ for any i and Δ_i is a log-concave function. □

Lemma A.2. *p_i is log-concave.*

Proof. Since Δ_i is a discrete function, we know that $\Delta_{i+1}\Delta_1 - \Delta_i\Delta_2 + \Delta_{i+1}\Delta_2 - \Delta_i\Delta_3 + ... + \Delta_{i+1}\Delta_{i-1} - \Delta_i\Delta_i < 0$ so $\Delta_{i+1}\Delta_1 - \Delta_i\Delta_2 + \Delta_{i+1}\Delta_2 - \Delta_i\Delta_3 + ... + \Delta_{i+1}\Delta_{i-1} - \Delta_i\Delta_i - \Delta_i\Delta_1 < 0$. In other words, we know $\Delta_{i+1}(\Delta_1+...+\Delta_i) - \Delta_i(\Delta_1+...+\Delta_i) - \Delta_i\Delta_{i+1}$, or $\Delta_{i+1}p_i - \Delta_i p_i - \Delta_i\Delta_{i+1} < 0$ or $p_i^2 > (p_i+\Delta_{i+1})(p_i-\Delta_i) = p_{i+1}p_{i-1}$, as desired. □

Lemma A.3. *$\frac{p_i}{p_i-p_0}$ is log-convex.*

Proof. Since p is log concave, we know that $p_i^2 > p_{i+1}p_{i-1} = (p_i+\Delta_{i+1})(p_i-\Delta_i)$ or in other words $\Delta_i\Delta_{i+1} + \Delta_i p_i - \Delta_{i+1}p_i > 0$, which gives us that $p_0(p_i-p_0)(\Delta_i\Delta_{i+1} + \Delta_i p_i - \Delta_{i+1}p_i > 0)$ and that $p_0((2p_i-p_0)\Delta_i\Delta_{i+1} - (p_i-p_0)(\Delta_i p_i - \Delta_{i+1}p_i)) > 0$, which means that $p_i^2 - (p_i-p_0)^2)\Delta_i\Delta_{i+1} - p_0p_i(p_i-p_0)(\Delta_i p_i - \Delta_{i+1}p_i) > 0$. Therefore we have: $(p_i-p_0)^2(\Delta_{i+1}p_i - \Delta_i p_i) - p_i^2(\Delta_{i+1}(p_i-p_0) - \Delta_i(p_i-p_0)) + p_i^2 - (p_i-p_0)^2)\Delta_i\Delta_{i+1} > 0$, so $(p_i-p_0)^2(p_i^2 + \Delta_{i+1}p_i - \Delta_i - \Delta_i\Delta_{i+1}) > p_i^2(p_i-p_0)^2 + \Delta_{i+1}(p_i-p_0-\Delta_i)p_i^2 - \Delta_i(p_i-p_0)p_i^2$, or $(p_i-p_0)^2(p_i+\Delta_{i+1})(p_i-\Delta_i) > p_i^2(p_i-p_0+\Delta_{i+1})(p_i-p_0-\Delta_i)$, which gives us $(\frac{p_i-p_0}{p_i})^2 > \frac{p_{i+1}-p_0}{p_{i+1}}\frac{p_{i-1}-p_0}{p_{i-1}}$, as desired. □

B Proofs from Section 4

Lemma 4.2 For the OR technology, the price of unaccountability occurs at a transition in the agency case, as opposed to a transition in the non-strategic case.

Proof. It suffices to show that for a fixed agency contract, the social welfare ratio is increasing as v increases. First consider the OR technology. We know that for all $i < j$, $\frac{p_j-p_0}{j} < \frac{p_i-p_0}{i}$. Therefore we have that for all $i < j$, $\frac{p_j}{j} < \frac{p_i}{i}$. If $ip_j < jp_i$, we have:

$$
\begin{aligned}
jp_i(v'-v) &> ip_j(v'-v) \text{ for any } v' > v \\
-jp_iv - ip_jv' &> -jp_iv' - ip_jv \\
-jcp_iv - icp_jv' &> -jcp_iv' - icp_jv \\
p_ivp_jv' - jcp_iv - icp_jv' + icjc &> p_iv'p_jv - jcp_iv' - icp_jv + icjc \\
(p_jv' - jc)(p_iv - ic) &> (p_jv - jc)(p_iv' - ic) \\
\frac{p_jv' - jc}{p_iv' - ic} &> \frac{p_jv - jc}{p_iv - ic} \text{ for any } v' > v
\end{aligned}
$$

Therefore, we know for fixed non-strategic and fixed agency contracts, the social welfare ratio increases and v increases. Finally, we know that $\frac{p_{j+1}v^* - (j+1)c}{p_iv^* - ic} = \frac{p_jv^* - jc}{p_iv^* - ic}$, where v^* is the point at which a principal is indifferent between contracting between j agents and contracting with $j+1$ agents. Therefore we know, for fixed agency contract, the social welfare ratio is increasing as v increases. □

Theorem 4.5 The POU for the OR technology is bounded by 2 for all $\alpha, \beta = 1 - \alpha$ and n.

Proof. To establish this result, it suffices to show that the social welfare ratio is bounded everywhere by 2. Given Lemma 4.2, it suffices to consider only transition points in the agency case, so let us consider the social welfare ratio at a transition in the agency case. Let us consider the social welfare ratio at $v_{i,i+1}$, where the principal is indifferent between contracting with i agents and $i+1$ agents in the agency case. Also suppose at $v_{i,i+1}$, the optimal non-strategic contract is j, where $n \geq j > i$. Therefore, we can write the social welfare ratio as: $\frac{p_jv_{i,i+1} - j}{p_iv_{i,i+1} - i}$. We want to show that $\frac{p_jv_{i,i+1} - j}{p_iv_{i,i+1} - i} \leq 2$. If $2p_i - p_j > 0$, then this statement is equivalent to $v_{i,i+1} \geq \frac{2i - j}{2p_i - p_j}$. If $2p_i - p_j < 0$, then this statement is equivalent to $v_{i,i+1} \leq \frac{j - 2i}{p_j - 2p_i}$.

First we consider the case that $2p_i - p_j > 0$. First suppose that $2i - j \leq 0$, then we know that $\frac{2i-j}{2p_i - p_j} < 0$, so $v_{i,i+1} \geq \frac{2i-j}{2p_i - p_j}$. Therefore it suffices to consider the case that $2i - j > 0$. Since the optimal non-strategic contract is j at $v_{i,i+1}$, we know that $\frac{1}{\Delta_{j+1}} \geq v_{i,i+1} \geq \frac{1}{\Delta_j}$. Therefore, it suffices to show that $\frac{1}{\Delta_j} \geq \frac{2i-j}{2p_i - p_j}$. Since $2i - j < j$ and $\Delta_j < \Delta_{j-1} < ... < \Delta_1$, we know that $\Delta_j < \frac{p_{2i-j}}{2i-j} = \frac{p_i - (p_i - p_{2i-j})}{2i-j} = \frac{p_i - (\Delta_i + \Delta_{i-1} + ... + \Delta_{2i-j+1})}{2i-j} < \frac{p_i - (\Delta_j + \Delta_{j-1} + ... + \Delta_{i+1})}{2i-j} = \frac{p_i - (p_j - p_i)}{2i-j} = \frac{2p_i - p_j}{2i-j}$.

Now we consider the case that $2p_i - p_j < 0$. Since the optimal non-strategic contract is j at $v_{i,i+1}$, we know that $\frac{1}{\Delta_{j+1}} \geq v_{i,i+1} \geq \frac{1}{\Delta_j}$. Therefore it suffices to show that $\frac{1}{\Delta_{j+1}} \leq \frac{j-2i}{p_j - 2p_i}$ or $p_j - 2p_i \leq (j-2i)(p_{j+1} - p_j)$. We can write $p_j = 1 - \theta^j q_0$, where $\theta = \frac{1-\beta}{1-\alpha}$ and $q_0 = (1-\alpha)^n$. Therefore it suffices to show that $(1 - \theta^j q_0) - 2(1 - \theta^i q_0) \leq (j - 2i)(1 - \theta^{j+1} q_0 - (1 - \theta^j q_0))$ or equivalently $-1 + 2\theta^i q_0 - \theta^j q_0 \leq (j - 2i)\theta^j q_0(1 - \theta)$ or equivalently, $\theta^i q_0(2 - \theta^{j-i} - (j - 2i)\theta^{j-i}(1 - \theta)) \leq 1$. Therefore it suffices to show that $\theta^i q_0 \leq \frac{1}{2}$. $\theta^i q_0 = (1 - \beta)^i (1 - \alpha)^{n-i}$. Since $1 - \beta = \alpha < \frac{1}{2}$, we know that $(1 - \beta)^i (1 - \alpha)^{n-i} < \frac{1}{2}$ for all $i > 0$.

Therefore the only remaining case is $i = 0$ and $2p_0 - p_j < 0$. We note that when $i = 0$, $2i - j < 0$, for any value of j. We also notice that if $2p_0 - 1 > 0$, then $2p_0 - p_j > 0$. In other words if $\alpha \geq 1 - \sqrt[n]{\frac{1}{2}}$, then $2p_0 - p_j > 0$. For $n \geq 3$, the RHS is at most 0.207, therefore if $\alpha > 0.207$ and $n \geq 3$, we know from above that $\frac{p_jv_{0,1} - j}{p_0v_{0,1}} \leq 2$. (The $n = 2$ case is established in [7]).

Now suppose that $\alpha \leq \frac{5-\sqrt{5}}{10}$. This means that $\alpha < 0.276$. If $\alpha \leq \frac{5-\sqrt{5}}{10}$, then:
$0 \leq 5 \cdot \alpha^2 - 5 \cdot \alpha + 1$

$2 \cdot \alpha - 2 \cdot \alpha^2 \leq \alpha^2 - \alpha + \alpha^2 + 1 - 2 \cdot \alpha + \alpha^2$
$2 \cdot \alpha \cdot (1-\alpha) \leq \alpha^2 - \alpha \cdot (1-\alpha) + (1-\alpha)^2$
$\frac{\alpha \cdot (1-\alpha)}{\alpha^2 - \alpha \cdot (1-\alpha) + (1-\alpha)^2} \leq \frac{1}{2}$
$\frac{\theta}{\theta^2 - \theta + 1} \leq \frac{1}{2}$, where $\theta = \frac{\alpha}{1-\alpha} = \frac{1-\beta}{1-\alpha}$

Since $q_0 > \frac{1}{2} \geq \frac{\theta}{\theta^2 - \theta + 1}$, so
$\theta \leq q_0 \cdot (\theta^2 - \theta + 1)$
$\theta - \theta^2 \cdot q_0 \leq q_0 - \theta \cdot q_0$
$(1 - \theta \cdot q_0) \cdot \theta \leq (1-\theta) \cdot q_0$
$(1 - \theta \cdot q_0)\theta(1-\theta)q_0 \leq (1-\theta)q_0(1-\theta)q_0$
$p_1(p_2 - p_1) \leq (p_1 - p_0)^2$ (eq. *)

We now use eq. * to prove inductively that $p_j p_1 - i \cdot (p_1 - p_0)^2 \leq 2p_1p_0$ Base case (j = 2): We combine $p_1^2 - (p_1-p_0)^2 \leq 2p_1p_0$ with eq. *, we get $p_1^2 - (p_1-p_0)^2 + p_1(p_2-p_1) \leq (p_1-p_0)^2 + 2p_1p_0$, or $p_1p_2 - 2(p_1-p_0)^2 \leq 2p_1p_0$.

Inductive Step: We are given that $p_jp_1 - j(p_1-p_0)^2 \leq 2p_1p_0$. If $p_1(p_2-p_1) \leq (p_1-p_0)^2$, $p_1(p_{j+1}-p_j) \leq (p_1-p_0)^2$ for all $j \geq 2$, because $p_{j+1} - p_j < p_2 - p_1$ (diminishing returns to scale of the OR function). Combining $p_1(p_{j+1}-p_j) \leq (p_1-p_0)^2$ with $p_jp_1 - j(p_1-p_0)^2 \leq 2p_1p_0$, we get $p_1p_j - j(p_1-p_0)^2 + p_1(p_{j+1}-p_j) \leq (p_1-p_0)^2 + 2p_1p_0$, or $p_1p_{j+1} - (j+1)(p_1-p_0)^2 \leq 2p_1p_0$. □

Theorem 4.7 The Price of Unaccountability for the threshold technology is not bounded for all values of $k \geq 2$ and n. More specifically, when $\alpha \to 0$, $POU \to \infty$.

Proof. We show that there exists a social welfare ratio that approaches ∞ as $\alpha \to 0$. This is sufficient to establish the desired result since the POU is described as the maximum social welfare ratio (where the maximum is taken over v). We focus on the social welfare ratio at the point of the first transition in the agency case. We can write this social welfare ratio as $\frac{p_j \cdot v - j \cdot c}{p_0 \cdot v}$, where v is the point of the first transition in the agency case and j is the optimal contract for the non-strategic case at v. We know from Lemma 4.8, that the first transition in the agency case jumps from 0 to $i \geq k$. We write $v = \frac{p_i \cdot i \cdot c}{(p_i - p_{i-1})(p_i - p_0)}$, where v is the point at where the principal is indifferent between contracting with 0 agents and i agents in the agency case. We know from Lemma 4.8 that $k \leq i \leq j$. Therefore, we can write the social welfare ratio at v as $\frac{p_jp_iic - jc(p_i-p_{i-1})(p_i-p_0)}{p_0p_iic}$. Note that this ratio is at least as big as $\frac{p_ip_iic - ic(p_i-p_{i-1})(p_i-p_0)}{p_0p_iic}$ or $\frac{p_ip_{i-1} + p_ip_0 - p_{i-1}p_0}{p_ip_0} = \frac{p_{i-1}}{p_0} + 1 - \frac{p_{i-1}}{p_i}$. As $\alpha \to 0$, we see that $\frac{p_{i-1}}{p_0} \to \infty$ and $1 \geq \frac{p_{i-1}}{p_i} \geq 0$, therefore this ratio approaches ∞. Hence, as $\alpha \to 0$, $\frac{p_jp_iic - jc(p_i-p_{i-1})(p_i-p_0)}{p_0p_iic} \to \infty$. Hence, the POU approaches ∞. □

Lemma 4.8 As $\alpha \to 0$, we know that $k \leq l_a \leq l_{ns}$, where l_a is the first transition in the agency case and l_{ns} is the first transition in the non-strategic case.

Proof. Recall that the first transition in the non-strategy case occurs to a value l_{ns} that satisfies $\underset{i}{\operatorname{argmin}} \frac{i \cdot c}{p_i - p_0}$. As $\alpha \to 0$, $p_i - p_0$ approaches 0, for all $i < k$. Therefore, $\frac{i \cdot c}{p_i - p_0} \to \infty$. As $\alpha \to 0$, $\frac{i \cdot c}{p_i - p_0}$ approaches a constant for $i \geq k$. Thus for sufficiently small α, the minimum occurs at an $l_{ns} \geq k$.

Now we focus on the agency case. Recall that the first transition occurs to a value l_a that satisfies $\underset{i}{\operatorname{argmin}} \frac{p_i \cdot i \cdot c}{(p_i - p_{i-1}) \cdot (p_i - p_0)}$. We note that as $\alpha \to 0$, $\frac{p_i}{p_i - p_0} \to 1$ for any i. Therefore as $\alpha \to 0$, this quantity approaches ∞ for all $i < k$. When $i = k$, as $\alpha \to 0$, this quantity becomes $\frac{k \cdot c}{p_k}$, which is a constant. When $i > k$, as $\alpha \to 0$, this quantity approaches $\frac{i \cdot c}{p_i - p_{i-1}}$. Thus the minimum value occurs at a value $l_a \geq k$. Combining this with Lemma 3.18, we know that $k \leq l_a \leq l_{ns}$. □

C Proofs from Section 5

Lemma 5.1 If the principal decides to contract with $j \cdot m + a$ agents for some $j \in Z^+$ and some $0 \leq a < m$, the probability of success is maximized by fully contracting j AND gates and contracting with a remaining agents on the same AND gate.

Proof. Consider an arbitrary contract of exactly $j \cdot m + a$ agents (that is not the case described above). Suppose that this contract specifies the following number of agents contracted on each AND gate $\hat{x} = (x_1, ..., x_q)$. Suppose that there exists an i and j such that $x_i \leq x_j < m$. We show that the contract $\hat{y} = (x_1, ..., x_i - 1, ..., x_j + 1, ..., x_q)$ has strictly greater success probability. We can write the success probability of the contract $\hat{y}$ as follows: $p(\hat{y}) = \beta^{x_j+1}\alpha^{m-x_j-1}\beta^{x_i-1}\alpha^{m-x_i+1}P(n-2, k-1) + \beta^{x_j+1}\alpha^{m-x_j-1}(1 - \beta^{x_i-1}\alpha^{m-x_i+1})P(n-2, k-1) + \beta^{x_i-1}\alpha^{m-x_i+1}(1 - \beta^{x_j+1}\alpha^{m-x_j-1})P(n-2, k-1) + (1 - \beta^{x_j+1}\alpha^{m-x_j-1})(1 - \beta^{x_i-1}\alpha^{m-x_i+1})P(n-2, k)$ And we can similarly write the probability of success for $\hat{x}$. We know that $\beta^{x_j}\alpha^{m-x_j-1}(\beta - \alpha)(P(n-2, k-1) - P(n-2, k)) > \beta^{x_i-1}\alpha^{m-x_i}(\beta - \alpha)(P(n-2, k-1) - P(n-2, k))$ for any $x_i \leq x_j$. Therefore, we know that $p(\hat{y}) > p(\hat{x})$, as desired. □

Lemma 5.2 For any principal's value v, the optimal contract involves a set of fully contracted AND gate.

Proof. Suppose that there exists a principal's value $v > 0$ and a j, where $j = i \cdot m + a$ for some $a > 0$, such that j is the optimal contract. In an abuse of notation, we say that the probability of success for this contract is $p(j)$. (Note that the preceding Lemma tells us exactly how these agents are contracted). If there exists a $v > 0$ such that this contract is optimal then $p(j)v - j > p(j-1)v - (j-1)$, or $v > \frac{1}{p(j)-p(j-1)}$, where $p(j-1)$ is the probability of success of contracting with $j-1$ agents. In what follows, we show that $p(j+1) - p(j) > p(j) - p(j-1)$. We can write $p(j) = \beta^a\alpha^{m-a}P(n-1, k-1) + (1 - \beta^a\alpha^{m-a})P(n-1, k) = P(n-1, k) + \beta^a\alpha^{m-a}(P(n-1, k-1) - P(n, k-1))$. Therefore, $p(j+1) = P(n-1, k) + \beta^{a+1}\alpha^{m-a-1}(P(n-1, k-1) - P(n, k-1))$ and $p(j-1) = P(n-1, k) + \beta^{a-1}\alpha^{m-a+1}(P(n-1, k-1) - P(n, k-1))$. Hence $p(j+1) - p(j) = \beta^{a+1}\alpha^{m-a-1}(P(n-1, k-1) - P(n, k-1)) - \beta^a\alpha^{m-a}(P(n-1, k-1) - P(n, k-1)) = \beta^a\alpha^{m-a-1}(\beta - \alpha)(P(n-1, k-1) - P(n, k-1))$ and $p(j) - p(j-1) = \beta^{a-1}\alpha^{m-a}(\beta - \alpha)(P(n-1, k-1) - P(n, k-1))$, which gives us $p(j+1) - p(j) > p(j) - p(j-1)$ as desired. Therefore if $v > \frac{1}{p(j)-p(j-1)}$, then this means $v > \frac{1}{p(j+1)-p(j)}$, and j cannot be the optimal contract. □

Theorem 5.3 The transition behavior for the majority-of-AND technology in the non-strategic case has a first transition to l fully contracted AND gates, where $1 \leq l \leq n$, followed by each subsequent transition of fully contracted AND gates.

Proof. From Lemma 5.2, it suffices to consider only the contracts that involve fully contracted AND gates. This is equivalent to a threshold function with a probability of success β^n for a high effort agent and a probability of success α^n for a low effort agent. Using Theorem 3.12, we get the desired result. □

Lemma 5.5 Consider an integer i such that $i = a \cdot j + b$, where $0 \leq b < j$. Fixing i, the probability of success for a majority-of-ORs function is maximized when $a+1$ agents are contracted on each of b OR gates and a agents are contracted on each of $j - b$ OR gates.

Proof. Consider an allocation of contracts to OR gates: $\hat{x} = (x_1, x_2, ..., x_m)$. If $\hat{x}$ is not the allocation of contracts in which $a + 1$ agents are contracted on each of b OR gates and a agents are contracted on each of $m-b$ OR gates, then there exists an i such that $x_i > a$ and a j such that $x_j < a$. Now consider the agents contracted for the i^{th} OR gate, label these agents $a_1, a_2, ..., a_{x_i}$. Likewise consider the agents contracted for the j^{th} OR gate, label these agents $a'_1, a'_2, ..., a'_{x_j}$. We use the notation $a_l = 1$ if agent l succeeds and $a_l = 0$ if agent l does not succeed. Consider the following three events: $A = (a_1 = 1 \vee a_2 = 1 \vee ... \vee a_{x_i-1} = 1)$, $B = (a_{x_i} = 1)$, and $C = (a'_1 = 1 \vee a'_2 = 1 \vee ... \vee a'_{x_j} = 1)$. We can write the probability of success of a profile $\hat{x}$ in terms of these three events. We can also write the probability of success of the profile $\hat{y}$ in terms of these three events, where $\hat{y}$ is the profile obtained from $\hat{x}$, by contracting one less agent on the i^{th} OR gate and one more agent on the j^{th} OR gate. $p(\hat{x}) = \Pr(A \wedge B \wedge C) \cdot p_{m-2,k-2} + \Pr(A \wedge \overline{B} \wedge C) \cdot p_{m-2,k-2} + \Pr(\overline{A} \wedge B \wedge C) \cdot p_{m-2,k-2} + \Pr(\overline{A} \wedge \overline{B} \wedge C) \cdot p_{m-2,k-1} + \Pr(A \wedge B \wedge \overline{C}) \cdot p_{m-2,k-1} + \Pr(A \wedge \overline{B} \wedge \overline{C}) \cdot p_{m-2,k-1} + \Pr(\overline{A} \wedge B \wedge \overline{C}) \cdot p_{m-2,k-1} + \Pr(\overline{A} \wedge \overline{B} \wedge \overline{C}) \cdot p_{m-2,k}$, where $p_{m,k}$ is the probability that at least k agents succeed out of m. Similarly, $p(\hat{y}) = \Pr(A \wedge B \wedge C) \cdot p_{m-2,k-2} + \Pr(A \wedge \overline{B} \wedge C) \cdot p_{m-2,k-2} + \Pr(\overline{A} \wedge B \wedge C) \cdot p_{m-2,k-1} + \Pr(\overline{A} \wedge \overline{B} \wedge C) \cdot p_{m-2,k-1} + \Pr(A \wedge B \wedge \overline{C}) \cdot p_{m-2,k-2} + \Pr(A \wedge \overline{B} \wedge \overline{C}) \cdot p_{m-2,k-1} + \Pr(\overline{A} \wedge B \wedge \overline{C}) \cdot p_{m-2,k-1} + \Pr(\overline{A} \wedge \overline{B} \wedge \overline{C}) \cdot p_{m-2,k}$. Therefore, we can write $p(\hat{y}) - p(\hat{x}) = (\Pr(A \wedge B \wedge \overline{C}) - \Pr(\overline{A} \wedge B \wedge C)) \cdot (p_{m-2,k-2} - p_{m-2,k-1})$. It is easy to see that $p_{m-2,k-2} - p_{m-2,k-1} > 0$. Note that A, B, C are all independent events, therefore $\Pr(A \wedge B \wedge \overline{C}) = \Pr(A) \cdot \Pr(B) \cdot \Pr(\overline{C})$ and $\Pr(\overline{A} \wedge B \wedge C) = \Pr(\overline{A}) \cdot \Pr(B) \cdot \Pr(C)$. Since $x_i - 1 > x_j$, $\Pr(A) > \Pr(C)$ and $\Pr(\overline{C}) > \Pr(\overline{A})$, which gives us $(\Pr(A \wedge B \wedge \overline{C}) - \Pr(\overline{A} \wedge B \wedge C)) \cdot (p_{m-2,k-2} - p_{m-2,k-1}) > 0$ and $p(\hat{y}) - p(\hat{x}) > 0$ as desired. □

Lemma 5.6 The first transition for the non-strategic case of the majority-of-OR technology jumps from contracting with 0 agents to l agents, where $1 \leq l \leq k$, followed by all remaining transitions, where the transitions proceed in such a way so that no OR gate has more than 1 more agent contracted as compared to any other OR gate.

Proof. First we show that the first transition jumps from 0 to a value of at most k. The majority of OR function which is equivalent to a majority function with $\alpha' = 1-(1-\alpha)^j$ and $\beta' = 1-\alpha(1-\alpha)^{j-1}$. As $\alpha \to 0$, note that the first transition jumps from 0 to k. As α increases, so does $\alpha' + \beta'$, as does the expected number of OR gates that succeed when you contract with k agents. Therefore we know for all α and $\beta = 1 - \alpha$, first transition occurs to a value l that is at most k. Since the first transition jumps from a value 0 to l (where $l \leq k$), there exists a v such that $p_l v - l > p_{l-1} v - (l-1)$ and $p_{l+1} v - (l+1) < p_l v - l$, or in other words, there exists a v such that $\frac{1}{p_{l+1}-p_l} > v > \frac{1}{p_l - p_{l-1}}$. This means that $p_l - p_{l-1} > p_{l+1} - p_l$. From Lemma 3.11, we know that $(p_{l+1} - p_l) - (p_l - p_{l-1}) = (\beta' - \alpha')^2 (P(m-2, l-1, = k-2) - P(m-2, l-1, = k-1))$, where $\alpha' = 1 - (1-\alpha)^j$ and $\beta' = 1 - (1-\beta)(1-\alpha)^{j-1}$. If this is < 0, it means that $P(m-2, l-1, = k-2) < P(m-2, l-1, = k-1)$ and the expected number of agents that succeed when you have $l-1$ agents succeed with probability β' and $m-l-1$ agents succeed with probability α' is greater than $k-1$. Lemma 3.11 also tells us that $p_{l+i} - p_{l+i-1} > p_{l+i+1} - p_{l+i}$, for all $1 \leq i \leq m-l-1$.

Now we consider the marginal increase in probability of success of contracting with the $m+1^{st}$ agent. We can write the marginal contribution of the $m+1^{st}$ agent as $p_{m+1} - p_m = P(m, 1, \geq k) - P(m, 0, \geq k) = (\beta'' - \alpha'')(P(m-1, 0, \geq k-1) - P(m-1, 0, \geq k)) = (\beta'' - \alpha'') P(m-1, 0, = k-1)$ (with $\alpha'' = 1 - (1-\beta)(1-\alpha)^{j-1}$ and $\beta'' = 1 - (1-\beta)^2 (1-\alpha)^{j-2}$). Observe that $p_m - p_{m-1} = P(m, m, \geq k) - P(m, m-1, \geq k) = (\beta' - \alpha')(P(m-1, m-1, \geq k-1) - P(m-1, m-1, \geq k)) = (\beta' - \alpha') P(m-1, m-1, = k-1)$ (with $\alpha' = 1 - (1-\alpha)^j$ and $\beta' = 1 - (1-\beta)(1-\alpha)^{j-1}$). Note that $(1-\beta)(1-\alpha)^{j-2}(\beta - \alpha) < (1-\alpha)^{j-1}(\beta - \alpha)$, so $\beta'' - \alpha'' < \beta' - \alpha'$. Therefore, we know that $p_{m+1} - p_m < p_m - p_{m-1}$. Using the same arguments as above we get that contracting with each successive agent has a smaller marginal increase in probability of success than the previous agent, which gives us the desired result. □